Categorical Semantics of Linear Logic

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Representations in group theory

Imagine that one wants to study the properties of a specific group G.

One well-known and important technique is to look at

the **representations** of the group G

where a representation is defined as:

- ▷ a finite (or infinite) dimensional **vector space** V,
- ▷ a **linear action**

$$- \bullet - \quad : \quad G \times V \longrightarrow V$$

of the group G on the vector space V.

Linear actions

Definition. A **linear action** is a function

$$- \bullet - \quad : \quad G \times V \longrightarrow V$$

defining an **action** of the group $(G, \cdot, e)$ on the vector space V

$$\forall g, g' \in G, \forall v \in V \qquad (g' \cdot g) \bullet v = g' \bullet (g \bullet v) \qquad e \bullet u = u$$

such that the **action** of any element $g \in G$

$$g \bullet - \quad : \quad V \longrightarrow V$$

defines a **linear map** from the vector space V to itself:

$$\forall v, w \in V, \qquad g \bullet (v + w) = (g \bullet v) + (g \bullet w) \qquad g \bullet 0 = 0$$

Linear actions

Equivalently, a **linear action**

$$\lambda \quad : \quad G \times V \longrightarrow V$$

is a family of **linear maps** from the vector space V to itself

$$\lambda_g \quad : \quad V \longrightarrow V$$

parameterized by $g \in G$ and satisfying the two equations:

$$\lambda_{g' \cdot g} = \lambda_{g'} \circ \lambda_g \qquad\qquad \lambda_e = id_V$$

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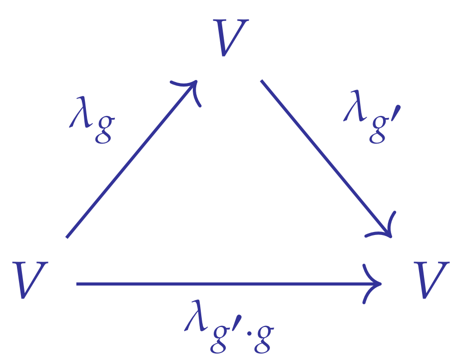

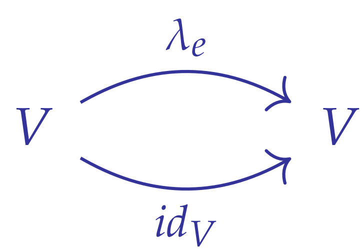

Illustration

The **group of rotations** of the three-dimensional Euclidean space $V = \mathbb{R}^3$

$$G = SO(3)$$

where a rotation

$$M : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

is an **isometry** preserving the **origin** as well as the **orientation** of $V = \mathbb{R}^3$.

Equivalently, a rotation is a real-valued 3×3-matrix

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

satisfying the equation:

$$\langle Mv, Mw \rangle = \langle v, w \rangle$$

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satisfying the equation:

$$\langle v, M^t M w \rangle \quad = \quad \langle v, w \rangle$$

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satisfying the equation:

$$M^t M = M M^t = Id_V$$

Illustration

A fruitful observation in algebra:

The **natural representation** in the algebra $\mathbb{C}[X,Y,Z]$ of polynomials

$$SO(3) \times \mathbb{C}[X,Y,Z] \longrightarrow \mathbb{C}[X,Y,Z]$$

defined by the **algebra maps** induced from the rotation $g \in SO(3)$

$$\lambda_g \quad : \quad \mathbb{C}[X,Y,Z] \longrightarrow \mathbb{C}[X,Y,Z]$$

can be decomposed as an **infinite sum** of representations

$$\mathbb{C}[X,Y,Z] \quad \cong \quad \bigoplus_{i \in I} V_i$$

which contains all the **irreducible representations** of $SO(3)$.

Denotational semantics

What is traditionally called

denotational semantics of proofs and programs

can be seen as

a representation theory for proofs and programs

based on the three fundamental concepts of

1. category **2. functor** **3. natural transformation**

A brief introduction to

Categories

Functors

Natural transformations

First steps in the functorial language

Categories

A category $\mathscr{A}$ is an oriented graph

- ▷ whose nodes are called **objects**
- ▷ whose edges are called **maps** or **arrows** or **morphisms**

Given two objects A and A', we write

$$\mathbf{Hom}_{\mathscr{A}}(A, A') \quad \text{or more simply} \quad \mathbf{Hom}(A, A')$$

for the **set of maps** from the object A to the object A' in the category $\mathscr{A}$.

Categories

A category $\mathscr{A}$ is moreover equipped with

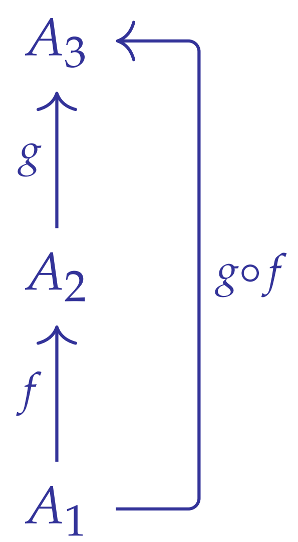
a composition law

defined as a family of functions:

$$\circ_{A_1, A_2, A_3} \; : \; \mathbf{Hom}(A_2, A_3) \times \mathbf{Hom}(A_1, A_2) \longrightarrow \mathbf{Hom}(A_1, A_3)$$

indexed by objects A_1, A_2, A_3 of the category $\mathscr{A}$.

Diagrammatically:



Categories

A category $\mathscr{A}$ is moreover equipped with

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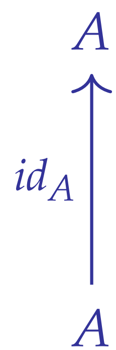
defined as a family of maps:

$$id_A \quad \in \quad \mathbf{Hom}(A, A)$$

indexed by the objects A of the category $\mathscr{A}$. Diagrammatically:



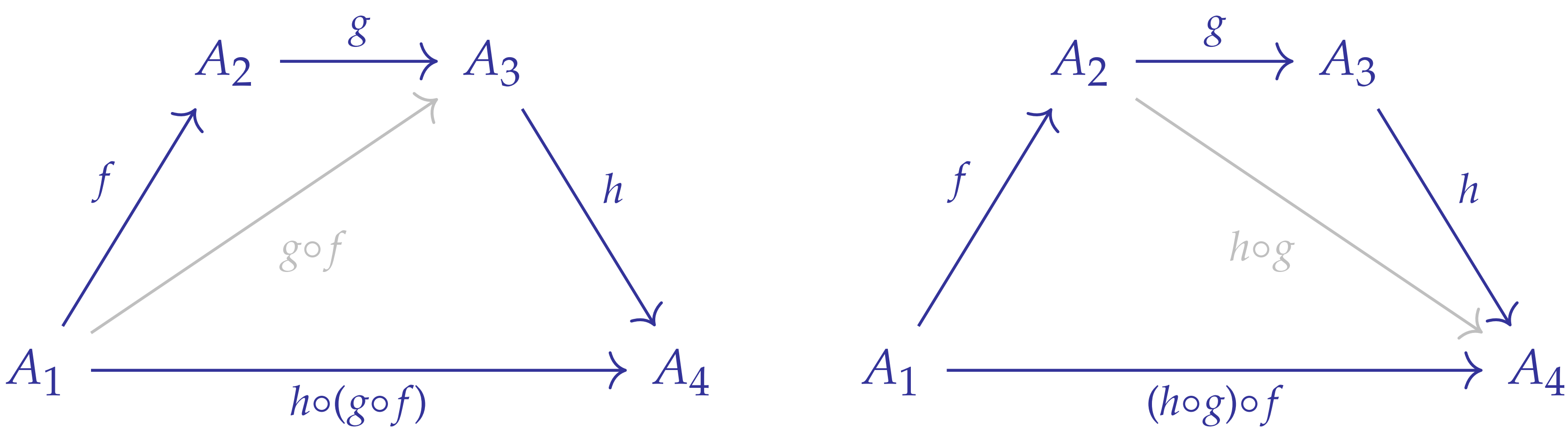
Categories

Finally, one requires the following two properties:

Associativity: the following equation is satisfied

$$(h \circ g) \circ f \quad = \quad h \circ (g \circ f)$$

for every path of length 3 in the category, as depicted below:



Categories

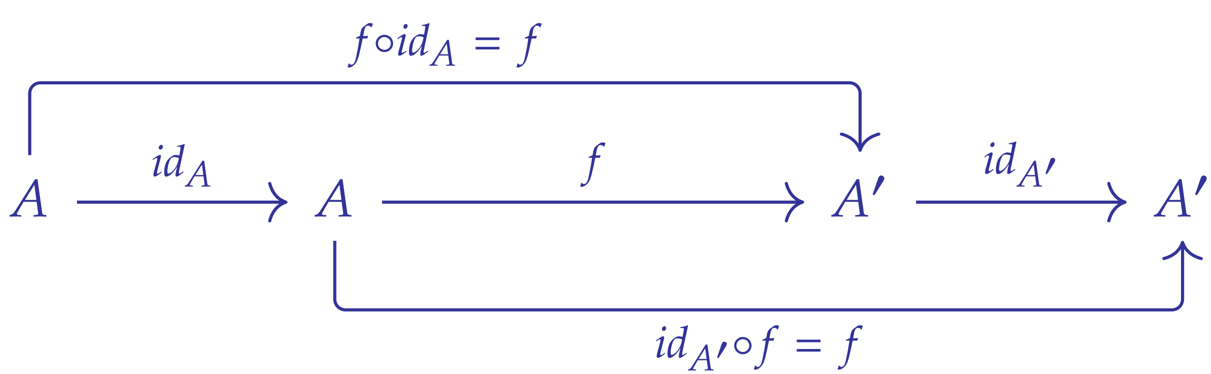
Neutrality: the two equations

$$f \circ id_A \quad = \quad f \quad = \quad id_{A'} \circ f$$

are satisfied for every map

$$A \xrightarrow{\;f\;} A'$$

in the category, as depicted below:



Large categories

A bestiary of examples given by **large categories** such as:

- ▷ the category $\mathbf{Set}$ with **sets as objects** and **functions as maps**
- ▷ the category $\mathbf{Rel}$ with **sets as objects** and **relations as maps**
- ▷ the category $\mathbf{Grp}$ of **groups** and **group homomorphisms**
- ▷ the category $\mathbf{Vec}$ of **vector spaces** and **linear maps**
- ▷ the category $\mathbf{Top}$ of **topological spaces** and **continuous functions**
- ▷ the category $\mathbf{Coh}$ of **coherence spaces** and **linear maps**
- ▷ the category $\mathbf{Stab}$ of **coherence spaces** and **stable maps**

Preorders as small categories

There is also a wide variety of **small categories** defined as preorders:

▷ a category $\mathscr{A}$ such that

the set $\mathbf{Hom}(A, A')$ is **empty** or **singleton** for all objects A, A'

is the same thing as a **preorder** $\leq_{\mathscr{A}}$ on the objects of $\mathscr{A}$.

The preorder relation $\leq_{\mathscr{A}}$ on the objects of $\mathscr{A}$ is defined as follows:

$A \leq_{\mathscr{A}} A'$ precisely when there exists a map $f : A \longrightarrow A'$ in $\mathscr{A}$

Monoids as small categories

▷ every monoid $M = (M, \cdot_M, e_M)$ may be equivalently seen as

a category ΣM with one **single object** noted $*$

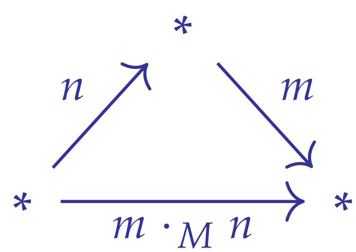
whose maps $* \longrightarrow *$ are the elements of the monoid:

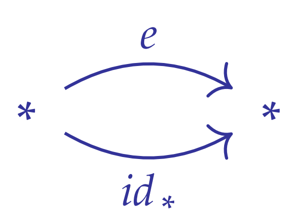
$$\mathbf{Hom}_{\Sigma M}(*, *) \quad = \quad M$$

equipped with the induced composition and identity laws:

$$m \circ_{\Sigma M} n \ := \ m \cdot_M n \qquad\qquad id_* \ := \ e_M$$

Diagrammatically:





— a little exercise just for the fun of it —

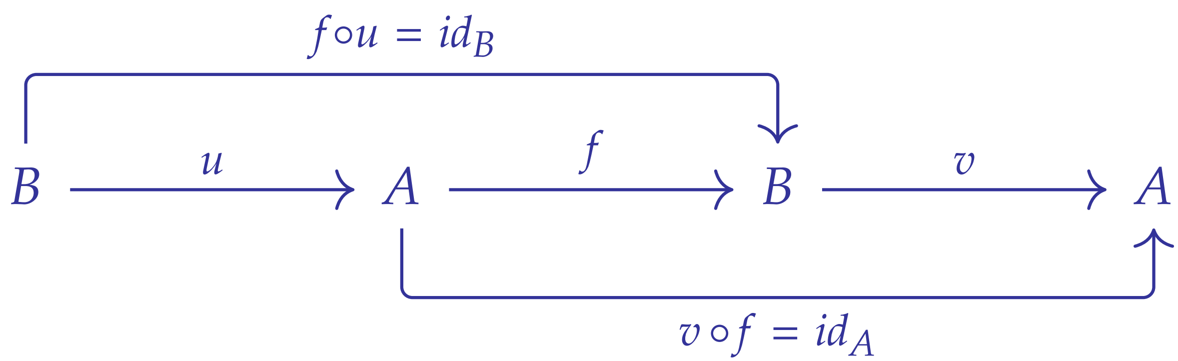
Definition A map in a category $\mathscr{A}$

$$f \quad : \quad A \longrightarrow B$$

is called an **isomorphism** when there exists a pair of maps

$$u\,,\,v \;:\; B \longrightarrow A$$

such that the two equations hold:



Exercise: Show that the two maps u and v are equal in that case.

