CIRM LL School
January 2022

Differential Linear Logic extended to Differential Operators

Marie Kerjean

CNRS
Université Sorbonne Paris Nord
LIPN, LoVe Team

Syntax and semantics

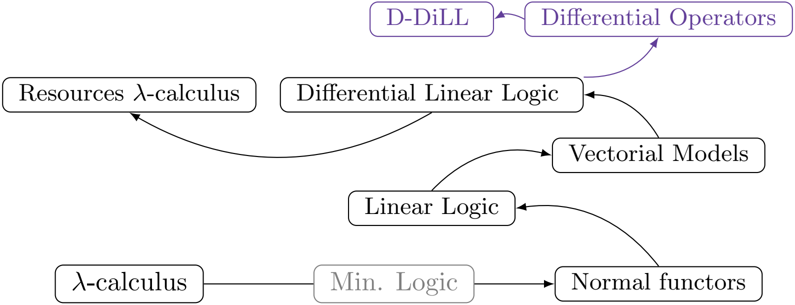
The syntax mirrors the semantics.

Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.
Types	Formulas	Objects
Execution	Cut-elimination	Equality

Syntax and semantics

The syntax mirrors the semantics.

Programs	**Logic**	**Semantics**
`fun (x:A)-> (t:B)`	Proof of $A \vdash B$	$f : A \to B$.
Types	Formulas	Objects
Execution	Cut-elimination	Equality



1. Syntax and semantics of (Differential) Linear Logic

2. Semantics with Distributions

3. Syntax with Differential Operators

Syntax and semantics of Linear Logic

Linear Logic

Syntax : $A \Rightarrow B = !\, A \multimap; B$
Semantics : $\mathcal{C}^\infty(A, B) \simeq \mathcal{L}(!A, B)$

A focus on linearity

- Higher-Order is about *Seely's isomoprhism.*

$$!A \otimes !B \simeq !(A \,\&\, B)$$

$$!A \hat{\otimes} !B \simeq !(A \times B)$$

- Classicality is about a linear involutive negation :

$$[\![A^\perp]\!] = \mathcal{L}([\![A]\!], \mathbb{R})$$

$A^{\perp\perp} \simeq A$ $\qquad$ $A \simeq A''$

Just a glimpse at Differential Linear Logic

$$A, B := A \otimes B | 1 | A ⅋ B | \bot | A \oplus B | 0 | A \times B | \top | !A | !A$$

Exponential rules of DiLL$_0$

$$\frac{!A, !A \vdash \Gamma}{!A \vdash \Gamma}\, c \qquad \frac{\vdash \Gamma}{!A \vdash \Gamma}\, w \qquad \frac{A \vdash \Gamma}{?A \vdash \Gamma, ?A}\, d$$

$$\frac{\vdash \Gamma, !A, \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\, \bar{c} \qquad \frac{\vdash}{\vdash !A}\, \bar{w} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A}\, \bar{d}$$

- *Normal functors, power series and λ-calculus.* Girard, APAL(1988)
- *Differential interaction nets*, Ehrhard and Regnier, TCS (2006)

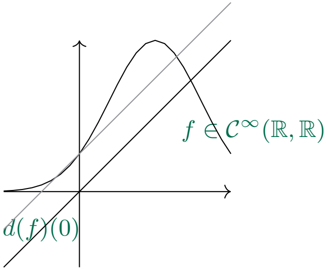
Differential Linear Logic

$$\frac{\ell : A \vdash B}{\ell : !A \vdash B}\ d$$

linear $\hookrightarrow$ *non-linear.*

$$\frac{f : !A \vdash B}{D_0(f) : A \vdash B}\ \bar{d}$$

non-linear $\hookrightarrow$ *linear*



Differential interaction nets, Ehrhard and Regnier, TCS (2006)

