

# Heterodox Exponential Modalities in Linear Logic

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CIRM, 28 January 2022

# The Perfect World (or, Linear Logic without Exponential Modalities)

## classical

$\otimes, 1, \wp, \perp$

$\&, \top, \oplus, 0$

\*-autonomous categories  
with fin. products  
(e.g.  $\mathbf{Vect}_k$ )

## intuitionistic

$\otimes, 1, \multimap$

$\&, \top, \oplus, 0$

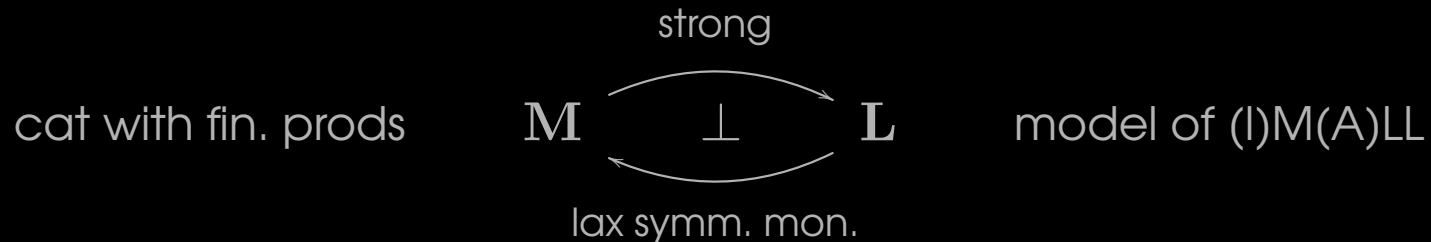
symmetric closed monoidal cats  
with fin. prods and fin. coprods  
(e.g.  $\mathbf{CMon}$ )

Everything is decidable:

- the space of proof search is finite;
- the size of proofs shrinks under cut-elimination (not quite in MALL...).

	provability	(untyped) cut-elimination (equality of normal forms)
MLL	NP-complete	P-complete
MALL	PSPACE-complete	coNP-complete

## Imperfection (or, Orthodox Exponential Modalities)



Examples:

- $(-)^* : \mathbf{Set} \rightleftarrows \mathbf{CMon} : U$
- $U : \mathbf{Set} \rightleftarrows \mathbf{Rel} : P$

Infinity steps in:

	provability	(untyped) cut-elimination
MELL	???	(undecidable) non-elementary
LL	undecidable	(undecidable) non-elementary

Not so “God-given”:

- who had heard of LNL adjunctions before linear logic?
- Not determined by  $\otimes$  (consider  $U : \mathbf{MRel} \rightleftarrows \mathbf{Rel} : M_{\text{fin}}$ )

## An Alternative Presentation of Linear Logic

Sequents with an “exponential part”:  $\vdash \Theta; \Gamma$

$$\begin{array}{c}
 \frac{}{\vdash \Theta; A^\perp, A} \quad \frac{\vdash \Theta; \Gamma, A^\perp \quad \vdash \Theta; \Delta, A}{\vdash \Theta; \Gamma, \Delta} \\
 \\
 \frac{}{\vdash \Theta; 1} \quad \frac{\vdash \Theta; \Gamma, A \quad \vdash \Theta; \Delta, B}{\vdash \Theta; \Gamma, \Delta, A \otimes B} \quad \frac{\vdash \Theta; \Gamma}{\vdash \Theta; \Gamma, \perp} \quad \frac{\vdash \Theta; \Gamma, A, B}{\vdash \Theta; \Gamma, A \wp B} \\
 \\
 \frac{}{\vdash \Theta; \Gamma, \top} \quad \frac{\vdash \Theta; \Gamma, A \quad \vdash \Theta; \Gamma, B}{\vdash \Theta; \Gamma, A \& B} \quad \frac{\vdash \Theta; \Gamma, A_i}{\vdash \Theta; \Gamma, A_1 \oplus A_2} \quad i \in \{1, 2\} \\
 \\
 \frac{\vdash \Theta; A}{\vdash \Theta; !A} \quad \frac{\vdash \Theta, A; \Gamma, A}{\vdash \Theta, A; \Gamma} \quad \frac{\vdash \Theta, A; \Gamma}{\vdash \Theta; \Gamma, ?A}
 \end{array}$$

(First considered by Andreoli for proof search).