Functors

A **functor** between categories $\mathscr{A}$ and $\mathscr{B}$

$$F \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

is an operation

- ▷ which transports every object $A \in \mathscr{A}$ to an object $F(A) \in \mathscr{B}$
- ▷ which transports every map

$$f \quad : \quad A \longrightarrow A'$$

of the category $\mathscr{A}$ to a map

$$F(f) \quad : \quad F(A) \longrightarrow F(A')$$

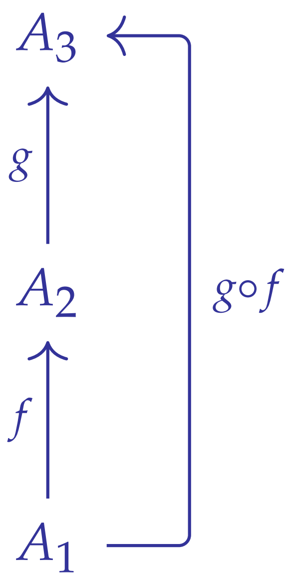
of the category $\mathscr{B}$.

Functors

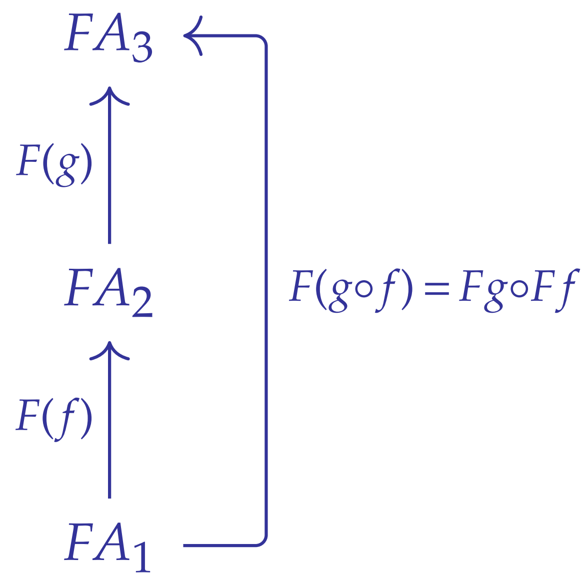
One requires moreover that

the image of the composite = the composite of the images

which means diagrammatically that



is mapped into

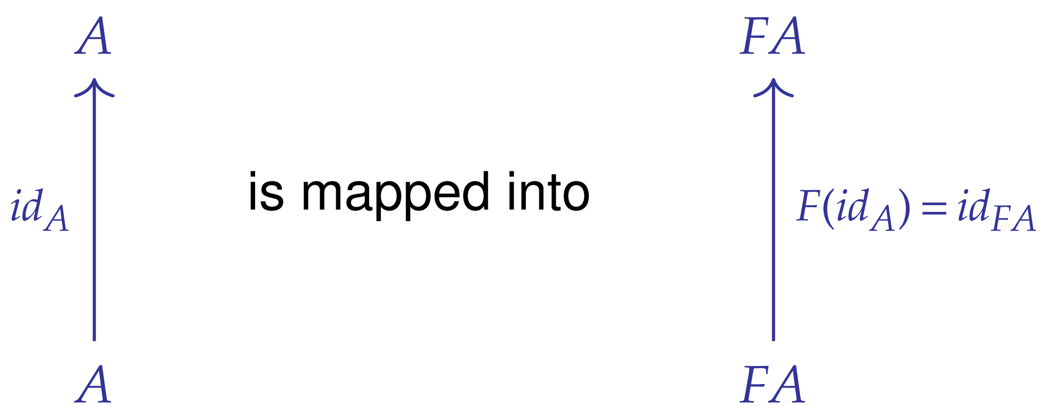


Functors

One also requires that

the image of the identity map = the identity map of the image

which means diagrammatically that



Natural transformations

A **natural transformation**

$$\theta \quad : \quad F \Longrightarrow G \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

between two functors F and G of the same source and target:

$$F\,,\,G \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

is a **family of maps** in the category $\mathscr{B}$

$$\theta_A \quad : \quad FA \longrightarrow GA$$

indexed by the objects of the category $\mathscr{A}$.

Natural transformations

One also requires that the **diagram commutes** in the category $\mathscr{B}$

$$\begin{array}{ccc} FA & \xrightarrow{\quad \theta_A \quad} & GA \\ {\scriptstyle Ff}\big\downarrow & & \big\downarrow{\scriptstyle Gf} \\ FA' & \xrightarrow{\quad \theta_{A'} \quad} & GA' \end{array}$$

in the sense that the equation below holds:

$$\boxed{\theta_{A'} \circ Ff \quad = \quad Gf \circ \theta_A}$$

for every map $f : A \longrightarrow A'$ of the category $\mathscr{A}$.

The 2-category $\mathbf{Cat}$ of categories

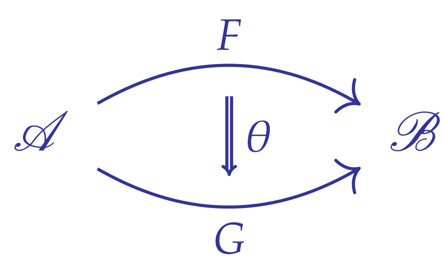
Categories, functors and natural transformations organize themselves into

a 2-category $\mathbf{Cat}$

where every natural transformation

$$\theta \quad : \quad F \Longrightarrow G \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

defines a **2-dimensional cell** between functors



seen themselves as **1-dimensional cells** between categories.

An intermezzo on 2-categories

Second steps in the functorial language

The notion of 2-category in four slides

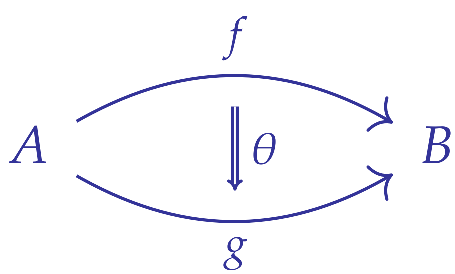
A 2-category $\mathcal{K}$ is defined just as a category except that the set

$$\mathbf{Hom}(A, B)$$

is now replaced by a **category** whose objects are 1**-dimensional cells**

$$f\,,\,g \quad : \quad A \longrightarrow B$$

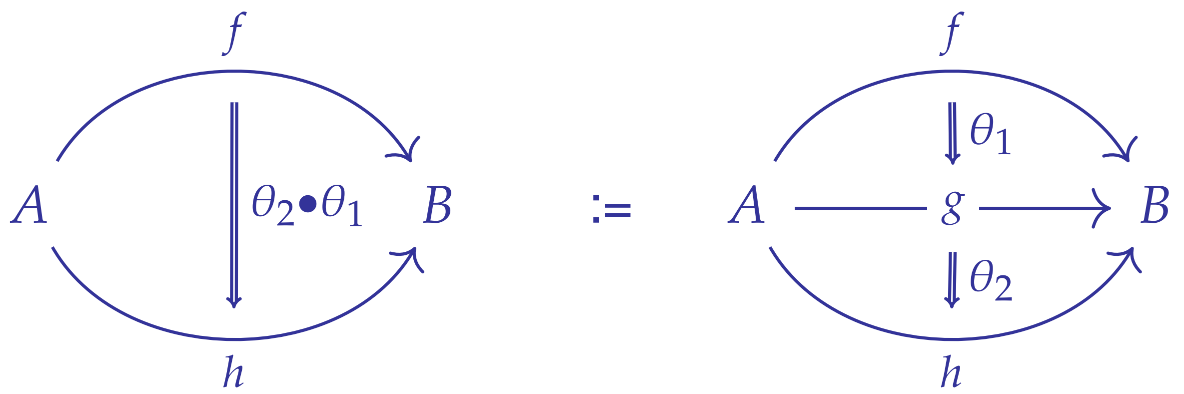
and whose maps $\theta : f \to g$ are 2**-dimensional cells**



between the 1-dimensional cells $f, g : A \to B$ of the 2-category $\mathcal{K}$.

Vertical composition

This equips the 2-category with a **vertical composition**



where we write $\theta_2 \bullet \theta_1$ for the composite of the two maps

$$\theta_1 : f \longrightarrow g \qquad\qquad \theta_2 : g \longrightarrow h$$

in the category $\mathbf{Hom}(A, B)$ of 1- and 2-dimensional cells from A to B.

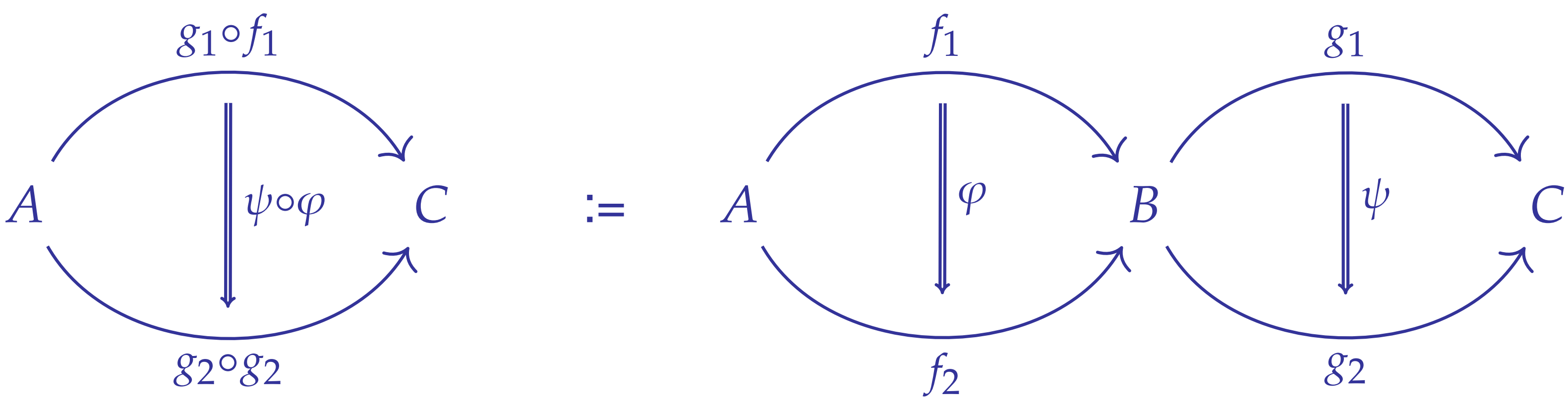
Horizontal composition

The **composition law** is defined as a **family of functors**

$$\circ_{A_1,A_2,A_3} \;:\; \mathbf{Hom}(A_2,A_3)\times\mathbf{Hom}(A_1,A_2)\longrightarrow\mathbf{Hom}(A_1,A_3)$$

between **hom-categories** of the 2-category.

This equips the 2-category with a **horizontal composition**



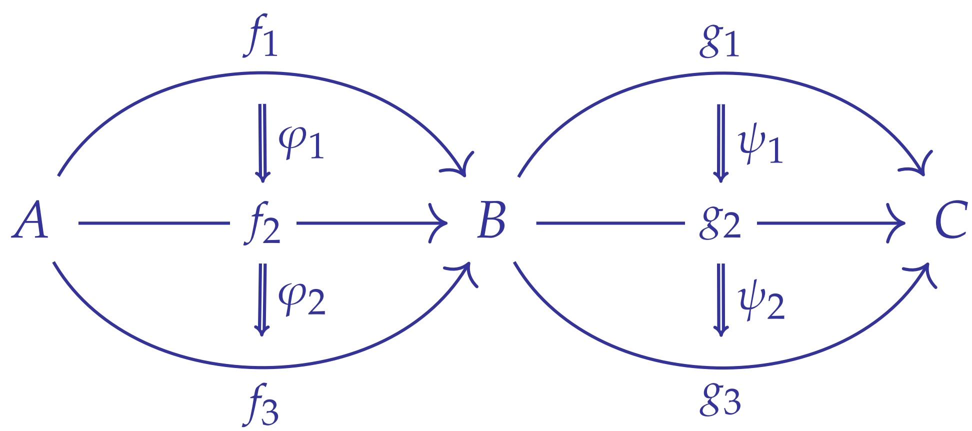
moreover **compatible with vertical composition** in the following sense.

