Focused proof systems

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Extended version of this talk is available in the paper "Focusing Gentzen's LK proof system" [Liang and Miller, to appear], in some slides, and in a YouTube video.

A useful reference: Stanford Encyclopedia of Philosophy

- *Linear Logic* by Roberto Di Cosmo and Dale Miller.
  `https://plato.stanford.edu/archives/sum2019/entries/logic-linear/`
- *Intuitionistic Logic* by Joan Moschovakis.
  `https://plato.stanford.edu/archives/fall2021/entries/logic-intuitionistic/`
- *Proof Theory* by Michael Rathjen and Wilfried Sieg.
  `https://plato.stanford.edu/archives/fall2020/entries/proof-theory/`.
- *The Development of Proof Theory* by Jan von Plato.
  `https://plato.stanford.edu/archives/win2018/entries/proof-theory-development/`
- *Church's Type Theory* by Christoph Benzmüller and Peter Andrews. `https://plato.stanford.edu/archives/sum2019/entries/type-theory-church/`
- *Logic and Games* by Wilfrid Hodges.
  `http://plato.stanford.edu/archives/spr2013/entries/logic-games/`

A brief history of focused proof systems

- 1935: Gentzen presents the LK and LJ sequent calculi
- 1987: Logic programming: a two phase construction of intuitionistic sequent proofs (goal-reduction and back-chaining) is complete. (M, Nadathur, Scedrov.)
- 1987: Girard introduces linear logic
- 1992: Andreoli gives first focused proof system (linear logic)
- 1993-2007: Focused proof systems for classical and intuitionistic logic (LJT, LJQ, LKT, LKQ, etc) [Danos et al; Laurent; Chaudhuri et al; Zeilburger; Dyckhoff et al; Wadler]
  - Functional programming: CBN, CBV, CBPV, etc
  - Logic programming: back-chaining, forward-chaining
- 2001: Girard introduces Ludics
- 2009: Liang & M present LJF and LKF.

This lecture will focus on LKF.

Gentzen's LK using two-sided sequents

Structural rules

$$\frac{\Gamma, B, B \vdash \Delta}{\Gamma, B \vdash \Delta}\ cL \qquad \frac{\Gamma \vdash \Delta, B, B}{\Gamma \vdash \Delta, B}\ cR \qquad \frac{\Gamma \vdash \Delta}{\Gamma, B \vdash \Delta}\ wL \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, B}\ wR$$

Identity rules

$$\frac{}{B \vdash B}\ init \qquad \frac{\Gamma \vdash \Delta, B \quad \Gamma, B \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut$$

Introduction rules

$$\frac{\Gamma, B_i \vdash \Delta}{\Gamma, B_1 \wedge B_2 \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, B \quad \Gamma \vdash \Delta, C}{\Gamma \vdash \Delta, B \wedge C} \qquad \frac{}{\Gamma \vdash \Delta, t}$$

$$\frac{\Gamma, B \vdash \Delta \quad \Gamma, C \vdash \Delta}{\Gamma, B \vee C \vdash \Delta} \qquad \frac{}{\Gamma, f \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, B_i}{\Gamma \vdash \Delta, B_1 \vee B_2}$$

$$\frac{\Gamma \vdash \Delta, B \quad \Gamma, C \vdash \Delta'}{\Gamma, \Gamma', B \supset C \vdash \Delta, \Delta'} \qquad \frac{\Gamma, B \vdash \Delta, C}{\Gamma \vdash \Delta, B \supset C}$$

$$\frac{\Gamma, Bs \vdash \Delta}{\Gamma, \forall x.Bx \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, By}{\Gamma \vdash \Delta, \forall x.Bx} \qquad \frac{\Gamma, By \vdash \Delta}{\Gamma, \exists x.Bx \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, Bs}{\Gamma \vdash \Delta, \exists x.Bx}$$

Observations about LK

The structural rule of exchange is built into this presentation.

The additive variants of conjunction and disjunction are used, not the multiplicative variants.

Implication is multiplicative (a kind of multiplicative disjunction).

Gentzen used negation $\neg B$ while here we use $B \supset f$. As a result, the LJ restriction on LK can be stated as either

- there is exactly one formula on the right, or
- the right is a *linear* context, the left is a *classical* context.

