Heterodox Exponential Modalities in Linear Logic

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The Perfect World (or, Linear Logic without Exponential Modalities)

classical	intuitionistic
$\otimes, 1, \parr, \bot$	$\otimes, 1, \multimap$
$\&, \top, \oplus, 0$	$\&, \top, \oplus, 0$
$*$-autonomous categories with fin. products (e.g. $\mathbf{Vect}_k$)	symmetric closed mononidal cats with fin. prods and fin. coprods (e.g. $\mathbf{CMon}$)

Everything is decidable:

- the space of proof search is finite;
- the size of proofs shrinks under cut-elimination (not quite in MALL...).

	provability	**(untyped) cut-elimination** (equality of normal forms)
MLL	NP-complete	P-complete
MALL	PSPACE-complete	coNP-complete

Imperfection (or, Orthodox Exponential Modalities)

cat with fin. prods $\begin{tikzcd} \mathbf{M} \arrow[r, bend left, "\text{strong}"] \arrow[r, phantom, "\perp"] & \mathbf{L} \arrow[l, bend left, "\text{lax symm. mon.}"] \end{tikzcd}$ model of (I)M(A)LL

Examples:

- $(-)^* : \mathbf{Set} \rightleftarrows \mathbf{CMon} : U$
- $U : \mathbf{Set} \rightleftarrows \mathbf{Rel} : P$

Infinity steps in:

	provability	**(untyped) cut-elimination**
MELL	???	(undecidable) non-elementary
LL	undecidable	(undecidable) non-elementary

Not so “God-given”:

- who had heard of LNL adjunctions before linear logic?
- Not determined by $\otimes$ (consider $U : \mathbf{MRel} \rightleftarrows \mathbf{Rel} : M_{\text{fin}}$)

An Alternative Presentation of Linear Logic

Sequents with an "exponential part": $\vdash \Theta; \Gamma$

$$\frac{}{\vdash \Theta; A^\perp, A} \qquad \frac{\vdash \Theta; \Gamma, A^\perp \quad \vdash \Theta; \Delta, A}{\vdash \Theta; \Gamma, \Delta}$$

$$\frac{}{\vdash \Theta; 1} \qquad \frac{\vdash \Theta; \Gamma, A \quad \vdash \Theta; \Delta, B}{\vdash \Theta; \Gamma, \Delta, A \otimes B} \qquad \frac{\vdash \Theta; \Gamma}{\vdash \Theta; \Gamma, \bot} \qquad \frac{\vdash \Theta; \Gamma, A, B}{\vdash \Theta; \Gamma, A \parr B}$$

$$\frac{}{\vdash \Theta; \Gamma, \top} \qquad \frac{\vdash \Theta; \Gamma, A \quad \vdash \Theta; \Gamma, B}{\vdash \Theta; \Gamma, A \,\&\, B} \qquad \frac{\vdash \Theta; \Gamma, A_i}{\vdash \Theta; \Gamma, A_1 \oplus A_2}\, i \in \{1,2\}$$

$$\frac{\vdash \Theta; A}{\vdash \Theta; !A} \qquad \frac{\vdash \Theta, A; \Gamma, A}{\vdash \Theta, A; \Gamma} \qquad \frac{\vdash \Theta, A; \Gamma}{\vdash \Theta; \Gamma, ?A}$$

(First considered by Andreoli for proof search).

The Polynomial Structure of Exponential Modalities

Decorate exponential part with $P_i \in \mathbb{N}[X]$: $\vdash P_1 \cdot A_1, \ldots, P_n \cdot A_n; \Gamma$

$$\frac{}{\vdash \vec{0} \cdot \Theta; A^\perp, A} \qquad \frac{\vdash \vec{P} \cdot \Theta; \Gamma, A^\perp \quad \vdash \vec{Q} \cdot \Theta; \Delta, A}{\vdash \vec{P} + \vec{Q} \cdot \Theta; \Gamma, \Delta}$$

$$\frac{}{\vdash \vec{0} \cdot \Theta; 1} \qquad \frac{\vdash \vec{P} \cdot \Theta; \Gamma, A \quad \vdash \vec{Q} \cdot \Theta; \Delta, B}{\vdash \vec{P} + \vec{Q} \cdot \Theta; \Gamma, \Delta, A \otimes B} \qquad \frac{\vdash \Theta; \Gamma}{\vdash \Theta; \Gamma, \bot} \qquad \frac{\vdash \Theta; \Gamma, A, B}{\vdash \Theta; \Gamma, A \parr B}$$