The interchange law

Horizontal and vertical composition are compatible in the sense that

$$(\psi_2 \bullet \psi_1) \circ (\varphi_2 \bullet \varphi_1) \quad = \quad (\psi_2 \circ \varphi_2) \bullet (\psi_1 \circ \varphi_1)$$

whenever we are in the following situation:



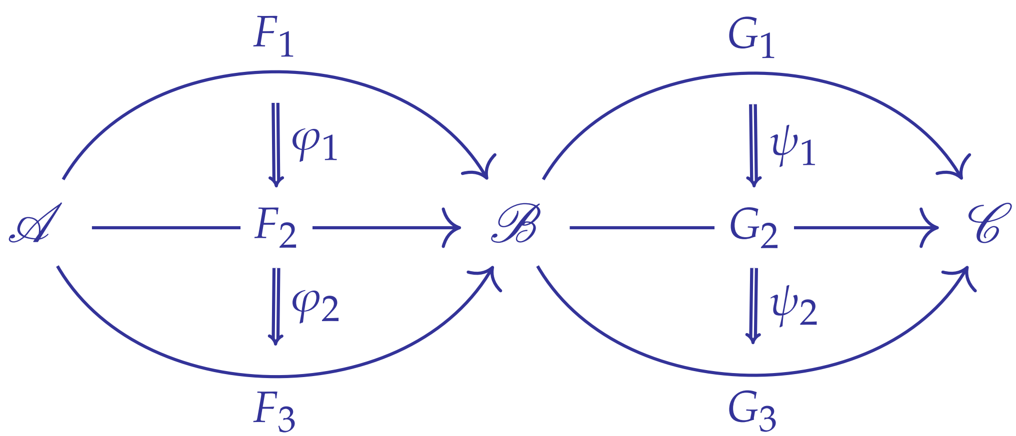
in the 2-category $\mathcal{K}$.

In the specific case of categories and functors

The **interchange law** of the 2-category $\mathcal{K} = \mathbf{Cat}$ ensures that

$$(\psi_2 \bullet \psi_1) \circ (\varphi_2 \bullet \varphi_1) \quad = \quad (\psi_2 \circ \varphi_2) \bullet (\psi_1 \circ \varphi_1)$$

whenever we have **natural transformations** of the following shape:



A brief introduction to

Categories

Functors

Natural transformations

First steps in the language of string diagrams

A brief [pictorial] introduction to

Categories

Functors

Natural transformations

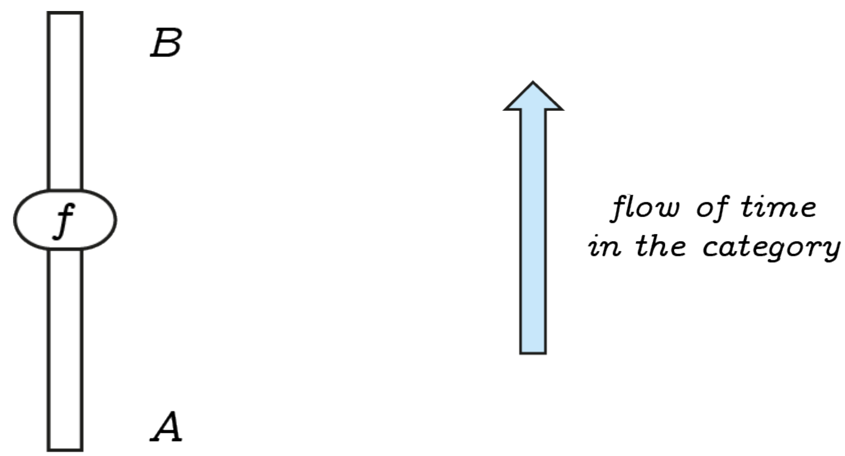
First steps in the language of string diagrams

Categories in string diagrams

The basic idea is to represent a map in a given category $\mathscr{A}$

$$f \quad : \quad A \longrightarrow B$$

as a **process** or as a **causal flow** going from bottom to top



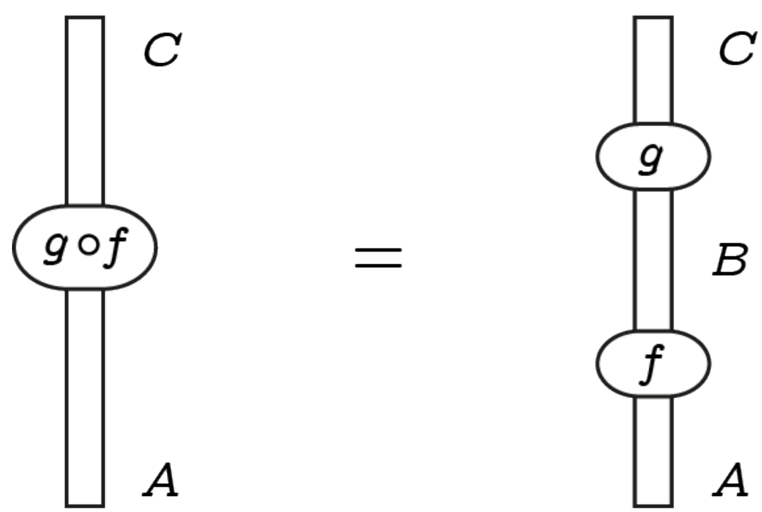
transforming an **input string** A into an **output string** B.

Categories in string diagrams

The composite of two maps in the category $\mathscr{A}$

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C$$

is represented by **composing vertically** the two string diagrams:

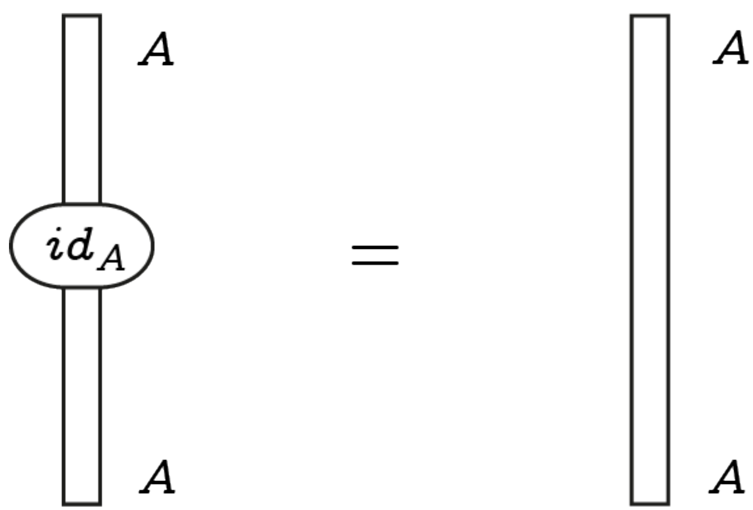


Categories in string diagrams

Accordingly, the identity map

$$id_A \quad : \quad A \longrightarrow A$$

is represented by the **trivial string** on which « nothing » happens:



Functors in string diagrams

By definition, a **functor**

$$F \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

transports every map of the category $\mathscr{A}$

$$f \quad : \quad A \longrightarrow A'$$

to a map of the category $\mathscr{B}$

$$Ff \quad : \quad FA \longrightarrow FA'$$

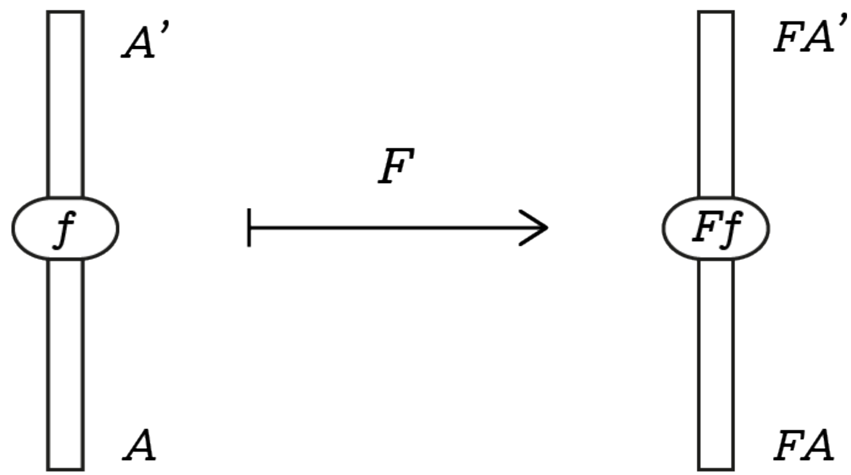
How shall we represent this operation using string diagrams?

Functors in string diagrams

In the language of string diagrams, a functor

$$F \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

behaves in the following way:

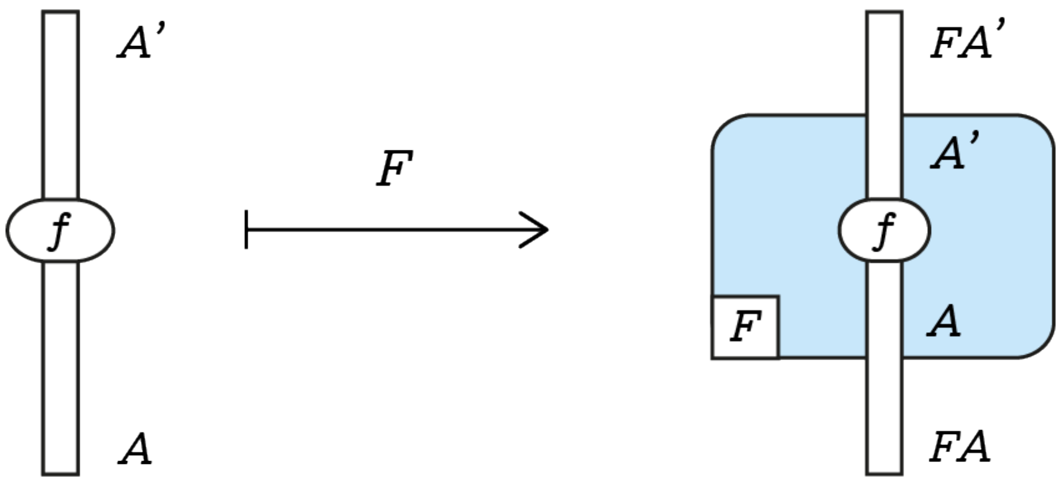


Functorial boxes

In the language of string diagrams, a functor

$$F \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

may be thus depicted as a **functorial box** in this way:



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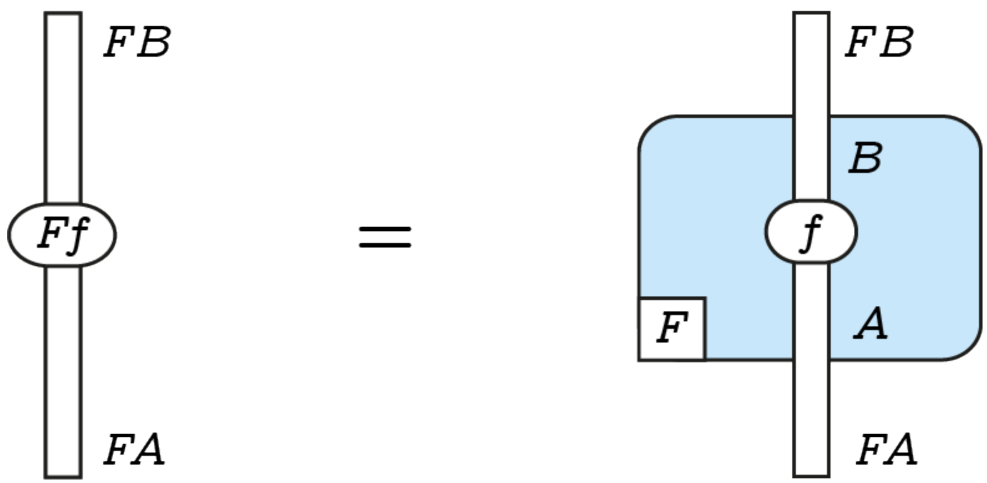
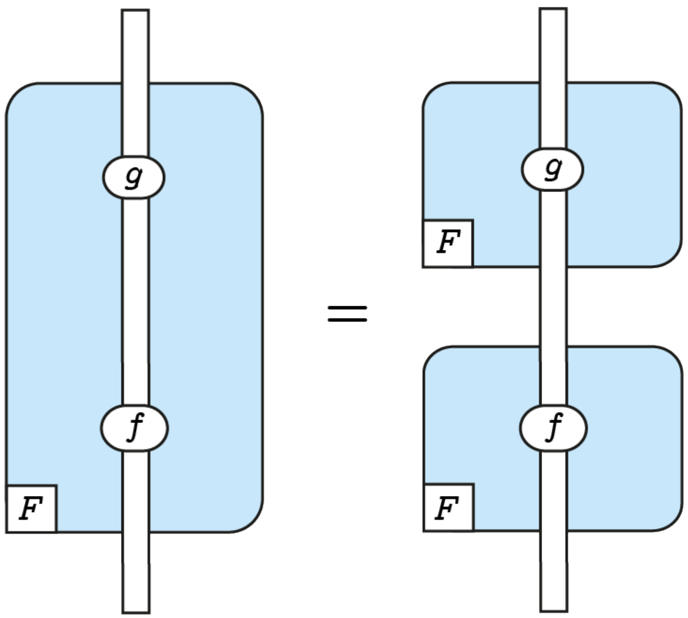
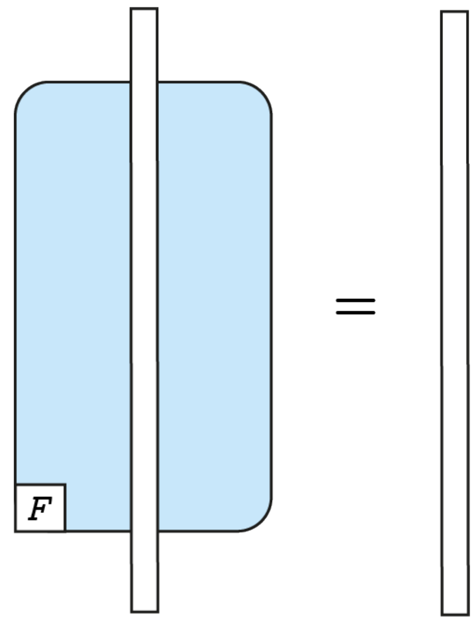