Intuitionistic logic is a hybridization of linear and classical logics.

Four shortcomings of the LK sequent calculus

1. The collision of cut and the structural rules
2. Permutations of inference rules
3. Chose either the additive or multiplicative versions of binary inference rules, but not both
4. No provision for synthetic inference rules

1: The collision of cut and the structural rules

Cut-elimination can be widely nondeterministic. Consider the collision of the cut rule with weakening.

$$\dfrac{\dfrac{\dfrac{\Xi_1}{\vdash B}}{\vdash C, B}\ wR \qquad \dfrac{\dfrac{\Xi_2}{\vdash B}}{C \vdash B}\ wL}{\dfrac{\vdash B, B}{\vdash B}\ cR}\ cut$$

Cut-elimination can yield either Ξ_1 or Ξ_2.

Examples like these are due to *Lafont* [Girard et al., 1989].

Polarization in classical logic will allow us to get rid of these (and similar) examples.

2. Permutations of inference rules

The following two deviations differ by permuting an inference rule.

$$\dfrac{\dfrac{\Gamma, B_i, C_j \vdash \Delta}{\Gamma, B_i, C_1 \wedge C_2 \vdash \Delta}}{\Gamma, B_1 \wedge B_2, C_1 \wedge C_2 \vdash \Delta} \qquad \dfrac{\dfrac{\Gamma, B_i, C_j \vdash \Delta}{\Gamma, B_1 \wedge B_2, C_j \vdash \Delta}}{\Gamma, B_1 \wedge B_2, C_1 \wedge C_2 \vdash \Delta}$$

These two derivations are different but probably should be considered equal.

Permutation of inference rules is a huge issue in trying to see structure in the sequent calculus.

The existence of such permutations is probably the main reason for the revolt against sequent calculus. Other popular choices are natural deduction, typed λ-calculi, expansion trees, proof nets, etc.

3. Choose only one among additive or multiplicative rules

Gentzen used the additive versions of conjunction and disjunction.

People in theorem proving usually use the invertible rules for the disjunction (which is a multiplicative rule). Things can be arranged so that the only non-invertible introduction rule is the $\exists$R rule.

Why not allow *both* the additive and multiplicative versions of these rules?

4. No provision for synthetic inference rules

Inference rules in LK are too small. Consider the axiom stating that the predicate *path* is transitive.

$$\forall x \forall y \forall z\, (path\ x\ y \supset path\ y\ z \supset path\ x\ z).$$

Using this axiom involves at least five LK introduction rules. It is more natural to view that formula as yielding an inference rule.

$$\frac{\Gamma \vdash \Delta,\ path\ x\ y \qquad \Gamma \vdash \Delta,\ path\ y\ z}{\Gamma \vdash \Delta, path\ x\ z}$$

$$\frac{path\ x\ y,\ path\ y\ z,\ path\ x\ z,\ \Gamma \vdash \Delta}{path\ x\ y,\ path\ y\ z, \Gamma \vdash \Delta}$$

One of these *synthetic rules* would be a more appropriate way to invoke the transitivity axiom.

How can we build such synthetic rules? Can we guarantee cut-elimination holds when we add them?

LKF: polarized formulas

Positive connectives are f^+, $\vee^+$, t^+, $\wedge^+$, and $\exists$.
Negative connectives are t^-, $\wedge^-$, f^-, $\vee^-$, and $\forall$.
Literals are atomic formulas and negated atomic formulas.

An *atomic bias assignment* is a function $\delta(\cdot)$ that maps atomic formulas to the set $\{+, -\}$.

We extended $\delta(\cdot)$ to literals by setting $\delta(\neg A)$ to the opposite polarity of $\delta(A)$.

A polarized formula is *positive* if its top-level connective is positive or its a literal L and $\delta(L) = +$.
A polarized formula is *negative* if its top-level connective is negative or its a literal L and $\delta(L) = -$.

We require that $\delta(\cdot)$ is stable under substitution: $\delta(A) = \delta(\theta A)$.
Thus, $\delta(A)$ is determined by the predicate symbol of A.

