Exponential as Substitutions and the Cost of Cut Elimination in Linear Logic

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LL, Resources, & Programs

LL is a resource-conscious logic.

Curry-Howard:
β-reduction decomposes through LL cut elimination.

Even the untyped λ-calculus can be represented.

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Given a λ-term t represented as a LL proof π_t.

π_t normalizes in n steps of cut-elimination.

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Let π be a proof of MELL.

At present, no relationships between:

The # of steps of cut elimination of π,

The complexity of eliminating cuts from π.

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Polynomial Cost Models

Ideal (first) solution:

A cut elimination strategy $\rightarrow_x$

of which the # of steps is a polynomial cost model.

That is:
The cost of implementing $\pi \rightarrow_x^n \pi'$ is polynomial in $|\pi|$ and n.

CAREFUL: n can be whatever, it need not being polynomial in $|\pi|$.

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Good understanding of time cost models for the λ-calculus.

Good news: now of space ones too.

Enabled by a perfect dissection of meta-level substitution.

Namely, Accattoli and Kesner's Linear Substitution Calculus (LSC).

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Decompose β:

$$(\lambda x.t)s \;\to_{\beta}\; t\{x \leftarrow s\}$$

via a new constructor $t[x \leftarrow s]$ and two steps:

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There are many (many!) possible sets of such rules.

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Rules for $\rightarrow_{\mathtt{S}}$ are always strongly normalizing.

Somewhat surprising:
Counting only $\rightarrow_{\mathtt{B}}$ steps gives a polynomial cost model.

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Similar idea in LL: exponentials decompose substitutions.

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Linear Substitution Calculus (LSC)

Terms and contexts:

$$t, s, u \quad := \quad x \quad | \quad \lambda x.t \quad | \quad ts \quad | \quad t[x \leftarrow s]$$

$$C \quad := \quad \langle \cdot \rangle \quad | \quad Ct \quad | \quad tC \quad | \quad \lambda x.C \quad | \quad C[x \leftarrow t] \quad | \quad t[x \leftarrow C]$$

Lists of substitutions:

$$L \quad := \quad \langle \cdot \rangle \quad | \quad L[x \leftarrow t]$$

Rewriting rules (closed by contexts $C\langle \cdot \rangle$):

distant **B**eta	$L\langle \lambda x.t \rangle s$	$\to_{\mathtt{dB}}$	$L\langle t[x \leftarrow s] \rangle$	
linear **s**ubstitution	$C\langle x \rangle [x \leftarrow s]$	$\to_{\mathtt{ms}}$	$C\langle s \rangle [x \leftarrow s]$	
garbage **c**ollection	$t[x \leftarrow s]$	$\to_{\mathtt{gc}}$	t	if $x \notin \mathtt{fv}(t)$

Outline

The Sub-Term Property

Strategies of the LSC naturally have the sub-term property:

> all sub-terms duplicated or erased
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> are sub-terms of t.

This is the key property for the study of cost models.

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Sub-Term Property and Polynomial Cost Models

Suppose that $\to_x$ has the sub-term property and is micro-step.

Then $t \to_x^n s$ is implementable in time polynomial in $|t|$ and n.

Usually in bi-linear time.

Therefore:

sub-term property + micro-steps $\Rightarrow$ polynomial cost model.

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λ-Calculus *vs* Sub-Term Property

In the λ-calculus, no strategy has the sub-term property.

Consider $\tau_t := \lambda y.ytt$ and $\mathtt{I} := \lambda x.x$. Then:

$$\begin{array}{rll} s := & (\lambda x.x(\lambda z.\tau_z)\tau_x)\mathtt{I} & \to_\beta \\ & \mathtt{I}(\lambda z.\tau_z)\tau_{\mathtt{I}} & \to_\beta \\ & (\lambda z.\tau_z)\tau_{\mathtt{I}} & \to_\beta \quad \tau_{\tau_{\mathtt{I}}} \end{array}$$

There never is a choice, each term has only one redex.

The third step duplicates $\tau_{\mathtt{I}}$ which is not a sub-term of s.