## The Polynomial Structure of Exponential Modalities

Decorate exponential part with  $P_i \in \mathbb{N}[X]$ :  $\vdash P_1 \cdot A_1, \dots, P_n \cdot A_n; \Gamma$

$$\begin{array}{c}
 \frac{}{\vdash \vec{0} \cdot \Theta; A^\perp, A} \quad \frac{\vdash \vec{P} \cdot \Theta; \Gamma, A^\perp \quad \vdash \vec{Q} \cdot \Theta; \Delta, A}{\vdash \vec{P} + \vec{Q} \cdot \Theta; \Gamma, \Delta} \\
 \\
 \frac{}{\vdash \vec{0} \cdot \Theta; 1} \quad \frac{\vdash \vec{P} \cdot \Theta; \Gamma, A \quad \vdash \vec{Q} \cdot \Theta; \Delta, B}{\vdash \vec{P} + \vec{Q} \cdot \Theta; \Gamma, \Delta, A \otimes B} \quad \frac{\vdash \Theta; \Gamma}{\vdash \Theta; \Gamma, \perp} \quad \frac{\vdash \Theta; \Gamma, A, B}{\vdash \Theta; \Gamma, A \wp B} \\
 \\
 \frac{}{\vdash \vec{0} \cdot \Theta; \top} \quad \frac{\vdash \vec{P} \cdot \Theta; \Gamma, A \quad \vdash \vec{Q} \cdot \Theta; \Gamma, B}{\vdash \vec{P} + \vec{Q} \cdot \Theta; \Gamma, A \& B} \quad \frac{\vdash \Theta; \Gamma, A_i}{\vdash \Theta; \Gamma, A_1 \oplus A_2}^{i \in \{1,2\}} \\
 \\
 \frac{\vdash \vec{P} \cdot \Theta; A}{\vdash X\vec{P} \cdot \Theta; !A} \quad \frac{\vdash \Theta, P \cdot A; \Gamma, A}{\vdash \Theta, P + 1 \cdot A; \Gamma} \quad \frac{\vdash \Theta, P \cdot A; \Gamma}{\vdash \Theta; \Gamma, ?A}
 \end{array}$$

Making structure explicit yields **graded modalities** (*bounded LL & co.*).

## A Family of Heterodox Exponential Modalities

We obtain a subsystem of LL by **restricting** the shape of  $P$  in

$$\frac{\vdash \Theta, P \cdot A; \Gamma}{\vdash \Theta; \Gamma, ?A}$$

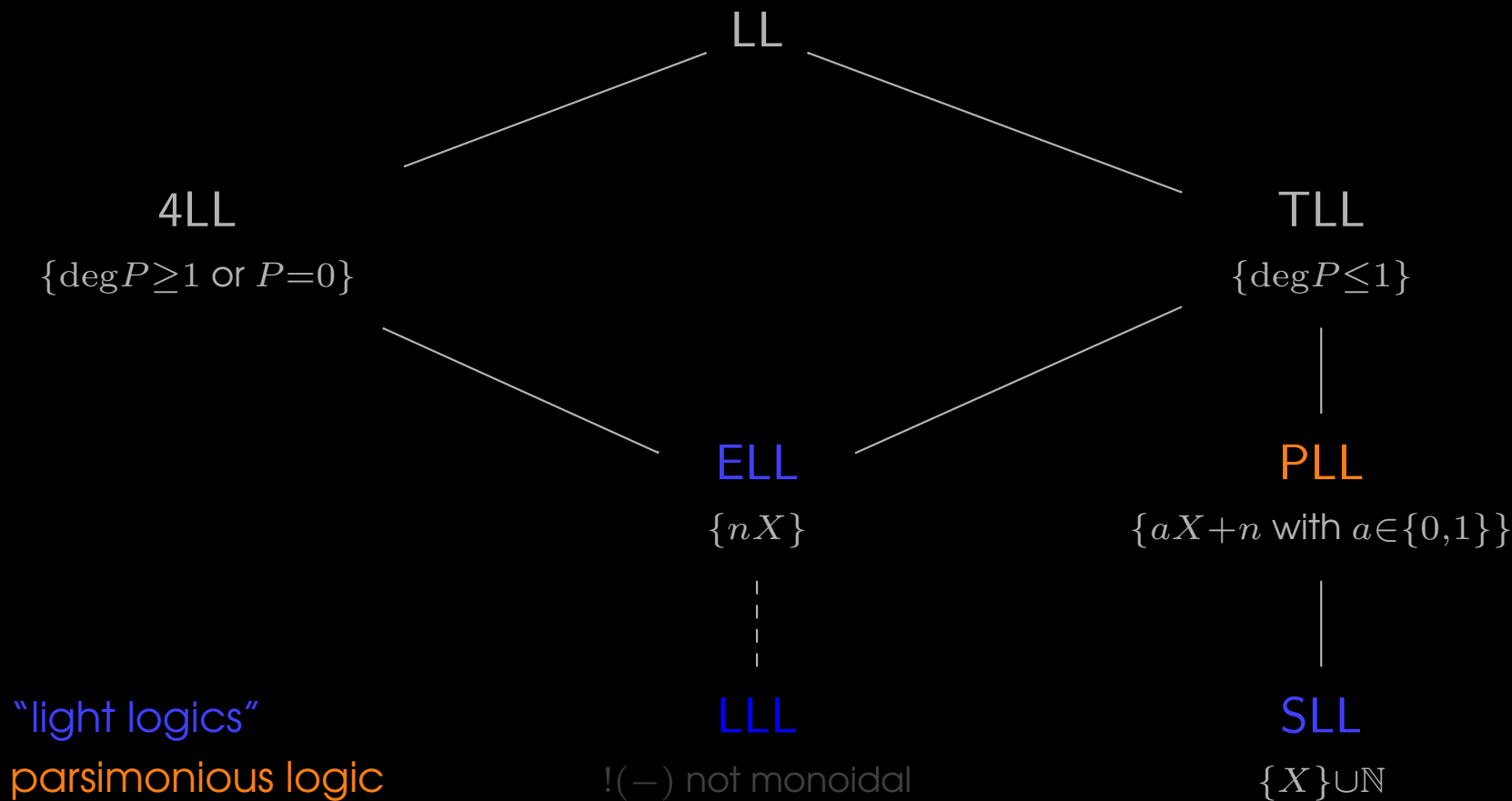
**Theorem.** *For every submonoid  $M$  of  $(\mathbb{N}[X], \circ, X)$ , the subsystem of LL defined by restricting the above rule to  $P \in M$  enjoys  $\eta$ -expansion and cut-elimination (also,  $!(-)$  is always lax monoidal).*