LKF: polarized formulas (continued)

Linear logic has other names for the polarized connectives.

	conjunction	true	disjunction	false
multiplicative	$\wedge^+$, $\otimes$	t^+, 1	$\vee^-$, $\parr$	f^-, $\bot$
additive	$\wedge^-$, &	t^-, $\top$	$\vee^+$, $\oplus$	f^+, 0

Logical connectives have *four attributes:*
arity, additive/multiplicative, polarity, conjunction/disjunction.

De Morgan duality flips the last two but leaves the first two unchanged.

LKF: negation normal form

Polarized formulas are in *negation normal form* (nnf), meaning that there is no occurrences of implication $\supset$ and that the negation symbol $\neg$ has only atomic scope.

The negation symbol $\neg$ is extended also to non-atomic polarized formulas.

- $\neg\neg A = A$ for atomic formula A
- $\neg(A \wedge^+ B) = \neg A \vee^- \neg B, \quad \neg(A \vee^- B) = \neg A \wedge^+ \neg B$
- $\neg(A \vee^+ B) = \neg A \wedge^- \neg B, \quad \neg(A \wedge^- B) = \neg A \vee^+ \neg B$
- $\neg\exists x.A = \forall x.\neg A, \quad \neg\forall x.A = \exists x.\neg A$

Let B be an unpolarized formula (in nnf) and let $\hat{B}$ result from annotating the propositional connectives in B with a $+$ or $-$. Let $\delta(\cdot)$ be an atomic bias assignment for the predicates in B. The pair $\langle\delta(\cdot), \hat{B}\rangle$ is a *polarization* of B.

LKF: sequent

LKF uses one-sided sequents of two varieties, namely,

$$\vdash \Gamma \Uparrow \Theta \quad \text{and} \quad \vdash B \Downarrow \Theta,$$

where Γ is a multiset of formulas, Θ is a set of formulas, and B is a single formula.

The Θ context is called *storage*.

Introductions take place on formulas between $\vdash$ and the $\Uparrow$ or $\Downarrow$.

Key facts:

- The right-introduction rules for negative connectives are **invertible**.
- The right-introduction rules for positive connectives are generally **not invertible**.

LKF: proof rules

Negative introduction rules

$$\frac{}{\vdash t^-, \Gamma \Uparrow \Theta} \qquad \frac{\vdash B, \Gamma \Uparrow \Theta \quad \vdash C, \Gamma \Uparrow \Theta}{\vdash B \wedge^- C, \Gamma \Uparrow \Theta}$$

$$\frac{\vdash \Gamma \Uparrow \Theta}{\vdash f^-, \Gamma \Uparrow \Theta} \qquad \frac{\vdash B, C, \Gamma \Uparrow \Theta}{\vdash B \vee^- C, \Gamma \Uparrow \Theta} \qquad \frac{\vdash [y/x]B, \Gamma \Uparrow \Theta}{\vdash \forall x.B, \Gamma \Uparrow \Theta}$$

Positive introduction rules

$$\frac{}{\vdash t^+ \Downarrow \Theta} \qquad \frac{\vdash B \Downarrow \Theta \quad \vdash C \Downarrow \Theta}{\vdash B \wedge^+ C \Downarrow \Theta} \qquad \frac{\vdash B_i \Downarrow \Theta}{\vdash B_1 \vee^+ B_2 \Downarrow \Theta} \qquad \frac{\vdash [s/x]B \Downarrow \Theta}{\vdash \exists x.B \Downarrow \Theta}$$

Non-introduction rules

$$\frac{}{\vdash L \Downarrow \neg L, \Theta}\ \mathit{init} \qquad \frac{\vdash \Gamma \Uparrow Q, \Theta}{\vdash Q, \Gamma \Uparrow \Theta}\ \mathit{store} \qquad \frac{\vdash N \Uparrow \Theta}{\vdash N \Downarrow \Theta}\ \mathit{release}$$

$$\frac{\vdash P \Downarrow P, \Theta}{\vdash \cdot \Uparrow P, \Theta}\ \mathit{decide}$$

Here: L is a positive literal, P is positive, N is negative, Q is positive or a literal.

Observations about LKF proof rules

We say that the polarized formula B has an LKF proof if the sequent $\vdash B \Uparrow \cdot$ has an LKF proof.