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Lack of sub-term property leads to size explosion.

$$s_n \to_\beta^n u_n \qquad \text{with } |s_n| = \mathcal{O}(n) \text{ and } |u_n| = \Omega(2^n)$$

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Mandatory also for logarithmic space.

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Linear logic models micro-step well.

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At present, no strategies with the sub-term property.

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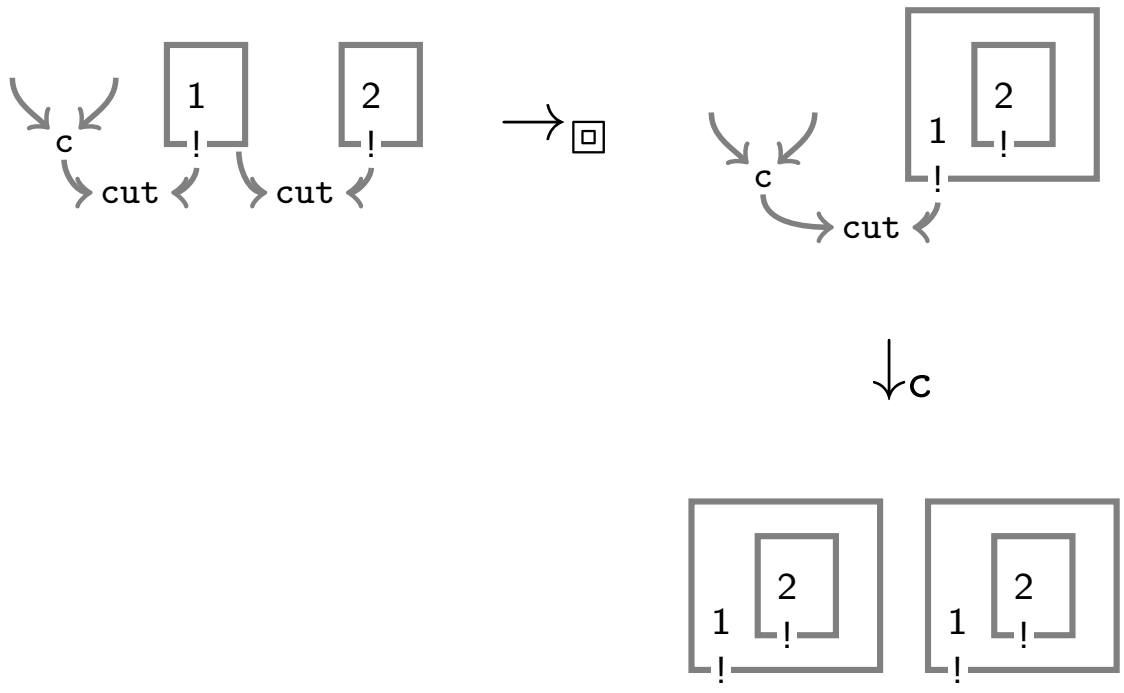