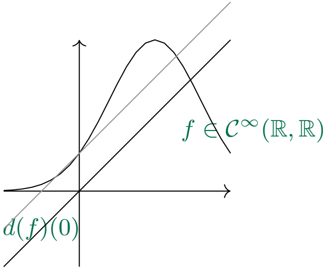
Differential Linear Logic

$$\frac{\vdash \Gamma, \ell : A^{\perp}}{\vdash \Gamma, \ell : ?A^{\perp}}\ d$$

linear $\hookrightarrow$ *non-linear.*

$$\frac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A}\ \bar{d}$$

non-linear $\hookrightarrow$ *linear*



Differential interaction nets, Ehrhard and Regnier, TCS (2006)

Differential Linear Logic

$$\frac{\vdash \Gamma, \ell : A^{\perp}}{\vdash \Gamma, \ell : ?A^{\perp}}\, {\color{red} d}$$

linear $\hookrightarrow$ *non-linear.*

$$\frac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A}\, {\color{red} \bar{d}}$$

non-linear $\hookrightarrow$ *linear*

Cut-elimination:

$$\frac{\dfrac{\vdash \Gamma, v : !A}{\vdash \Gamma, !A}\,{\color{red}\bar{d}} \qquad \dfrac{\vdash \Delta, A^{\perp}}{\vdash \Delta, ?A^{\perp}}\,{\color{red}d}}{\vdash \Gamma, \Delta}\,\mathrm{cut} \quad \rightsquigarrow \quad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\Gamma, \Delta}\,\mathrm{cut}$$

Differential interaction nets, Ehrhard and Regnier, TCS (2006)

Differential Linear Logic

$$\dfrac{\vdash \Gamma, \ell : A^{\perp}}{\vdash \Gamma, \ell : ?A^{\perp}}\ d$$

linear ↪ non-linear.

$$\dfrac{\vdash \Delta, v : A}{\vdash \Delta, (f \mapsto D_0(f)(v)) : !A}\ \bar{d}$$

non-linear ↪ linear

Cut-elimination:

$$\dfrac{\dfrac{\vdash \Gamma, v : A}{\vdash \Gamma, D_0(\_)(v) : !A}\ \bar{d} \qquad \dfrac{\vdash \Delta, \ell : A^{\perp}}{\vdash \Delta, \ell : ?A^{\perp}}\ d}{\Gamma, \Delta}\ \text{cut}$$

$$\rightsquigarrow \quad \dfrac{\vdash \Gamma, v : A \qquad \vdash \Delta, \ell : A^{\perp}}{\vdash \Gamma, \Delta, D_0(\ell)(x) = \ell(x) : \mathbb{R} = \perp}\ \text{cut}$$

Differential interaction nets, Ehrhard and Regnier, TCS (2006)

That's the resolution of a Differential Equation !

Smooth and classical models of Differential Linear Logic

What's the good category in which we interpret formulas ?

- Exponentials are distributions
- From reflexivity to polarities

Exponential as Distributions

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\begin{array}{lll} \phi_f : & g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \mapsto \int fg \\ \delta_a : & g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \mapsto g(a) \end{array}$$

Théorie des distributions, Schwartz, 1947.

Exponential as Distributions

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\begin{array}{lll} \phi_f : & g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \mapsto \int fg \\ \delta_a : & g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \mapsto g(a) \end{array}$$

Théorie des distributions, Schwartz, 1947.

- In a classical models of Differential Linear Logic, the exponential is a space of distributions with compact support.

$$!A \multimap \bot = A \Rightarrow \bot$$

Exponential as Distributions

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\begin{array}{llll} \phi_f : & g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \mapsto \int fg \\ \delta_a : & g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \mapsto g(a) \end{array}$$

Théorie des distributions, Schwartz, 1947.

- In a classical models of Differential Linear Logic, the exponential is a space of distributions with compact support.

$$!A \multimap \bot = A \Rightarrow \bot$$
$$\mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$

Exponential as Distributions

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\begin{array}{lll} \phi_f : & g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \mapsto \int fg \\ \delta_a : & g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \mapsto g(a) \end{array}$$

Théorie des distributions, Schwartz, 1947.

- In a classical models of Differential Linear Logic, the exponential is a space of distributions with compact support.

$$!A \multimap \perp = A \Rightarrow \perp$$
$$\mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$
$$(!E)'' \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$

Exponential as Distributions

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\begin{array}{lll} \phi_f : & g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \mapsto \int fg \\ \delta_a : & g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \mapsto g(a) \end{array}$$

Théorie des distributions, Schwartz, 1947.