Key observations:

1. *Contraction* occurs only in the *decide* rule and only for *positive* formulas. A negative formula is never contracted.
2. *Weakening* occurs only at the leaves (in the *init* and t^+ rules) and only on *positive formulas* and *negative literals*.

Theorem (Soundness and completeness of LKF)

Let B be an unpolarized formula.

1. *If $\hat{B}$ is a polarization of B and $\hat{B}$ has an LKF proof, the B has an LK proof.*
2. *If B has an LK proof and $\hat{B}$ is a polarization of B, then $\hat{B}$ has an LKF proof.*

See [Liang and Miller, to appear] for a direct proof.

The structure of (cut-free) focused proofs

A sequent of the form $\vdash \cdot \Uparrow \Theta$ is called a *border sequent*.

Such sequents can only be proved by using the *decide* rule.

A *synthetic inference rule* is defined as these two phases, with border sequents as the conclusion and the premises.

$$
\dfrac{\dfrac{\cdots \qquad \dfrac{\cdots \qquad \dfrac{\dfrac{\cdots \quad \vdash \cdot \Uparrow \Theta' \quad \cdots}{\vdash N \Uparrow \cdots}\ \text{neg phase}}{\vdash N \Downarrow \cdots}\ \textit{release} \qquad \cdots}{\vdash \cdots \Downarrow \cdots}\ \text{pos phase} \qquad \cdots}{\vdash P \Downarrow \Theta}}{\vdash \cdot \Uparrow \Theta}\ \textit{decide}\ P \in \Theta
$$

The cut rule for LKF

The *cut* rule operates on $\Uparrow$ sequents.

$$\frac{\vdash B \Uparrow \Theta \quad \vdash \neg B \Uparrow \Theta'}{\vdash \cdot \Uparrow \Theta, \Theta'}\ cut$$

During the proof of cut-elimination, the following four variants of the cut rule need to be considered and eliminated as well.

$$\frac{\vdash A, \Gamma \Uparrow \Theta \quad \vdash \neg A, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ cut_u \qquad \frac{\vdash A \Downarrow \Theta \quad \vdash \neg A, \Gamma' \Uparrow \Theta'}{\vdash \Gamma' \Uparrow \Theta, \Theta'}\ cut_f$$

$$\frac{\vdash \Gamma \Uparrow \Theta, P \quad \vdash \neg P, \Gamma' \Uparrow \Theta'}{\vdash \Gamma, \Gamma' \Uparrow \Theta, \Theta'}\ dcut_u \qquad \frac{\vdash B \Downarrow \Theta, P \quad \vdash \neg P \Uparrow \Theta'}{\vdash B \Downarrow \Theta, \Theta'}\ dcut_f$$

Here, A and B are arbitrary polarized formulas and P is a positive polarized formula.

Applications of LKF

1. Lafont's examples disappear since cut connects a positive and a negative formula and in LKF, only a positive formula can be weakened and contracted.
2. The completeness of LKF immediately yields Herbrand's theorem.
3. LKF can host other focused proof systems, such as the LKQ and LKT proof systems of [Danos et al., 1995]. Gentzen's LK can also be hosted in LKF. Here, polarizing allows the insertion of *delays* (1-ary propositional connectives).
4. Admissibility of *cut* in LKF implies its admissibility in LK.
5. The many negative translations of classical logic into intuitionistic logic can be seen as fixing a polarization of classical formulas.
6. Synthetic inference rules...

Application of LKF: Synthetic inference rules

Let Θ contain the negated and polarized transitivity axiom:

$$\exists x \exists y \exists z.(path\ x\ y \wedge^+ path\ y\ z \wedge^+ \neg path\ x\ z)$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Xi_1 \\ \vdash path\ r\ s \Downarrow \Theta\end{matrix} \qquad \begin{matrix}\Xi_2 \\ \vdash path\ s\ t \Downarrow \Theta\end{matrix} \qquad \begin{matrix}\Xi_3 \\ \vdash \neg path\ r\ t \Downarrow \Theta\end{matrix}}{\vdash path\ r\ s \wedge^+ path\ s\ t \wedge^+ \neg path\ r\ t \Downarrow \Theta}\ \wedge^+ \times 2}{\vdash \exists x \exists y \exists z.(path\ x\ y \wedge^+ path\ y\ z \wedge^+ \neg path\ x\ z) \Downarrow \Theta}\ \exists \times 3}{\vdash \cdot \Uparrow \Theta}\ decide$$

The shape of Ξ_1, Ξ_2, and Ξ_3 depends on the polarity of the *path* predicate.