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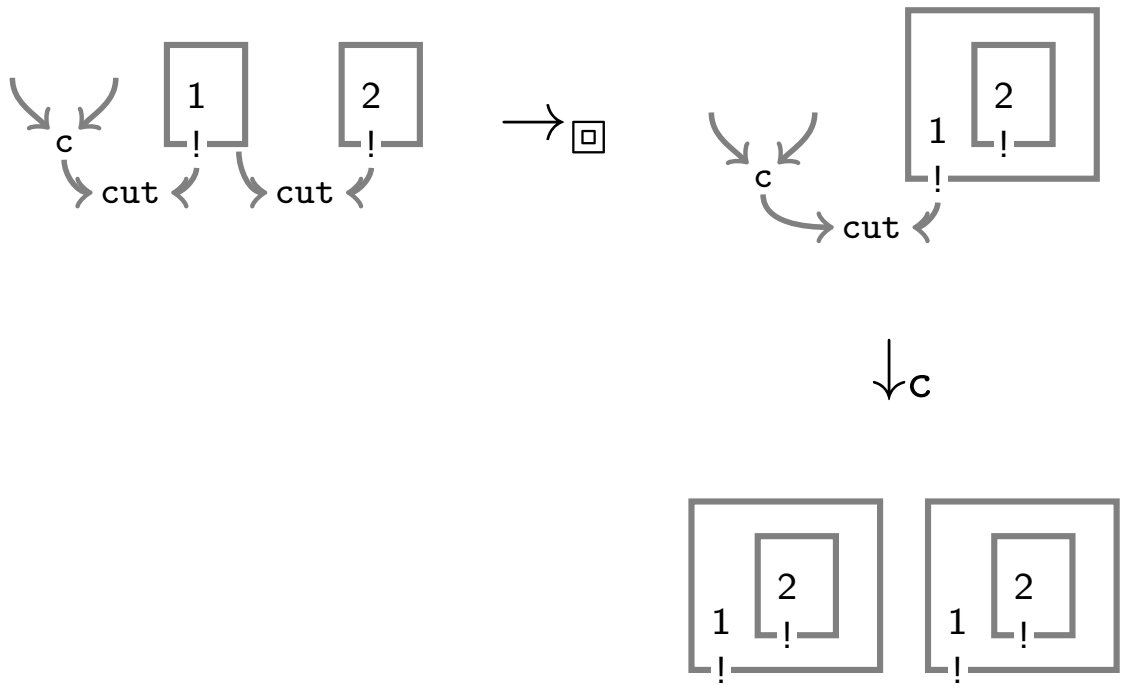

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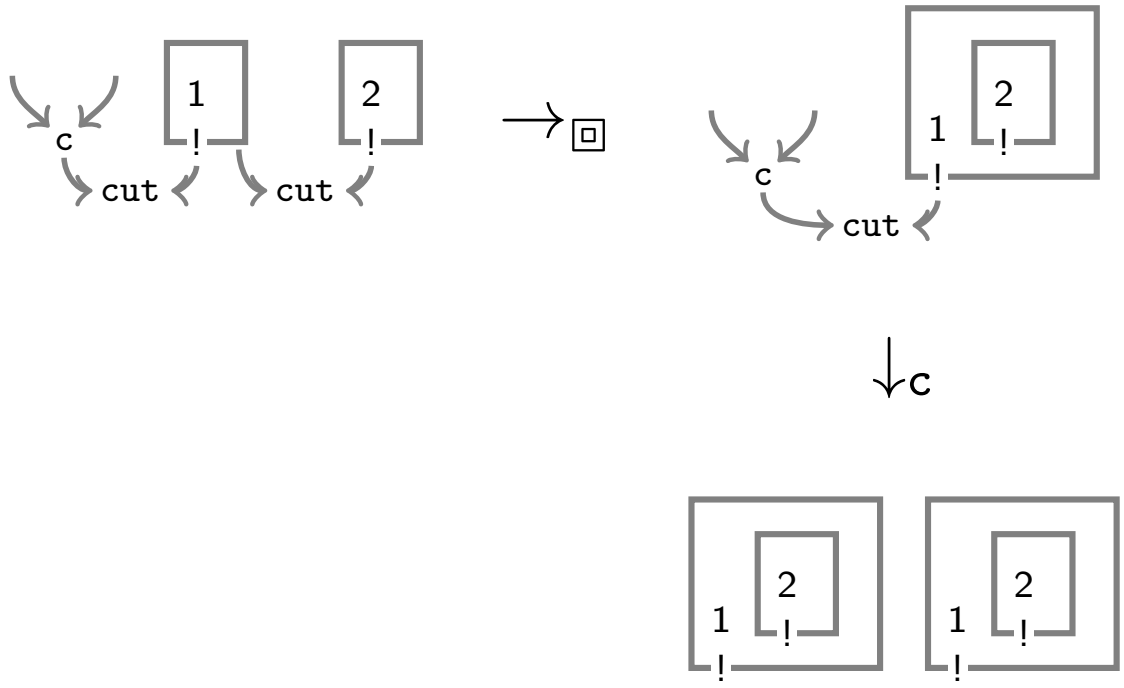