$$\frac{}{\vdash \vec{0} \cdot \Theta; \Gamma, \top} \qquad \frac{\vdash \vec{P} \cdot \Theta; \Gamma, A \quad \vdash \vec{Q} \cdot \Theta; \Gamma, B}{\vdash \vec{P} + \vec{Q} \cdot \Theta; \Gamma, A \,\&\, B} \qquad \frac{\vdash \Theta; \Gamma, A_i}{\vdash \Theta; \Gamma, A_1 \oplus A_2}\, i \in \{1,2\}$$

$$\frac{\vdash \vec{P} \cdot \Theta; A}{\vdash X\vec{P} \cdot \Theta; !A} \qquad \frac{\vdash \Theta, P \cdot A; \Gamma, A}{\vdash \Theta, P + 1 \cdot A; \Gamma} \qquad \frac{\vdash \Theta, P \cdot A; \Gamma}{\vdash \Theta; \Gamma, ?A}$$

Making structure explicit yields graded modalities (*bounded* LL & co.).

A Family of Heterodox Exponential Modalities

We obtain a subsystem of LL by restricting the shape of P in

$$\frac{\vdash \Theta, P \cdot A; \Gamma}{\vdash \Theta; \Gamma, ?A}$$

Theorem. *For every submonoid M of $(\mathbb{N}[X], \circ, X)$, the subsystem of LL defined by restricting the above rule to $P \in M$ enjoys η-expansion and cut-elimination* (*also, $!(-)$ is always lax monoidal*).
Moreover, if we define

$$0(A) := 1 \qquad 1(A) := A$$
$$(P+Q)(A) := P(A) \otimes Q(A) \qquad (PQ)(A) := P(Q(A))$$
$$X(A) := !A$$

then $P \in M$ implies $!A \multimap P(A)$ provable in the subsystem.

Examples of Systems with Heterodox Modalities

LL

4LL

$\{\deg P \geq 1$ or $P = 0\}$

TLL

$\{\deg P \leq 1\}$

ELL

$\{nX\}$

PLL

$\{aX + n$ with $a \in \{0, 1\}\}$

LLL

$!(-)$ not monoidal

SLL

$\{X\} \cup \mathbb{N}$

"light logics"

parsimonious logic

Main Properties

- 4LL, TLL: [Danos, Joinet 2003]. Stream computation in 4LL [Dal Lago 2016].
- Light logics: enjoy *untyped* normalization.
  - ELL: [Girard 1998] [Danos, Joinet 2003] characterizes elementary time.
  - SLL: [Lafont 2004] characterizes polynomial time.
  - LLL: [Girard 1998] [Danos, Joinet 2003] characterizes polynomial time.
- PLL: [M. 2014] Turing-complete if untyped. With $!A \cong A \otimes !A$:
  - propositional: characterizes logspace [M. 2015];
  - linear 2nd order: characterizes polytime [M. and Terui 2015].
- Two different approaches to control complexity:
  - stratification (light logics) vs. local control (parsimony);
  - parsimony enables *non-uniform complexity* via approximations.

Characterizing Complexity Classes: What and How

Typical Theorem. *For some types* Str *and* Bool, *terms of type*

$$\mathsf{Str} \multimap \mathsf{Bool}$$

decide exactly the problems in the complexity class C.

TYPICAL PROOF.

Soundness: (decidable by a term implies in C)

Find a parameter d such that:

- terms of size s and parameter d normalize in $O(f(d, s))$ time/space;
- terms of type Str have constant parameter d and size $O(n)$ where n is the length of the represented string;
- for constant k, the bound $O(f(k, n))$ ensures membership in C.

The proof may be combinatorial or semantic.
For light logics, d does *not* depend on the type of the term.
For logspace, use the GoI (normalization via traveling pointers).

Completeness: (in C implies decidable by a term)

A programming excercise (maybe non-trivial). $\square$

Approximations (or, Exponential Modalities are Limits)

Relation $t \sqsubset M$ between simple programs and programs
(and between simple computations and computations) with cost maps

$$
\begin{tikzcd}
\text{Apx terms} \arrow[dr, "c_0"'] & & \text{Computations} \arrow[dl, "c_1"] \\
& \text{Cost} &
\end{tikzcd}
$$

Such that

$$
\begin{array}{ccc}
\begin{array}{ccc} & & u \\ & & \sqcap \\ M & \xrightarrow[\rho]{} & N \end{array}
& \text{iff} &
\begin{array}{ccc} t & \longrightarrow & u \\ \sqcap & & \\ M & & \end{array}
\end{array}
$$

$$c_1(\rho) = c_0(t)$$

Conclusions

- Light logics are dead, long live heterodox exponentials!
- Categorical models?
- Limit constructions and approximations?
- Where do approximations come from?