Functorial boxes

Functorial boxes satisfy the following pictorial equations:





$$F(g \circ f) \quad = \quad Fg \circ Ff$$

$$F\,id_A \quad = \quad id_{FA}$$

Natural transformations in string diagrams

What about **natural transformations**

$$\theta \quad : \quad F \Longrightarrow G \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

which are (as we have just seen) defined as a family of maps

$$\theta_A \quad : \quad FA \longrightarrow GA$$

making the diagram commute:

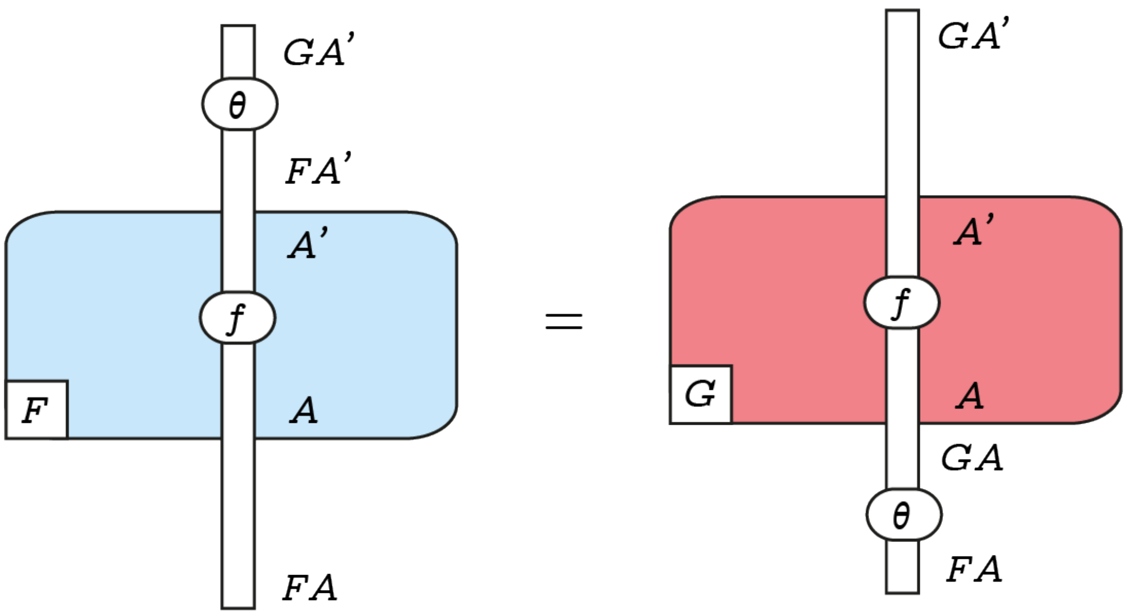
$$\begin{array}{ccc} FA & \xrightarrow{\quad \theta_A \quad} & GA \\ {\scriptstyle Ff}\Big\downarrow & & \Big\downarrow{\scriptstyle Gf} \\ FA' & \xrightarrow{\quad \theta_{A'} \quad} & GA' \end{array}$$

Natural transformations in string diagrams

Natural transformations

$$\theta \quad : \quad F \Longrightarrow G \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

thus satisfy the pictorial equation in string diagrams:



$$\theta_{A'} \circ Ff \quad = \quad Gf \circ \theta_A$$

Back to representation theory

On our way to the mathematical interpretation of linear logic

Representation theory for groups

We have seen that a **linear action** of a group $G = (G, \cdot_G, e_G)$

$$\lambda \quad : \quad G \times V \longrightarrow V$$

is a family of **linear maps** from the vector space V to itself

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parameterized by $g \in G$ and satisfying the two equations:

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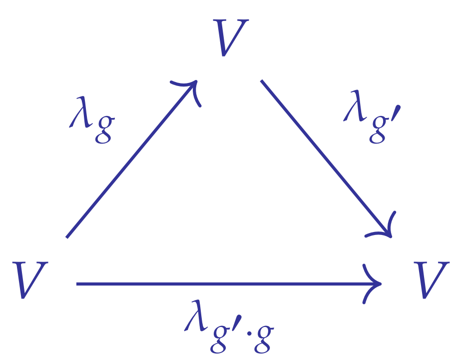
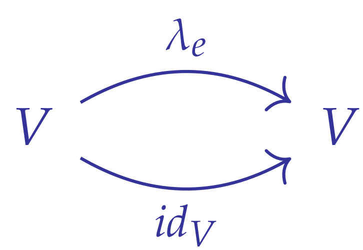




A functorial way to look at representation theory

Key observation: a **linear action**

$$\lambda \quad : \quad G \times V \longrightarrow V$$

is the same thing as a **functor**

$$F \quad : \quad \Sigma G \longrightarrow \mathbf{Vec}$$

from the category ΣG with one object $*$ to the category $\mathbf{Vec}$.

The functor $F : \Sigma G \to \mathbf{Vec}$ associated to the linear action $\lambda : G \times V \to V$

▷ transports the single object $* \in \Sigma G$ to the vector space $V \in \mathbf{Vec}$

▷ transports every map $g : * \to *$ to the linear map $\lambda_g : V \to V$.

Key insight

In order to define

a functorial interpretation of linear logic (as a whole!)

we need to pick in a consistent way:

- ▷ a mathematical interpretation for every formula and every proof.

To that purpose, we will design and investigate

categorified notions of boolean algebras

provided by notions of **monoidal categories** with dualities:

star-autonomous categories	**compact-closed categories**

Key insight

Every **boolean algebra** defines a partial order.

For that reason, there exists **at most** one map between two formulas:

$$A \text{ implies } B \qquad \Longleftrightarrow \qquad A \leq B$$

Categories will enable us to have **different maps** for different proofs:

$$\dfrac{\begin{array}{c}\pi \\ \vdots\end{array}}{A \vdash B} \qquad \Rightarrow \qquad A \xrightarrow{\pi} B$$

Proof theory appears here as a **categorification** of algebraic semantics!

Proof-nets and proof-structures

The distinction between **proof-structures** and **proof-nets** is at the heart of **categorical semantics** with the unifying idea of **free constructions**.

Indeed, as we will see very soon:

the free star-autonomous category

has **formulas of linear logic (MLL)** as objects and **proof-nets** as maps,

the free compact-closed category

has **sequences of atoms** as objects and **proof-structures** as maps.

The free star-autonomous category

Key idea: construct a category **star-autonomous** of a syntactic nature

- ▷ whose objects A, B, C are the **formulas** of linear logic,
- ▷ whose maps between formulas

$$\pi \quad : \quad A \longrightarrow B$$

are the **proofs** of linear logic, defined as **derivation trees**

$$\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{A \vdash B}$$

modulo an **equational equivalence** extending **cut-elimination**:

$$\pi \quad \cong \quad \pi'$$

A few examples of equations

The derivation tree

$$\dfrac{\text{Right}\otimes\ \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash A} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Delta \vdash B}}{\Gamma, \Delta \vdash A \otimes B} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Upsilon_1, A, B, \Upsilon_2 \vdash C}}{\Upsilon_1, A \otimes B, \Upsilon_2 \vdash C}\ \text{Left}\otimes}{\Upsilon_1, \Gamma, \Delta, \Upsilon_2 \vdash C}\ \text{Cut}$$

is equivalent to the derivation tree

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\end{array}}{\Gamma \vdash A} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_2\\ \vdots\end{array}}{\Delta \vdash B} \quad \dfrac{\begin{array}{c}\pi_3\\ \vdots\end{array}}{\Upsilon_1, A, B, \Upsilon_2 \vdash C}}{\Upsilon_1, A, \Delta, \Upsilon_2 \vdash C}\ \text{Cut}}{\Upsilon_1, \Gamma, \Delta, \Upsilon_2 \vdash C}\ \text{Cut}$$

A few examples of equations

The derivation tree

$$\text{Exchange}\ \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1,\cdots,A_n \vdash B}}{A_{\sigma(1)},\ldots,A_{\sigma(n)} \vdash B} \quad \dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, B, \Upsilon_2 \vdash C}$$
$$\dfrac{\qquad\qquad\qquad\qquad\qquad\qquad}{\Upsilon_1, A_{\sigma(1)},\ldots,A_{\sigma(n)}, \Upsilon_2 \vdash C}\ \text{Cut}$$

is equivalent to the derivation tree

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{A_1,\cdots,A_n \vdash B} \quad \dfrac{\dfrac{\begin{matrix}\pi_2\\ \vdots\end{matrix}}{\Upsilon_1, B, \Upsilon_2 \vdash C}}{\Upsilon_1, A_1,\cdots,A_n, \Upsilon_2 \vdash C}\ \text{Cut}}{\Upsilon_1, A_{\sigma(1)},\ldots,A_{\sigma(n)}, \Upsilon_2 \vdash C}\ \text{Exchange}$$

Proof invariants

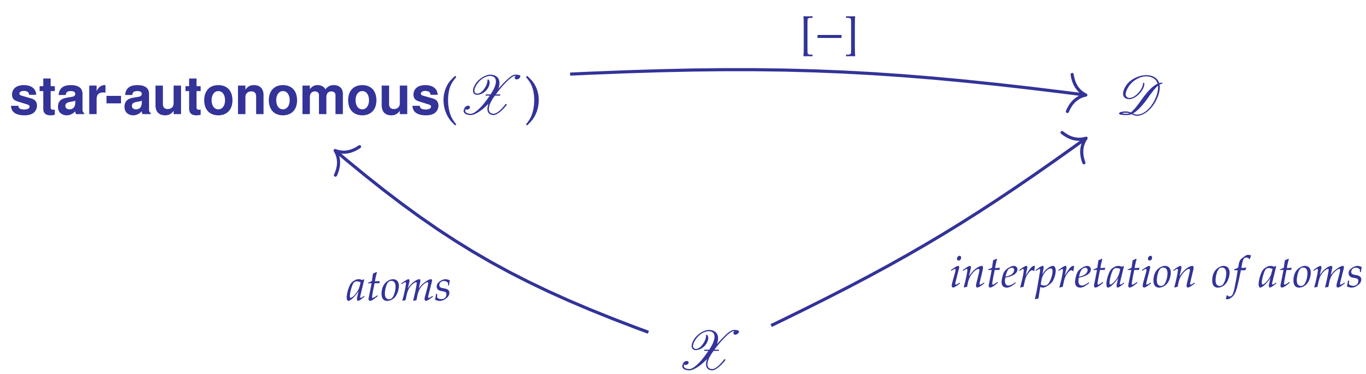
Key property. Every functor to a star-autonomous category $\mathscr{D}$

$$\mathscr{X} \longrightarrow \mathscr{D}$$

lifts uniquely (⋆) to a functor of star-autonomous categories

$$[-] \quad : \quad \textbf{star-autonomous}(\mathscr{X}) \longrightarrow \mathscr{D}$$

defining a **proof invariant** modulo cut-elimination:



(⋆) up to a unique iso

Translating proof-nets into proof-structures

In particular, the **canonical functor** from proof-nets to proof-structures

$$\textbf{star-autonomous}(\mathscr{X}) \longrightarrow \textbf{compact-closed}(\mathscr{X})$$

transports the two **different** maps (= proof-nets) in **star-autonomous**($\mathscr{X}$)

$$\textbf{id}, \textbf{sym} \quad : \quad \bot \otimes \bot \longrightarrow \bot \otimes \bot$$

represented by the derivation trees of linear logic:

$$\textbf{id} \;=\; \dfrac{\dfrac{\dfrac{}{\vdash 1, \bot} \qquad \dfrac{}{\vdash 1, \bot}\text{axiom}}{\vdash 1, 1, \bot \otimes \bot}\otimes\text{-intro}}{\vdash 1 \parr 1, \bot \otimes \bot}\parr\text{-intro} \qquad \textbf{sym} \;=\; \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash 1, \bot} \qquad \dfrac{}{\vdash 1, \bot}\text{axiom}}{\vdash 1, 1, \bot \otimes \bot}\otimes\text{-intro}}{\vdash 1, 1, \bot \otimes \bot}\text{exchange}}{\vdash 1 \parr 1, \bot \otimes \bot}\parr\text{-intro}$$

to the very same map (= proof-structure) in **compact-closed**($\mathscr{X}$).

Cartesian categories

A categorification of the notion of semilattice in order theory

Cartesian products

Suppose given two objects A and B in a category $\mathscr{C}$.