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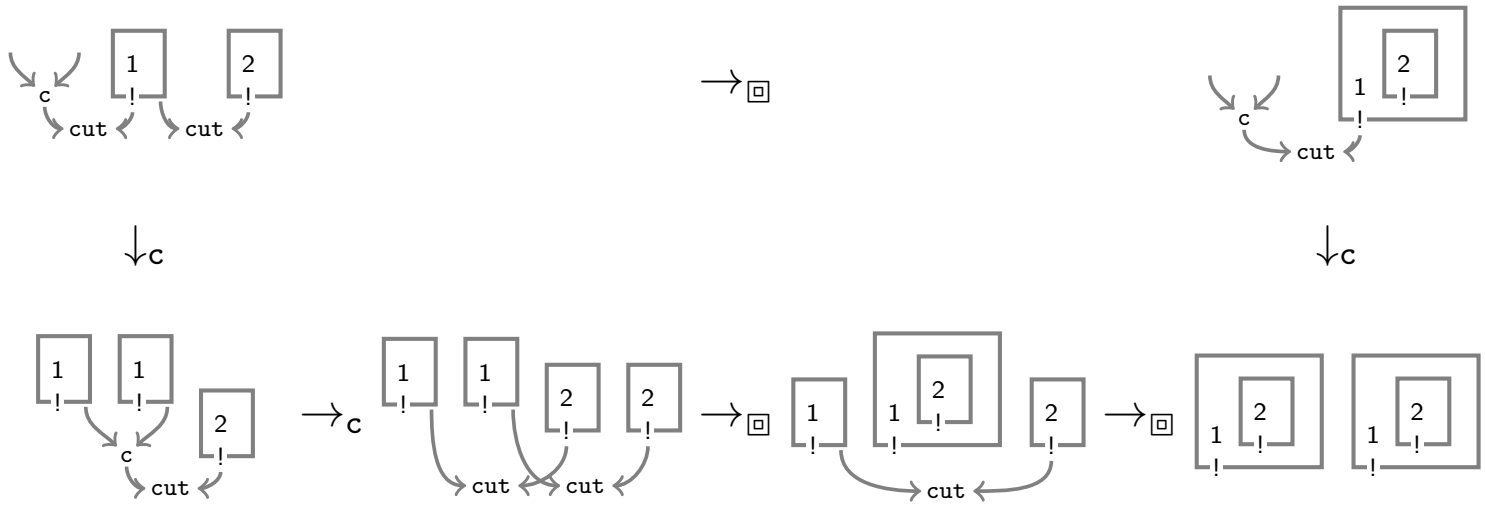

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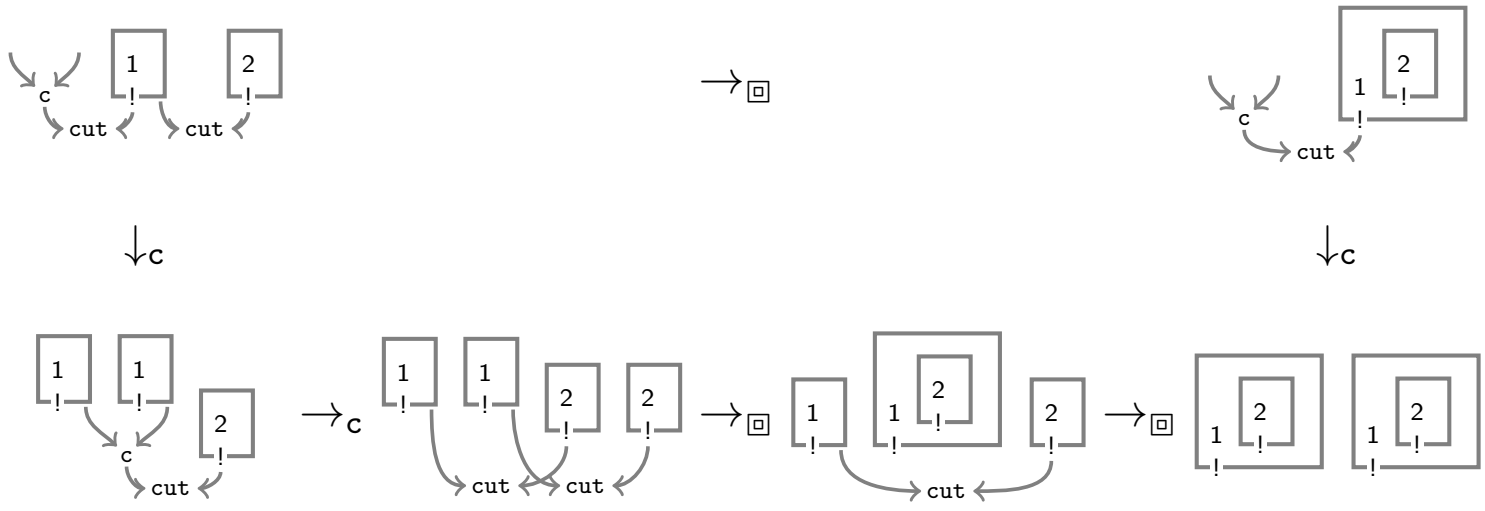