- In a classical models of Differential Linear Logic, the exponential is a space of distributions with compact support.

$$!A \multimap \perp = A \Rightarrow \perp$$
$$\mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$
$$(!E)'' \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$
$$!E \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$

Exponential as Distributions

- Distributions with compact support are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'$, seen as generalisations of functions with compact support:

$$\begin{array}{lll} \phi_f : & g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \mapsto \int fg \\ \delta_a : & g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \mapsto g(a) \end{array}$$

Théorie des distributions, Schwartz, 1947.

- In a classical models of Differential Linear Logic, the exponential is a space of distributions with compact support.

$$!A \multimap \bot = A \Rightarrow \bot$$
$$\mathcal{L}(!E, \mathbb{R}) \simeq \mathcal{C}^\infty(E, \mathbb{R})$$
$$(!E)'' \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$
$$!E \simeq \mathcal{C}^\infty(E, \mathbb{R})'$$

A convenient differential category, Blute, Ehrhard, Tasson 2012

- Seely's isomorphism corresponds to the *Kernel theorem*:

$$\mathcal{C}^\infty(E, \mathbb{R})' \tilde{\otimes} \mathcal{C}^\infty(F, \mathbb{R})' \simeq \mathcal{C}^\infty(E \times F, \mathbb{R})'$$

Smoothness and Duality

Smoothness

Spaces : E is a **locally convex** and **Haussdorf** topological vector space.
Functions: $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is infinitely and everywhere differentiable.

The two requirements works as opposite forces .

✓ A cartesian closed category with smooth functions.

$\rightsquigarrow$ Completeness, and a dual topology fine enough.

✓ Interpreting $(E^\perp)^\perp \simeq E$ without an orthogonality:

$\rightsquigarrow$ Reflexivity : $E \simeq E''$, and a dual topology coarse enough.

.

What's not working

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

$$\dim \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m) = \infty.$$

What's not working

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

$$\dim \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m) = \infty.$$

We can't restrict ourselves to finite dimensional spaces.

The tentative to have a normed space of analytic functions fails (Girard's Coherent Banach spaces).

- We want to use power series.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can't be bounded on an unbounded space (Liouville's Theorem).
- Thus functions must depart from an open ball, but arrive in a closed ball. Thus they do not compose.
- This is why Coherent Banach spaces don't work.

What's not working

A space of (non necessarily linear) functions between finite dimensional spaces is not finite dimensional.

$$\dim \mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m) = \infty.$$

We can't restrict ourselves to finite dimensional spaces.

The tentative to have a normed space of analytic functions fails (Girard's Coherent Banach spaces).

- We want to use power series.
- For polarity reasons, we want the supremum norm on spaces of power series.
- But a power series can't be bounded on an unbounded space (Liouville's Theorem).
- Thus functions must depart from an open ball, but arrive in a closed ball. Thus they do not compose.
- This is why Coherent Banach spaces don't work.

We can't restrict ourselves to normed spaces.

MLL in TopVect

Topological vector spaces.

It's a mess.

Duality is not an orthogonality in general :

- It depends of the topology E'_β, E'_c, E'_w, E'_μ on the dual.
- It is typically *not* preserved by $\otimes$.
- It is in the canonical case not an orthogonality : E'_β is not reflexive.
  $E \nrightarrow E^{\perp\perp}$

Monoidal closedness does not extends easily to the topological case :

- Many possible topologies on $\otimes$: $\otimes_\beta$, $\otimes_\pi$, $\otimes_\varepsilon$.
- $\mathcal{L}_\mathcal{B}(E \otimes_\mathcal{B} F, G) \simeq \mathcal{L}_\mathcal{B}(E, \mathcal{L}_\mathcal{B}(F, G))$
  $\Leftrightarrow$ "Grothendieck problème des topologies".