Definition. The **cartesian product** of A and B is a triple

$$(A \times B, \mathbf{fst}, \mathbf{snd})$$

consisting of an object $A \times B$ together with a pair of maps

$$A \xleftarrow{\mathbf{fst}} A \times B \xrightarrow{\mathbf{snd}} B$$

which is **universal** among all such **spans** (= pairs of maps)

$$A \xleftarrow{f} X \xrightarrow{g} B$$

in the category $\mathscr{C}$.

Universal property of the cartesian product

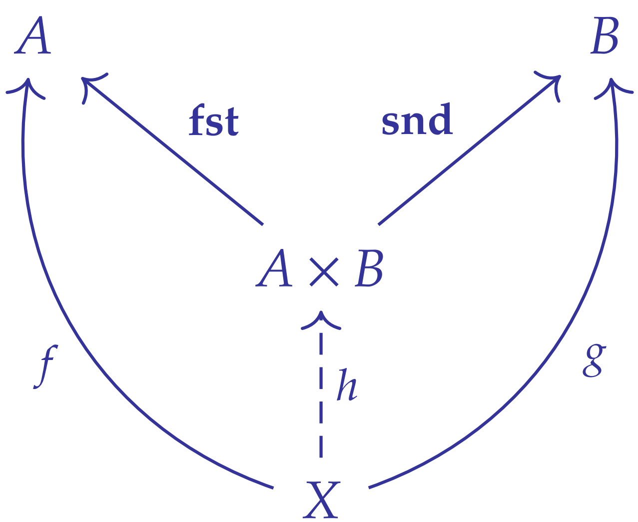
Property. For every object $X \in \mathscr{A}$ equipped with a span

$$f : X \longrightarrow A \qquad\qquad g : X \longrightarrow B$$

there exists a **unique** map

$$h : X \longrightarrow A \times B$$

making the diagram below commute:



$$\mathbf{fst} \circ h = f$$
$$\mathbf{snd} \circ h = g$$

Terminal object

Definition.

An object $\mathbf{1}$ is **terminal** in a category $\mathscr{A}$ when for every object A, there exists a **unique** map

$$A \longrightarrow \mathbf{1}$$

from the object A to the object $\mathbf{1}$.

Cartesian categories

Definition.

A **cartesian category** is a category $\mathscr{C}$ equipped with

- ▷ a cartesian product

$$A \xleftarrow{\ \mathbf{fst}\ } A \times B \xrightarrow{\ \mathbf{snd}\ } B$$

  for every pair of objects A and B of the category,

- ▷ a terminal object $\mathbf{1}$.

A bestiary of cartesian categories

- the category **Set** with the **cartesian product** $A, B \mapsto A \times B$
- the category **Rel** with the **disjoint sum** $A, B \mapsto A + B$
- the category **Grp** with the **cartesian product** $G, H \mapsto G \times H$
- the category **Vec** with the **sum** $V, W \mapsto V \oplus W$
- the category **Top** with the **cartesian product** $X, Y \mapsto X \times Y$
- the category **Coh** with the **with product** $A, B \mapsto A \& B$
- the category **Stab** with the **cartesian product** $D, E \mapsto D \times E$

Functoriality of the cartesian product

Key structural property.

The cartesian product of a cartesian category $\mathscr{C}$ induces a functor

$$\begin{array}{rcl} \mathscr{C} \times \mathscr{C} & \xrightarrow{\textbf{times}} & \mathscr{C} \\ (A, B) & \longmapsto & A \times B \end{array}$$

which transports every pair

$$(A, B) \quad \in \quad \mathscr{C} \times \mathscr{C}$$

to the cartesian product

$$A \times B \quad \in \quad \mathscr{C}$$

in the cartesian category.

Functoriality of the cartesian product

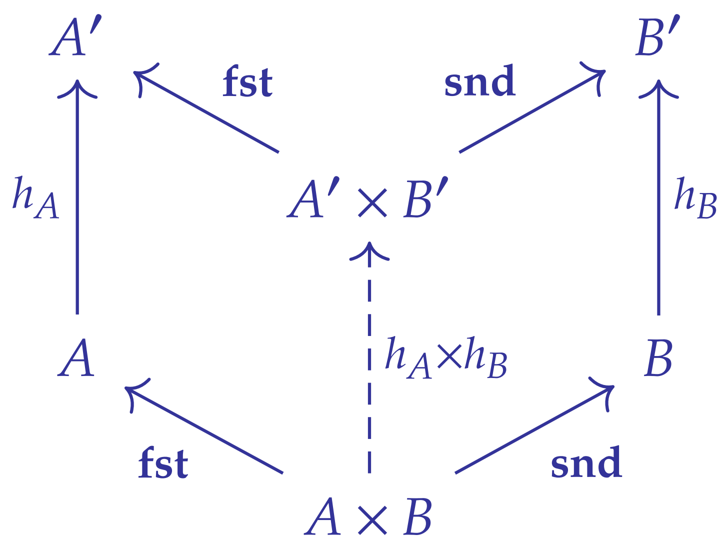
Sketch of the proof: every pair of maps

$$h_A \;:\; A \longrightarrow A' \qquad\qquad h_B \;:\; B \longrightarrow B'$$

induces a map

$$h_A \times h_B \;:\; A \times B \longrightarrow A' \times B'$$

defined as the **unique map** making the diagram below commute:



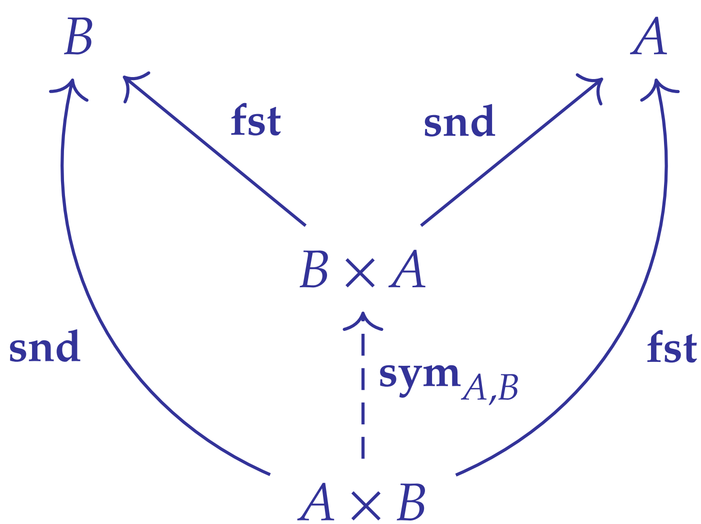
Symmetry maps

Key structural property.

In a cartesian category, every pair A, B comes equipped with a map

$$\mathbf{sym}_{A,B} \quad : \quad A \times B \longrightarrow B \times A$$

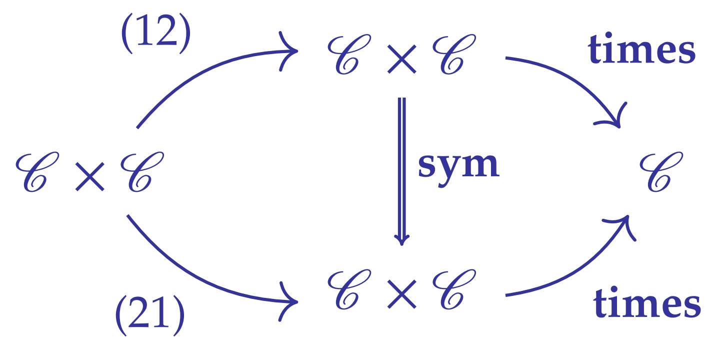
defined as the unique map making the diagram commute:



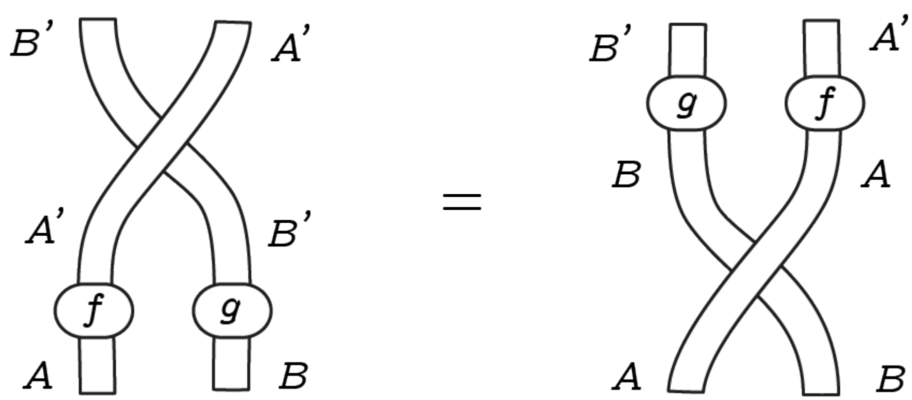
$$\mathbf{fst} \circ \mathbf{sym}_{A,B} = \mathbf{snd}$$
$$\mathbf{snd} \circ \mathbf{sym}_{A,B} = \mathbf{fst}$$

Symmetry maps = braiding = exchange

The family of **symmetry maps** defines a natural transformation



depicted as a **symmetry** in the language of string diagrams:



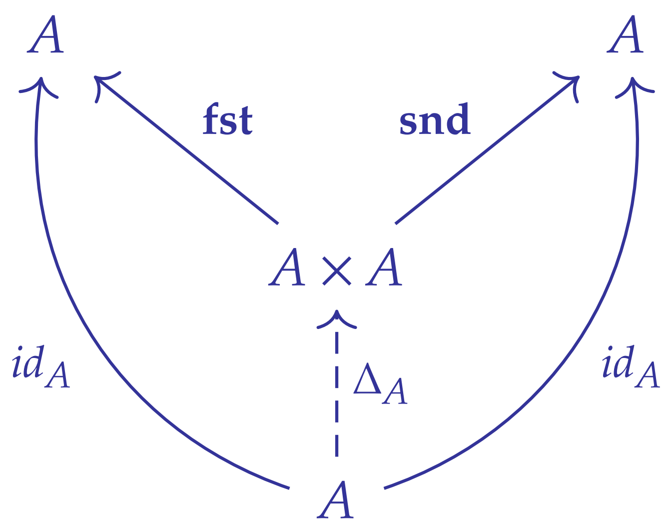
Diagonal maps

Key structural property.

In a cartesian category, every object A comes equipped with a map

$$\Delta_A \quad : \quad A \longrightarrow A \times A$$

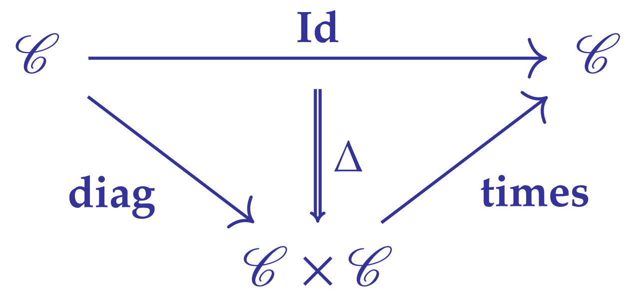
defined as the unique map making the diagram commute:



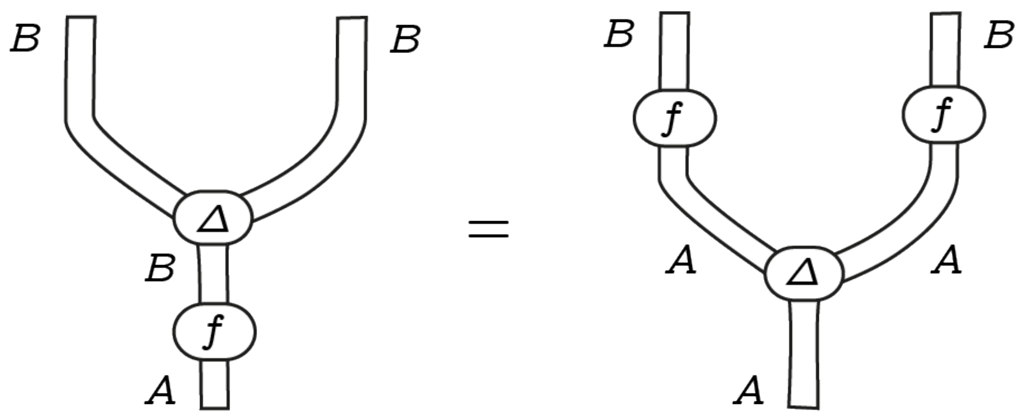
$$\begin{aligned} \mathbf{fst} \circ \Delta_A &= id_A \\ \mathbf{snd} \circ \Delta_A &= id_A \end{aligned}$$

Diagonal maps = duplication = contraction

The family of **diagonal maps** defines a natural transformation



depicted as a **duplicator** in the language of string diagrams:



Eraser maps

Key structural property.