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Reduction at level 0 is not diamond.

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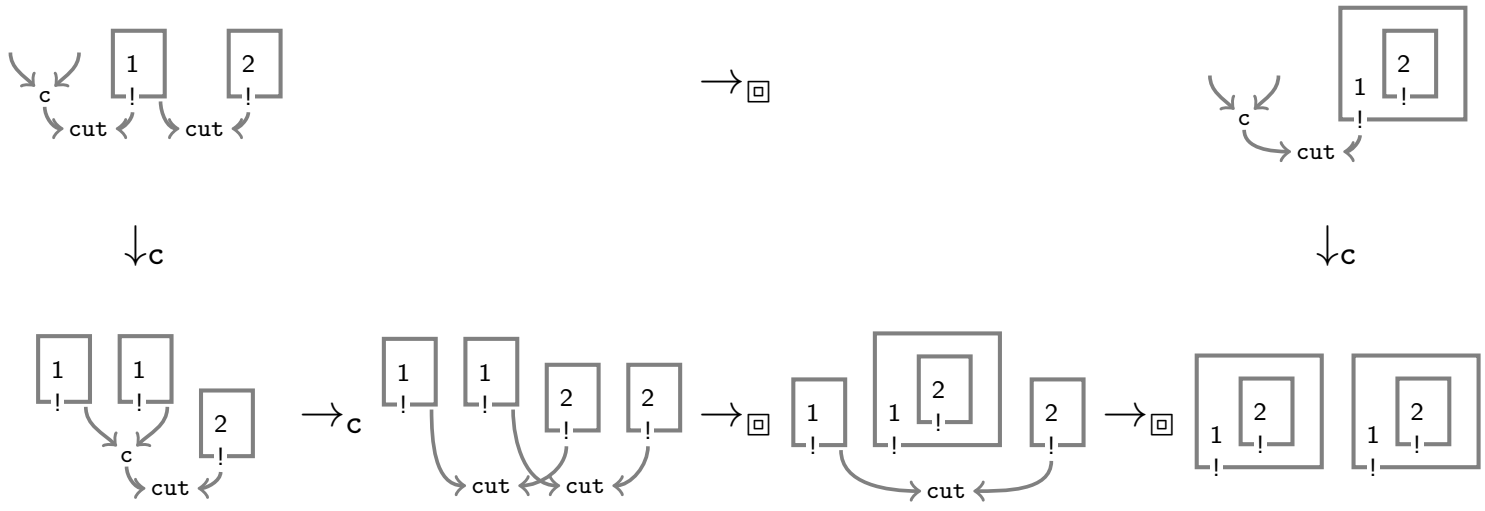

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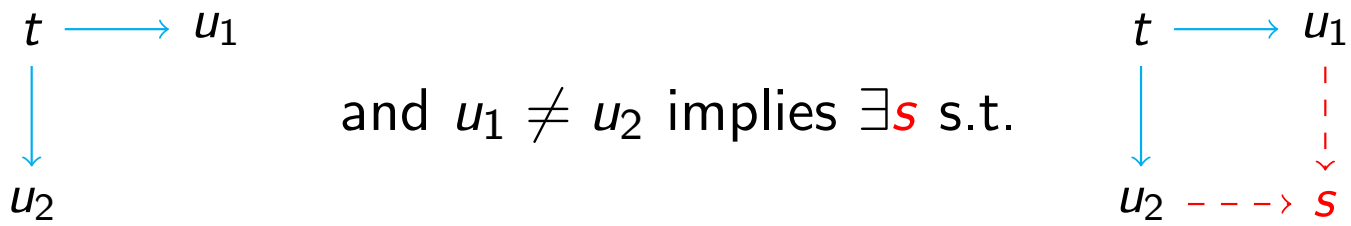