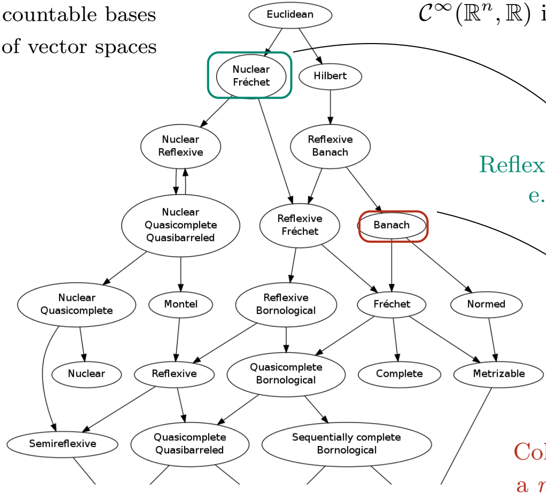
Topological models of DiLL

[Ehr02] [Ehr05] [DE08]

countable bases

of vector spaces

$\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ is not finite dimensional

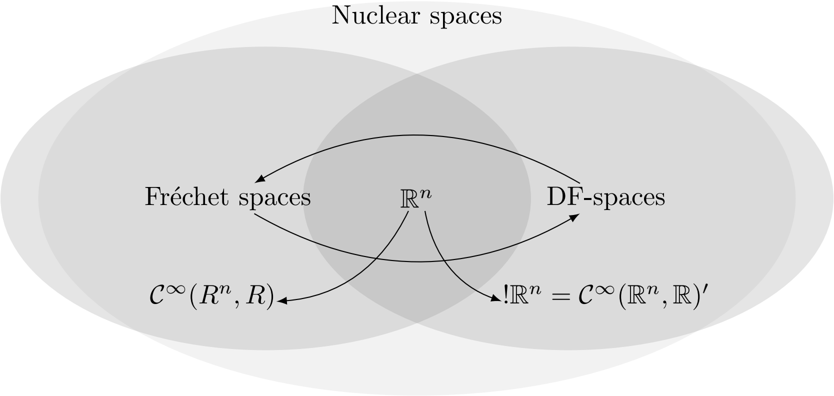


Reflexive anc complete :
e.g. $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$

Coherent Banach spaces [Gir99]
a *norm* is too restrictive

A polarized model of Smooth differential Linear Logic

Typical Nuclear Fréchet spaces are spaces of functions $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, $\mathcal{H}(\mathbb{R}^n, \mathbb{R})$.



And more : $\uparrow$ is the completion $\rightsquigarrow$ Chiralities *[Mellies]*.

A Logical account for Linear Partial Differential Equations

Linear functions as solutions to a Differential equation

Slogan : From Linearity/Non-linearity to
Solutions/Parameter of a differential equation.

$f \in \mathcal{C}^\infty(A, \mathbb{R})$ is linear

Linear functions as solutions to a Differential equation

Slogan : From Linearity/Non-linearity to Solutions/Parameter of a differential equation.

$f \in \mathcal{C}^\infty(A, \mathbb{R})$ is linear *iff* $\forall x, f(x) = D_0(f)(x)$

Linear functions as solutions to a Differential equation

Slogan : From Linearity/Non-linearity to
Solutions/Parameter of a differential equation.

$$f \in \mathcal{C}^\infty(A, \mathbb{R}) \text{ is linear} \quad \textit{iff } \forall x, f(x) = D_0(f)(x)$$
$$\textit{iff } \exists g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), f = \bar{d} g$$

Linear functions as solutions to a Differential equation

Slogan : From Linearity/Non-linearity to
Solutions/Parameter of a differential equation.

$$
\begin{aligned}
f \in \mathcal{C}^\infty(A, \mathbb{R}) \text{ is linear} \quad & \textit{iff } \forall x, f(x) = D_0(f)(x) \\
& \textit{iff } \exists g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), f = \bar{d}g \\
\phi \in A'' \simeq A \quad & \textit{iff } \exists \psi \in !A, \phi \circ D_0 = \psi
\end{aligned}
$$

Linear functions as solutions to a Differential equation

Slogan : From Linearity/Non-linearity to Solutions/Parameter of a differential equation.