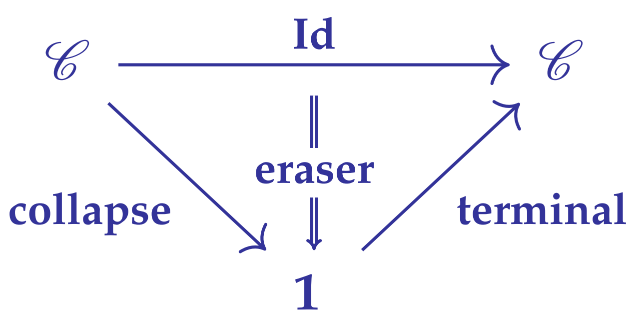
In a cartesian category, every object A comes equipped with a map

$$A \xrightarrow{\ \mathbf{eraser}\ } \mathbf{1}$$

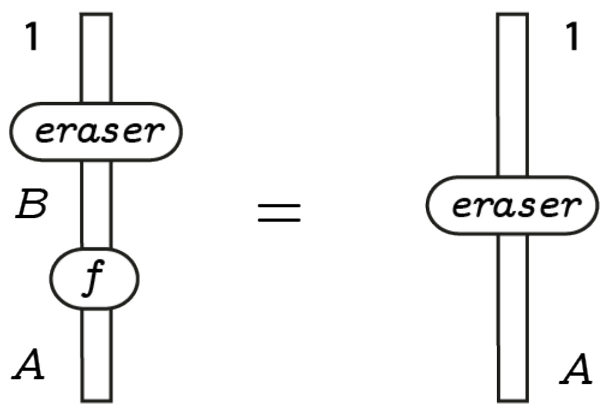
to the terminal object of the cartesian category $\mathscr{C}$.

Eraser maps = garbage collect = weakening

The family of **eraser maps** defines a natural transformation

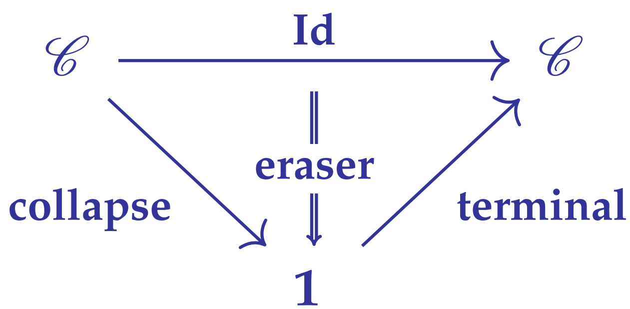


depicted as an **eraser** in the language of string diagrams:

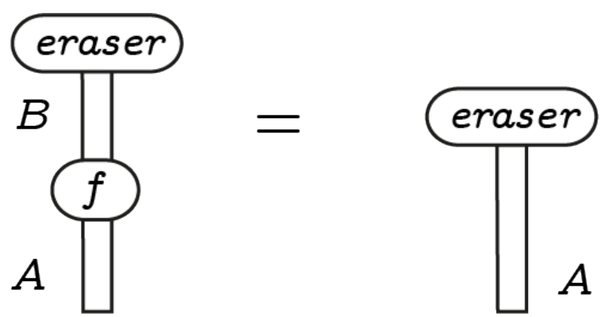


Eraser maps = garbage collect = weakening

The family of **eraser maps** defines a natural transformation



depicted as an **eraser** in the language of string diagrams:



Monoidal categories

The linear counterpart of cartesian categories

Monoidal categories

A **monoidal category** is a category $\mathscr{C}$ equipped with a functor

$$\otimes \quad : \quad \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$$

together with an object $I \in \mathscr{C}$ and three natural transformations:

$$(A \otimes B) \otimes C \xrightarrow{\alpha} A \otimes (B \otimes C)$$

$$I \otimes A \xrightarrow{\lambda} A \qquad\qquad A \otimes I \xrightarrow{\rho} A$$

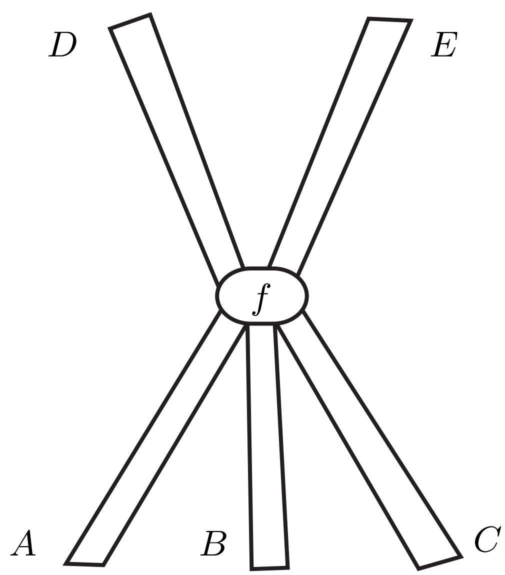
satisfying a series of coherence properties.

String diagrams in monoidal categories

A map in the monoidal category

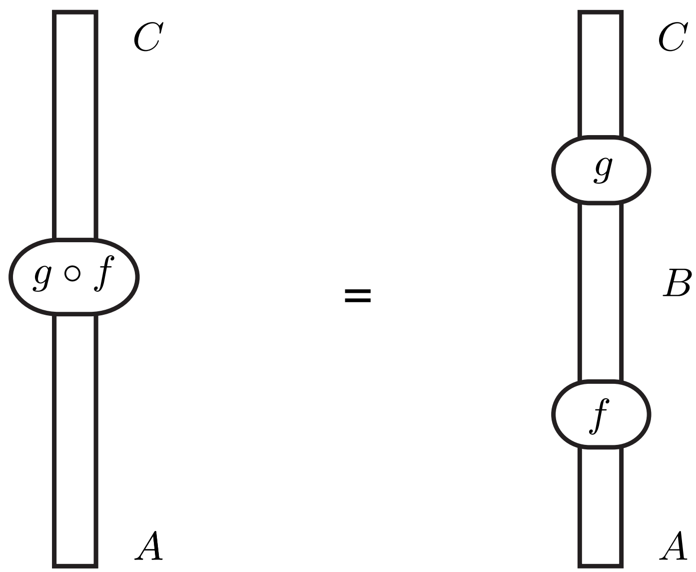
$$f \quad : \quad A \otimes B \otimes C \longrightarrow D \otimes E$$

is depicted as a process taking **three inputs** and producing **two outputs**:



Composition

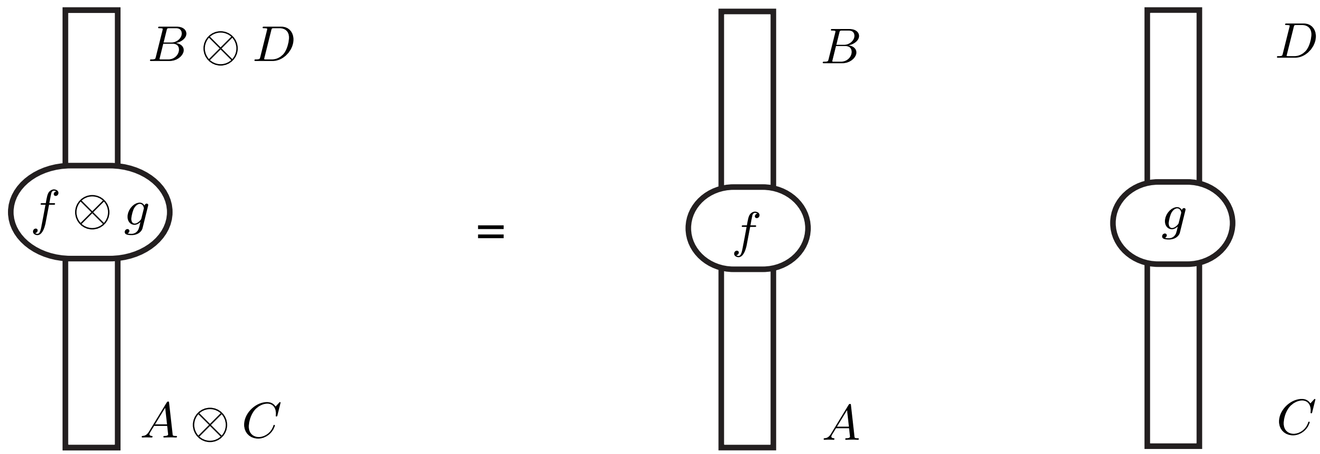
The map $A \xrightarrow{f} B \xrightarrow{g} C$ is depicted as



Vertical composition

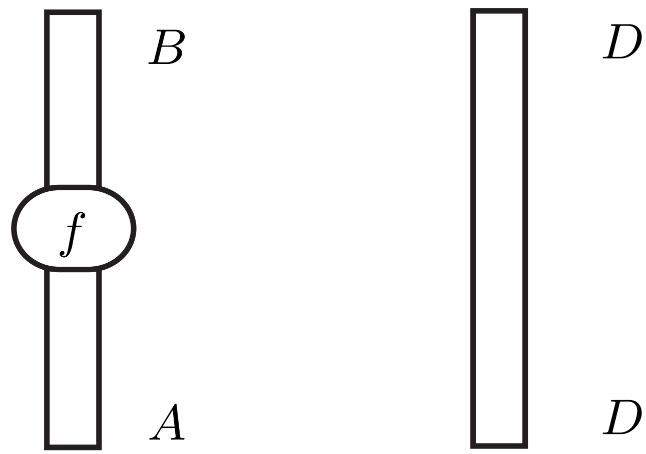
Tensor product

The map $(A \xrightarrow{f} B) \otimes (C \xrightarrow{g} D)$ is depicted as



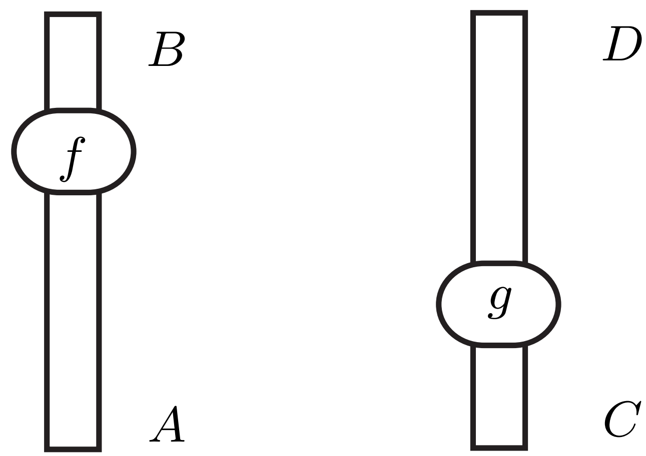
Horizontal tensor product

Example



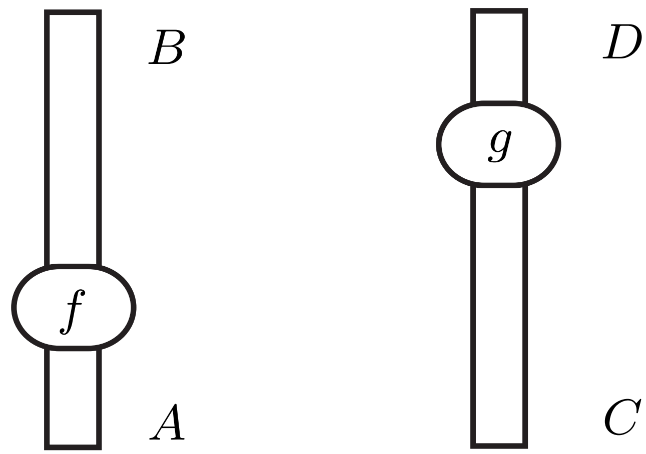
$f \otimes id_D$

Example



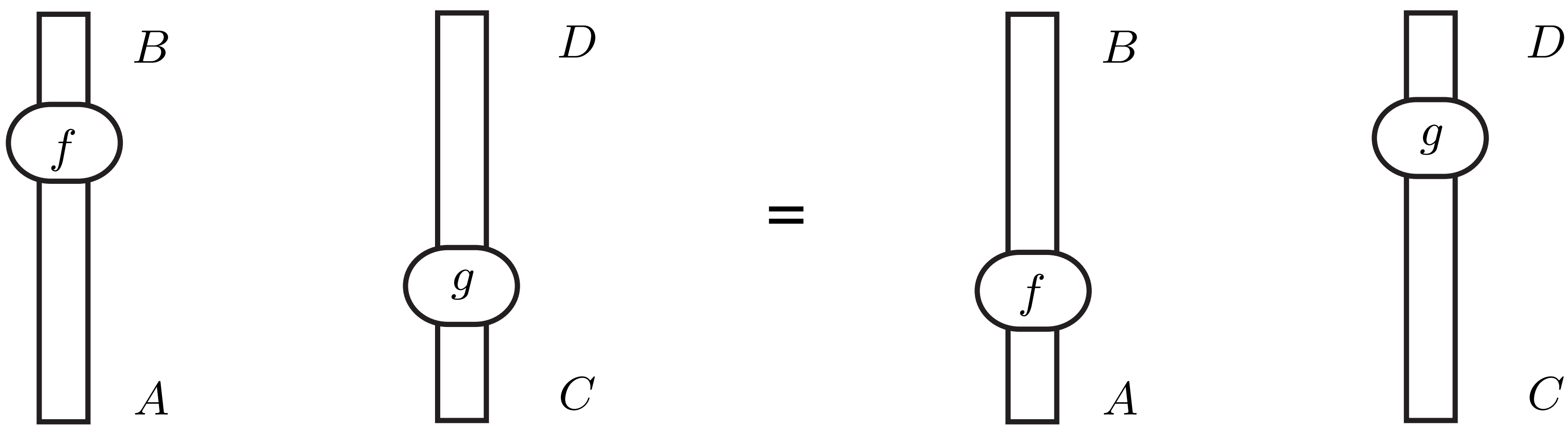
$$(f \otimes id_D) \circ (id_A \otimes g)$$