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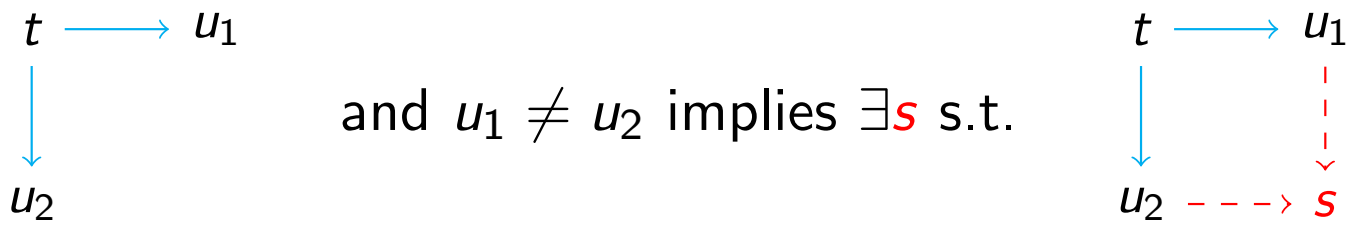

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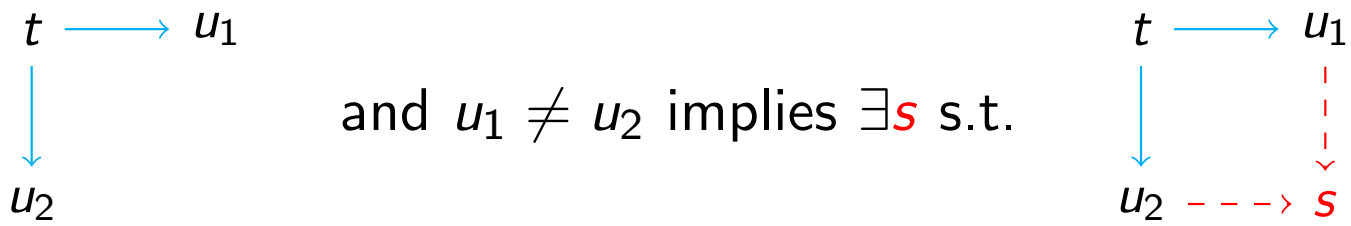

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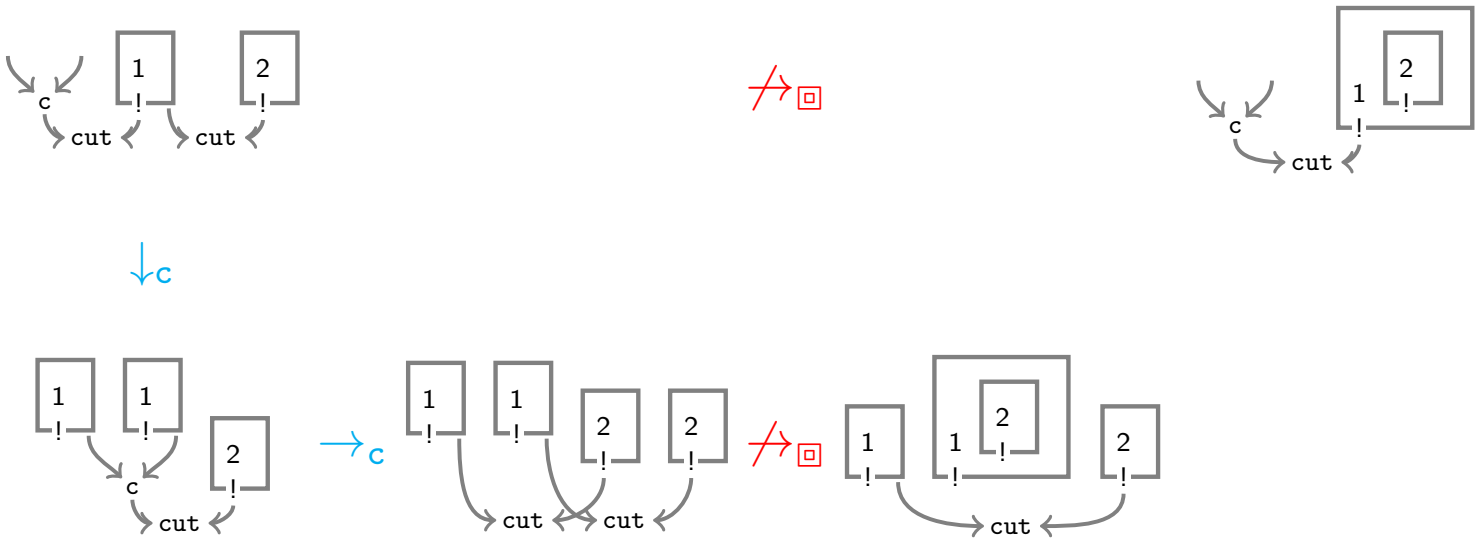