*Moreover, if we define*

$$\begin{aligned} 0(A) &:= 1 & 1(A) &:= A \\ (P + Q)(A) &:= P(A) \otimes Q(A) & (PQ)(A) &:= P(Q(A)) \\ X(A) &:= !A \end{aligned}$$

*then  $P \in M$  implies  $!A \multimap P(A)$  provable in the subsystem.*

# Examples of Systems with Heterodox Modalities



## Main Properties

- 4LL, TLL: [Danos, Joinet 2003]. Stream computation in 4LL [Dal Lago 2016].
- Light logics: enjoy *untyped* normalization.
  - ELL: [Girard 1998] [Danos, Joinet 2003] characterizes elementary time.
  - SLL: [Lafont 2004] characterizes polynomial time.
  - LLL: [Girard 1998] [Danos, Joinet 2003] characterizes polynomial time.
- PLL: [M. 2014] Turing-complete if untyped. With  $!A \cong A \otimes !A$ :
  - **propositional**: characterizes logspace [M. 2015];
  - **linear 2nd order**: characterizes polytime [M. and Terui 2015].
- Two different approaches to control complexity:
  - **stratification** (light logics) vs. **local control** (parsimony);
  - parsimony enables *non-uniform complexity* via approximations.



## Characterizing Complexity Classes: What and How

**Typical Theorem.** For some types *Str* and *Bool*, terms of type

$$\text{Str} \multimap \text{Bool}$$

decide exactly the problems in the complexity class *C*.

TYPICAL PROOF.

**Soundness:** (decidable by a term implies in *C*)

Find a parameter *d* such that:

- terms of size *s* and parameter *d* normalize in  $O(f(d, s))$  time/space;
- terms of type *Str* have constant parameter *d* and size  $O(n)$  where *n* is the length of the represented string;
- for constant *k*, the bound  $O(f(k, n))$  ensures membership in *C*.

The proof may be combinatorial or semantic.

For light logics, *d* does *not* depend on the type of the term.

For logspace, use the Gol (normalization via traveling pointers).

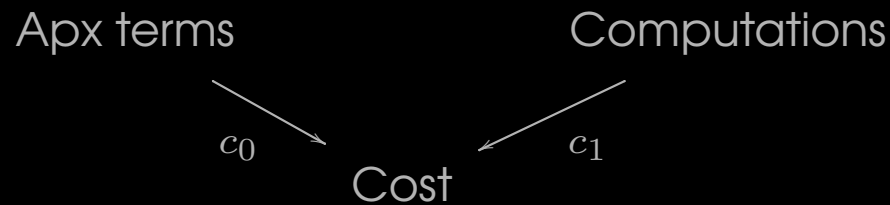
**Completeness:** (in *C* implies decidable by a term)

A programming exercise (maybe non-trivial).



# Approximations (or, Exponential Modalities are Limits)

Relation  $t \sqsubseteq M$  between simple programs and programs (and between simple computations and computations) with cost maps



Such that

$$\begin{array}{ccc}
 u & & t \longrightarrow u \\
 \sqcap & \text{iff} & \sqcap \\
 M \xrightarrow[\rho]{} N & & M \\
 c_1(\rho) = c_0(t) & & 
 \end{array}$$

## Conclusions

- Light logics are dead, long live heterodox exponentials!
- Categorical models?
- Limit constructions and approximations?
- Where do approximations come from?