$$\begin{aligned}
f \in \mathcal{C}^\infty(A, \mathbb{R}) \text{ is linear} \quad & \textit{iff } \forall x, f(x) = D_0(f)(x) \\
& \textit{iff } \exists g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), f = \bar{d} g \\
\phi \in A'' \simeq A \quad & \textit{iff } \exists \psi \in !A, \phi \circ D_0 = \psi \\
\phi \in A'' \simeq A \quad & \textit{iff } \exists \psi \in !A, D_0(\phi) = \psi
\end{aligned}$$

Linear functions as solutions to a Differential equation

Slogan : From Linearity/Non-linearity to Solutions/Parameter of a differential equation.

$$\begin{aligned}
f \in \mathcal{C}^\infty(A, \mathbb{R}) \text{ is linear} \quad & \textit{iff } \forall x, f(x) = D_0(f)(x) \\
& \textit{iff } \exists g \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}), f = \bar{d}g \\
\phi \in A'' \simeq A \quad & \textit{iff } \exists \psi \in !A, \phi \circ D_0 = \psi \\
\phi \in A'' \simeq A \quad & \textit{iff } \exists \psi \in !A, D_0(\phi) = \psi \\
\phi \in !_D A \quad & \textit{iff } \exists \psi \in !A, D(\phi) = \psi
\end{aligned}$$

Another exponential is possible

$$!_D E := (D(\mathcal{C}^\infty(E, \mathbb{R})') \subset (\mathcal{C}^\infty_c(E, \mathbb{R}))'$$

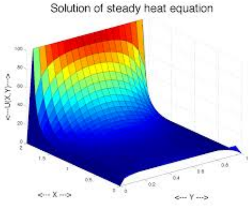
The exponential is the space of solutions to a differential equation.

- $!_{D_0} E := E'' \simeq E.$
- $!_{Id} E := !E = \mathcal{C}^\infty(E, \mathbb{R})'.$

Linear Partial Differential Equations with constant coefficients

Consider D a LPDO with constant coefficients: $D = \sum_{\alpha, |\alpha| \leq n} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}$.

The heat equation in $\mathbb{R}^2$
$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$
$u(x, y, 0) = f(x, y)$



Theorem (Malgrange 1956)

For any D LPDOcc, there is $E_D \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})'$ such that :

$$D(E_D) = \delta_0$$

and thus : output $D(E_D * \phi) = \phi$ input

D-DiLL

DiLL

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ w \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ c \qquad \frac{\vdash \Gamma, \ell : A}{\vdash \Gamma, \ell : ?A}\ d$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, !A}\ \bar{w} \qquad \frac{\vdash \Gamma, !A \qquad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A}\ \bar{c} \qquad \frac{\vdash \Gamma, x : A}{\vdash \Gamma, D_0(\_)(x)!A}\ \bar{d}$$

$D - DiLL$: dereliction solves / codereliction applies

$$\frac{\vdash \Gamma}{\vdash \Gamma, D(cst_1) : ?_D A}\ w_D \qquad \frac{\vdash \Gamma, ?A, ?_D A}{\vdash \Gamma, ?_D A}\ c \qquad \frac{\vdash \Gamma, f : ?_D A}{\vdash \Gamma, f * E_D : ?A}\ d_D$$

$$\frac{\vdash}{\vdash E_D : !_D A}\ \bar{w}_D \qquad \frac{\vdash \Gamma, \phi : !A \qquad \vdash \Delta, \psi : !_D A}{\vdash \Gamma, \Delta, !_D A}\ \bar{c}_D \qquad \frac{\vdash \Gamma, \psi : !_D A}{\vdash \Gamma, D\psi : !A}\ \bar{d}_D$$

A Logical Account for LPDEs, K. LICS 2018.

Conclusion

A few insights:

- To Linear Logic principles correspond Functional Analysis tools.
- Being dereliction and co-dereliction hides the application and resolution of some differential equation.

A LOT of questions:

- How do we mix LPDOs between them (S. Mirwasser, F. Breuvart)?
  - What's formal link with Indexed Logics ?
- Does it extend to other differential operators ?
  - Approximate methods of resolutions ?
  - Computational content of ODE's ?
- How do they act on higher-order functions ?