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Deep connections with games, GoI, intersection types, π-calculus,...

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It is defined only for (proofs representing) λ-terms.

Even then, linear head does not compute cut-free proofs.

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The LSC models linear head reduction elegantly.

Two possible routes:

1. Generalize LHR using proof nets.
2. Generalize the LSC to linear logic.

We follow 2.

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3. Fresh look at LL, importing ideas from sibling field.

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Outline

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Key property of explicit substitutions:

Substitution rules alone always terminate.

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It is an exponentials-only variant of $\delta\delta$.

Logical insight: involutive negation is dangerous!

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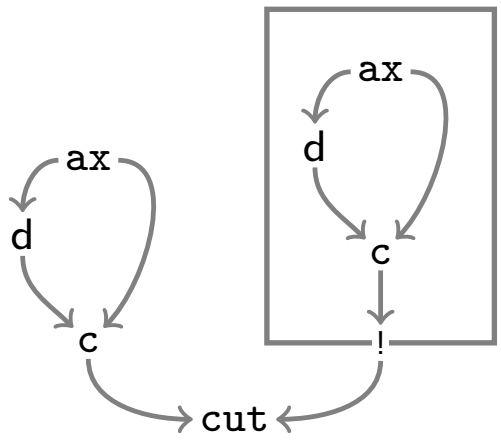

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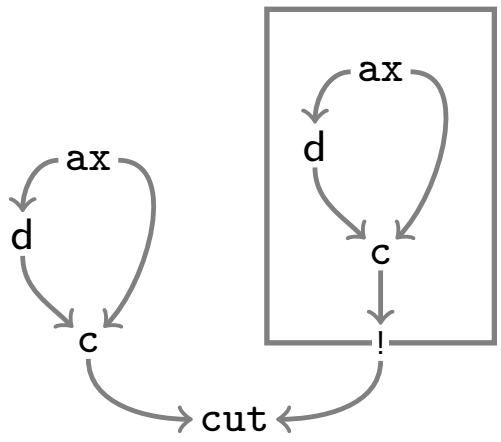

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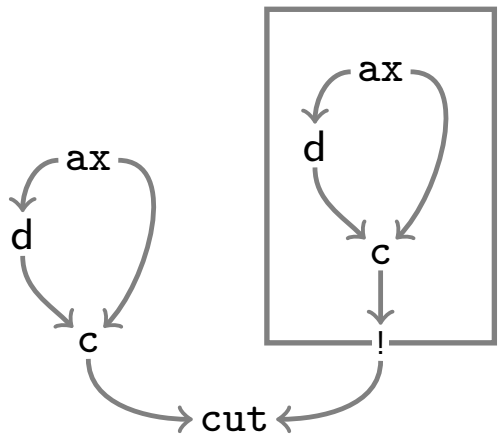

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Then we place ourselves in ILL.

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Solution for IMELL.

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