Application of LKF: Synthetic inference rules (continued)

If path-atoms are negative, then Ξ_1 and Ξ_2 end with the *release* and *store* rules while the proof Ξ_3 is trivial. This synthetic rule is

$$\frac{\vdash \cdot \Uparrow \mathit{path}\ r\ s, \Theta \qquad \vdash \cdot \Uparrow \mathit{path}\ s\ t, \Theta}{\vdash \cdot \Uparrow \mathit{path}\ r\ t, \Theta}$$

If path atoms are positive, then Ξ_3 end with the *release* and *store* rules while the proofs Ξ_1 and Ξ_2 are trivial. This synthetic rule is

$$\frac{\vdash \cdot \Uparrow \neg \mathit{path}\ r\ s, \neg \mathit{path}\ s\ t, \neg \mathit{path}\ r\ t, \Theta}{\vdash \cdot \Uparrow \neg \mathit{path}\ r\ s, \neg \mathit{path}\ s\ t, \Theta}$$

These synthetic inference rules are the one-sided version of the *back-chaining* and *forward-chaining* rules displayed earlier (see [Chaudhuri et al., 2008]).

Cut-elimination holds when synthetic inference rules are added [Marin et al., 2022].

Conclusion

The LKF proof system is proposed as an improvement on LK, especially for computer scientists interested in computational logic.

The LKF proof system is flexible and can mimic a range of proof systems and supports the inclusion of synthetic inference rules.

The proof theory of LKF can be applied to unpolarized proof systems as well (e.g., Herbrand's theorem).

Intuitionistic logic can similarly be given a focused proof system LJF [Liang and Miller, 2009].

Bibliography

Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *J. of Logic and Computation*, 2(3): 297–347, 1992. doi: 10.1093/logcom/2.3.297.

Kaustuv Chaudhuri, Frank Pfenning, and Greg Price. A logical characterization of forward and backward chaining in the inverse method. *J. of Automated Reasoning*, 40(2-3):133–177, March 2008. doi: 10.1007/s10817-007-9091-0.

V. Danos, J.-B. Joinet, and H. Schellinx. LKT and LKQ: sequent calculi for second order logic based upon dual linear decompositions of classical implication. In J.-Y. Girard, Y. Lafont, and L. Regnier, editors, *Advances in Linear Logic*, number 222 in London Mathematical Society Lecture Note Series, pages 211–224. Cambridge University Press, 1995.

Gerhard Gentzen. Investigations into logical deduction. In M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131. North-Holland, Amsterdam, 1935. doi: 10.1007/BF01201353. Translation of articles that appeared in 1934-35. Collected papers appeared in 1969.

Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–102, 1987. doi: 10.1016/0304-3975(87)90045-4.

Jean-Yves Girard. Locus solum: From the rules of logic to the logic of rules. *Mathematical Structures in Computer Science*, 11(3):301–506, June 2001.

Jean-Yves Girard, Paul Taylor, and Yves Lafont. *Proofs and Types*. Cambridge University Press, 1989.

Chuck Liang and Dale Miller. Focusing and polarization in linear, intuitionistic, and classical logics. *Theoretical Computer Science*, 410(46):4747–4768, 2009. doi: 10.1016/j.tcs.2009.07.041.

Chuck Liang and Dale Miller. Focusing Gentzen's LK proof system. In Thomas Piecha and Kai Wehmeier, editors, *Peter Schroeder-Heister on Proof-Theoretic Semantics*, Outstanding Contributions to Logic. Springer, to appear.

Sonia Marin, Dale Miller, Elaine Pimentel, and Marco Volpe. Synthetic inference rules for geometric theories. *Annals of Pure and Applied Logic*, pages 1–41, January 2022. doi: 10.1016/j.apal.2022.103091.

Dale Miller, Gopalan Nadathur, and Andre Scedrov. Hereditary Harrop formulas and uniform proof systems. In David Gries, editor, *2nd Symp. on Logic in Computer Science*, pages 98–105, Ithaca, NY, June 1987.