Example



$$(id_B \otimes g) \circ (f \otimes id_C)$$

Meaning preserved by deformation



$$(f \otimes id_D) \circ (id_A \otimes g) \quad = \quad (id_B \otimes g) \circ (f \otimes id_C)$$

Ribbon categories

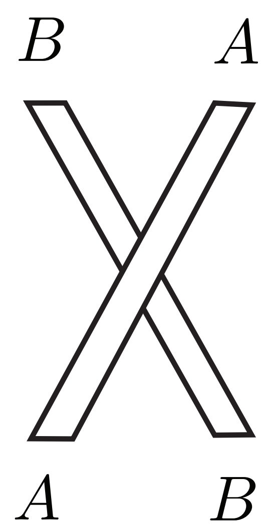
The functorial approach to knot invariants

Braided categories

A monoidal category $\mathscr{C}$ equipped with a family of isomorphisms

$$\gamma_{A,B} \quad : \quad A \otimes B \quad \longrightarrow \quad B \otimes A$$

natural in A and B, represented pictorially as the positive braiding

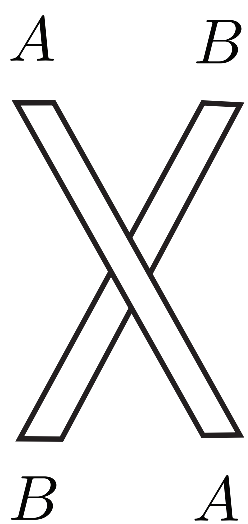


Braided categories

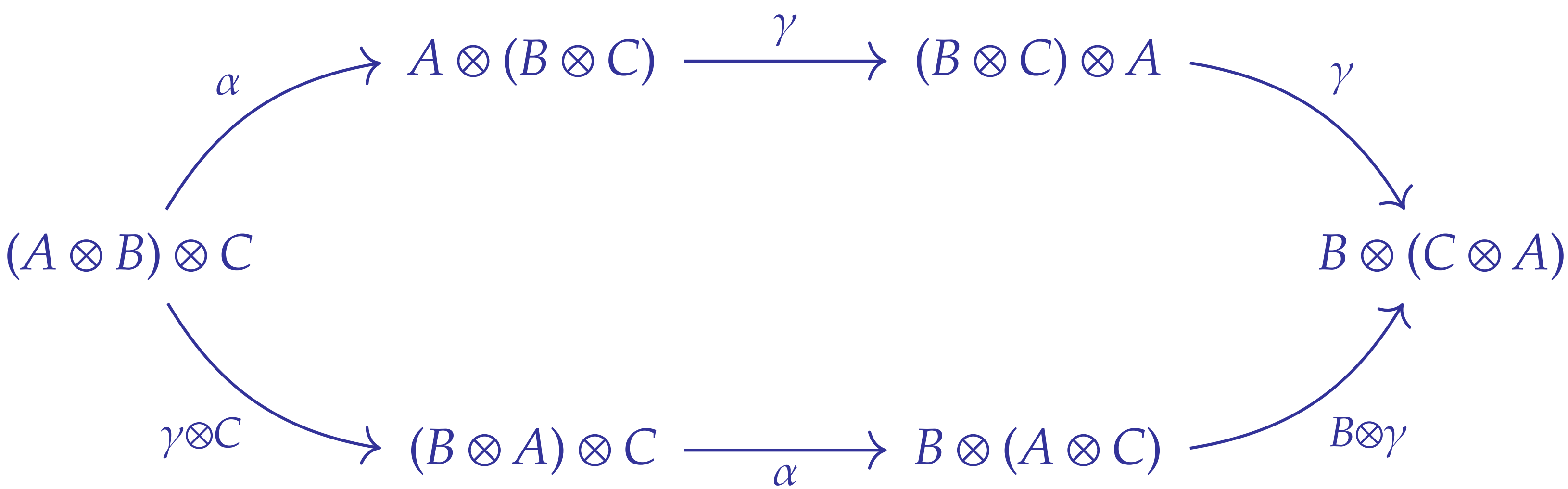
As expected, the inverse map

$$\gamma^{-1}_{A,B} \quad : \quad B \otimes A \quad \longrightarrow \quad A \otimes B$$

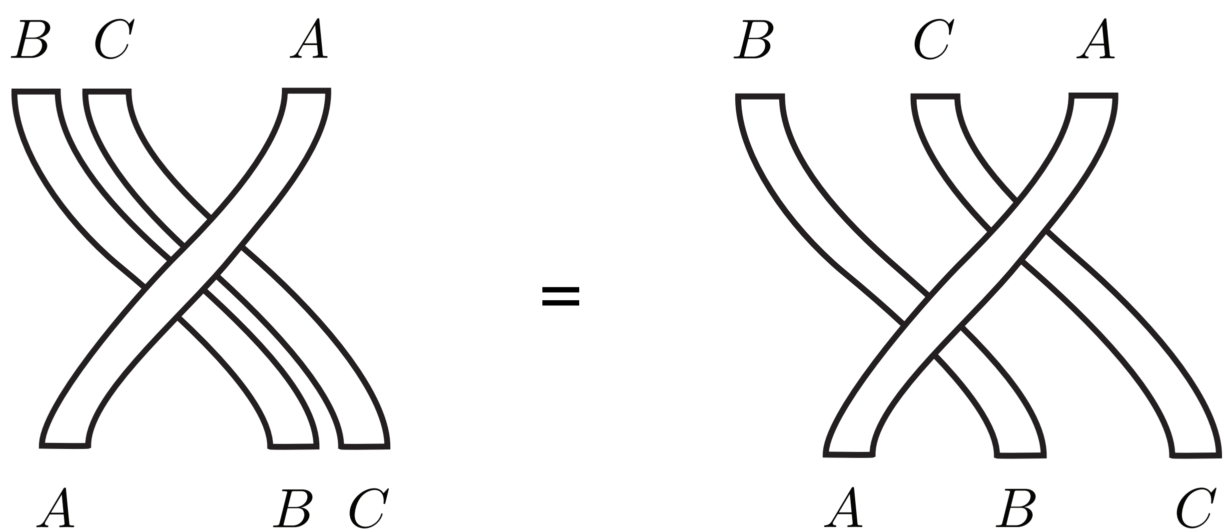
is represented pictorially as the negative braiding



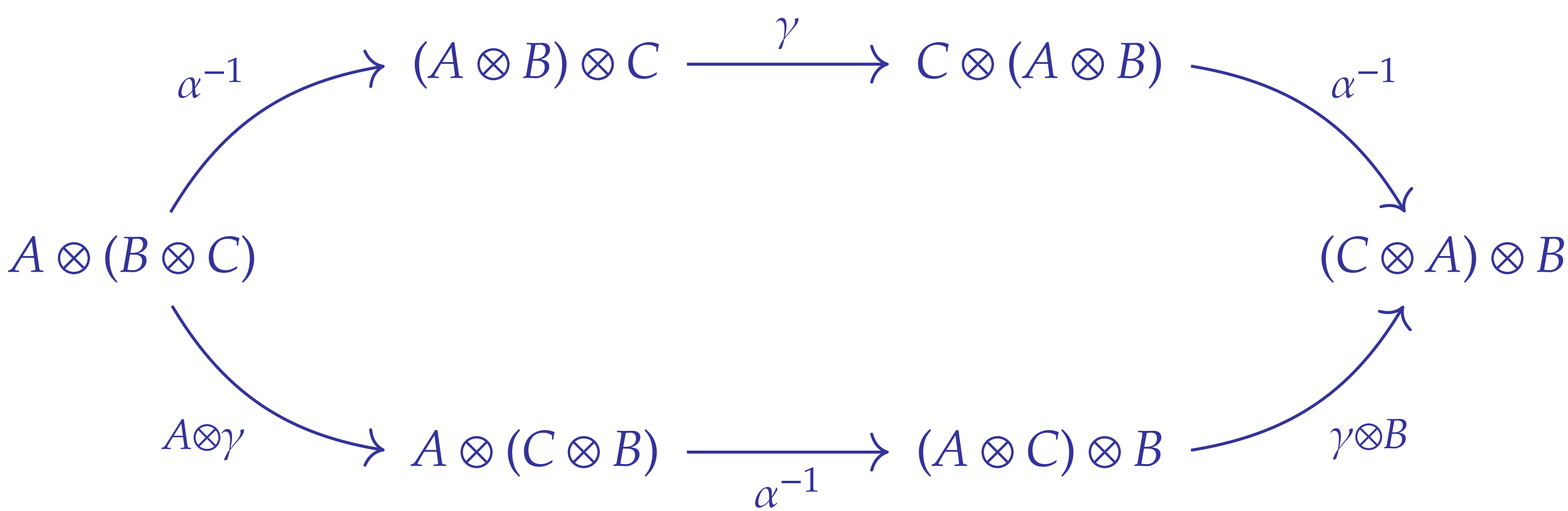
Coherence diagram for braids [1]



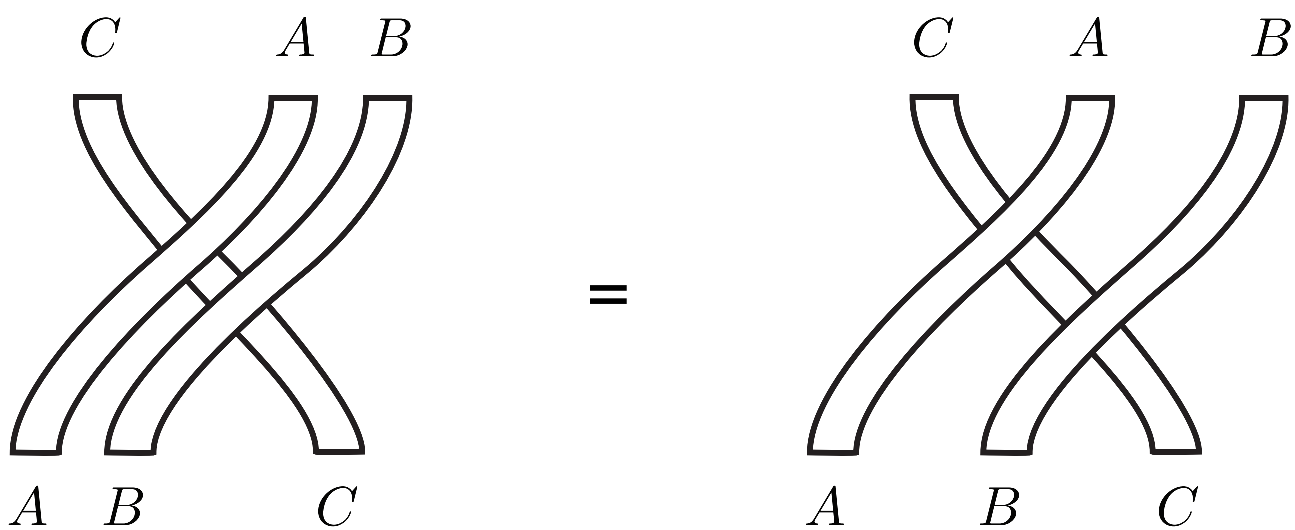
Same coherence diagram in string diagrams



Coherence diagram for braids [2]



Same coherence diagram in string diagrams



Balanced categories

A braided monoidal category $\mathscr{C}$ equipped with a **twist**

$$\theta_A \quad : \quad A \quad \longrightarrow \quad A$$

defined as a natural family of isomorphisms, and depicted as

Coherence for twists

The twist θ is required to satisfy the equality

$$\theta_I \quad = \quad id_I$$

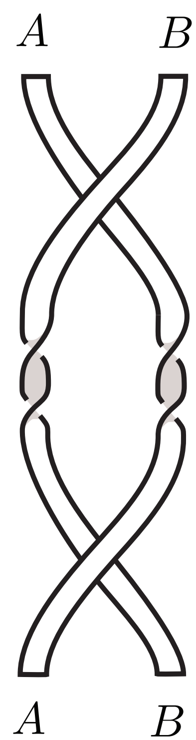
and to make the diagram

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\gamma_{A,B}} & B \otimes A \\ {\scriptstyle \theta_{A\otimes B}}\Big\downarrow & & \Big\downarrow{\scriptstyle \theta_B \otimes \theta_A} \\ A \otimes B & \xleftarrow[\gamma_{B,A}]{} & B \otimes A \end{array}$$

commute for all objects A and B.

Coherence for twists

$\theta_{A\otimes B}$ =

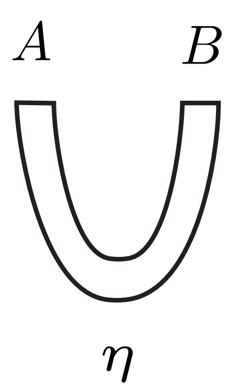


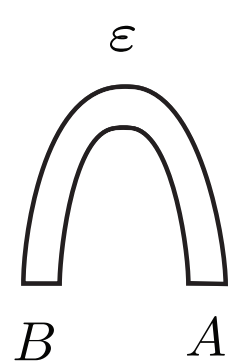
Duality

A **dual pair** $A \dashv B$ is defined as a pair of maps

$$\eta : I \longrightarrow A \otimes B \qquad\qquad \varepsilon : B \otimes A \longrightarrow I$$

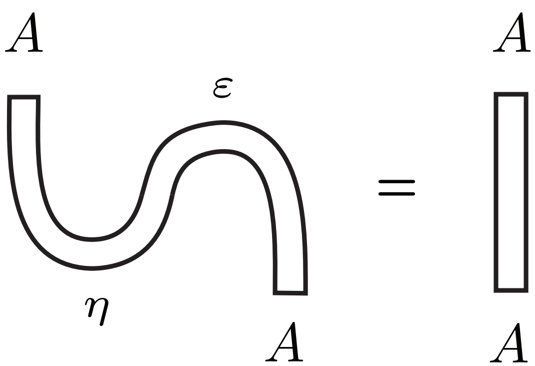
which are depicted as

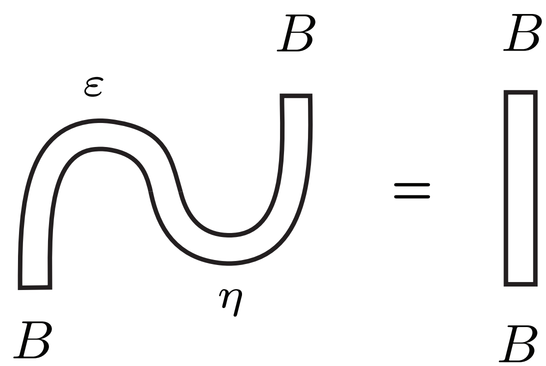




Coherence for duality

The two maps η and ε should satisfy the "zig-zag" equalities:



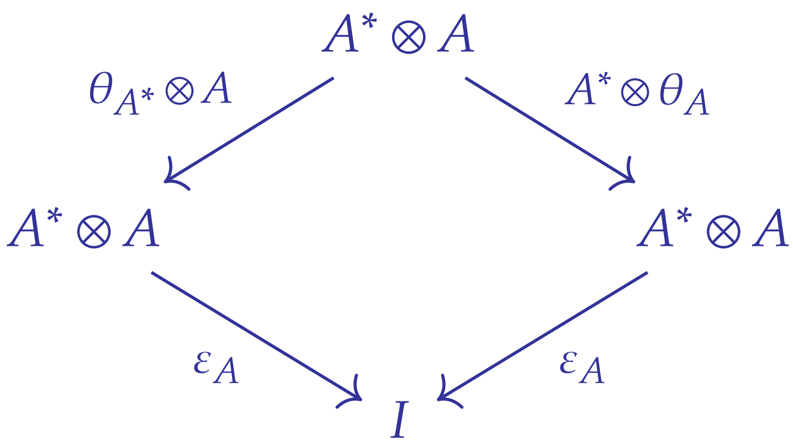


In that case, the object A is called a right dual of the object B.

Ribbon categories

Definition. A ribbon category is a balanced category $\mathscr{C}$ where

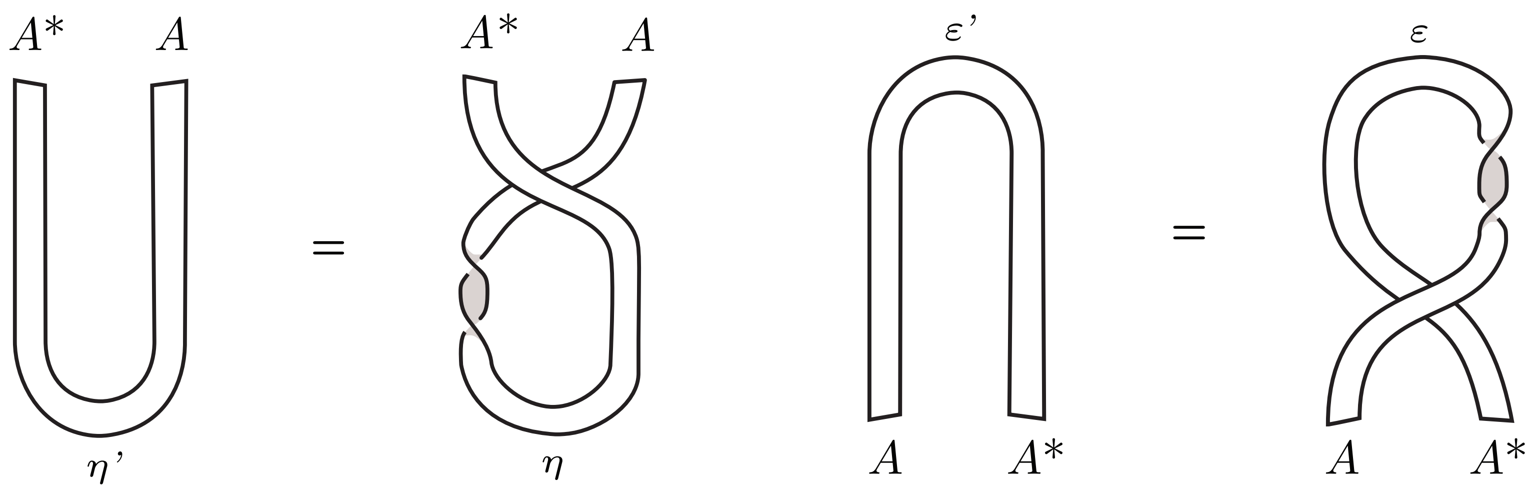
- ▷ every object A has a right dual A^*
- ▷ the diagram



commutes for all objects A.

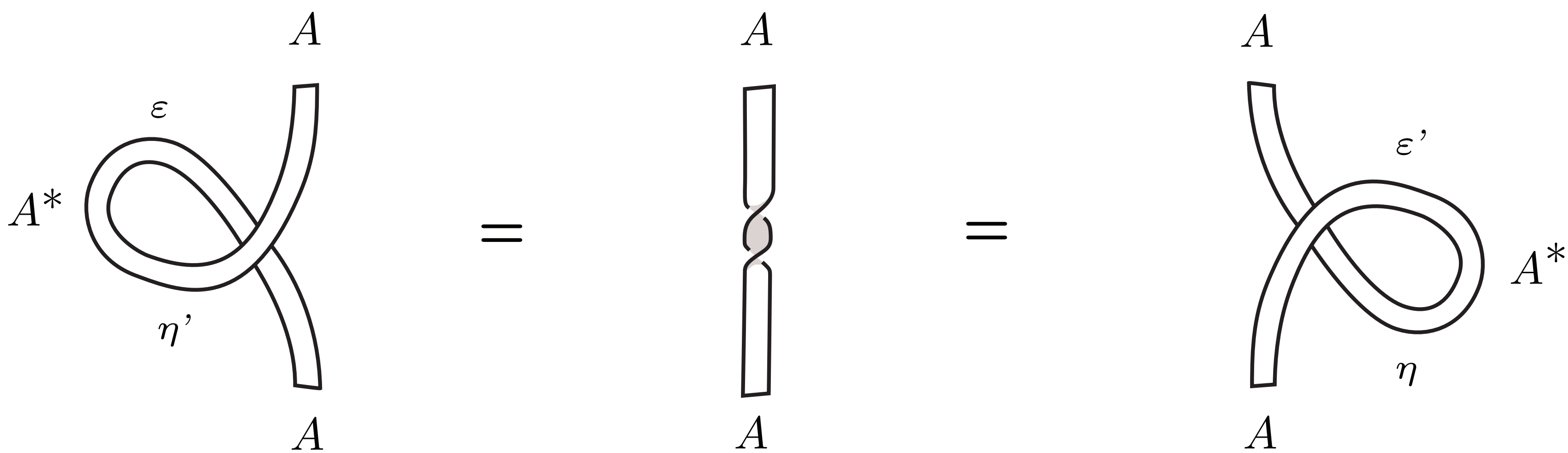
Ribbon categories

Remark. In a ribbon category, the object A^* is also a left dual of A.



Ribbon categories

Hence, the equations below are satisfied in every ribbon category



The free ribbon category

The next theorem offers a bridge between algebra and ribbon topology:

Theorem [Shum 1994]

The free ribbon category **free-ribbon**$(\mathscr{X})$ generated by a category $\mathscr{X}$ has

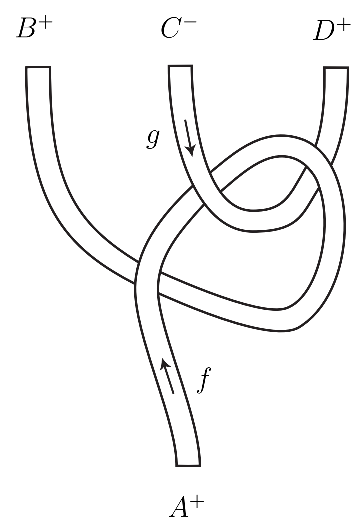
- ▷ **objects:** the signed sequences $(A_1^{\varepsilon_1}, \ldots, A_k^{\varepsilon_k})$ of objects of $\mathscr{X}$,
- ▷ **maps:** the **framed tangles** with links labelled by maps in $\mathscr{X}$.

The free ribbon category

So, a typical map in the category **free-ribbon**($\mathscr{X}$)

$$(A^+) \longrightarrow (B^+, C^-, D^+)$$

looks like this:



where $f : A \longrightarrow B$ and $g : C \longrightarrow D$ are maps in the original category $\mathscr{X}$.

Knot invariants

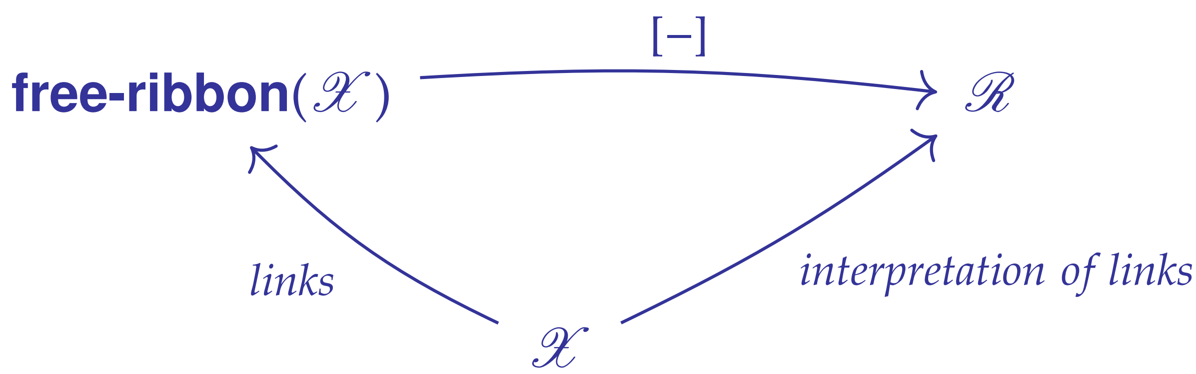
Theorem. Every functor to a ribbon category $\mathscr{R}$

$$\mathscr{X} \longrightarrow \mathscr{R}$$

lifts uniquely (⋆) to a functor of ribbon categories

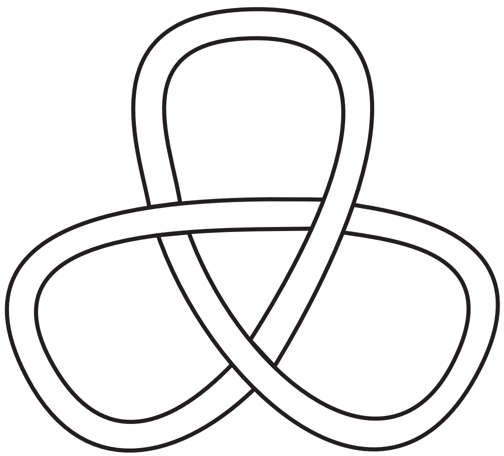
$$[-] \quad : \quad \textbf{ribbon}(\mathscr{X}) \longrightarrow \mathscr{R}$$

defining a **knot invariant** modulo topological deformation:

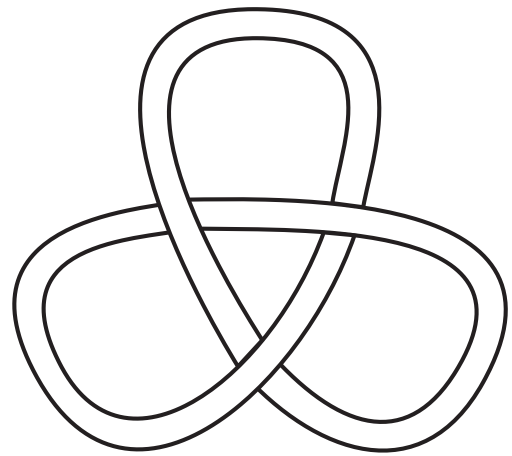


(⋆) up to a unique iso

The Jones polynomial invariant



$$\frac{2}{x^2} + \frac{1}{x^4} + \frac{y^2}{x^2}$$



$$2x^2 - x^4 + x^2y^2$$

Symmetries

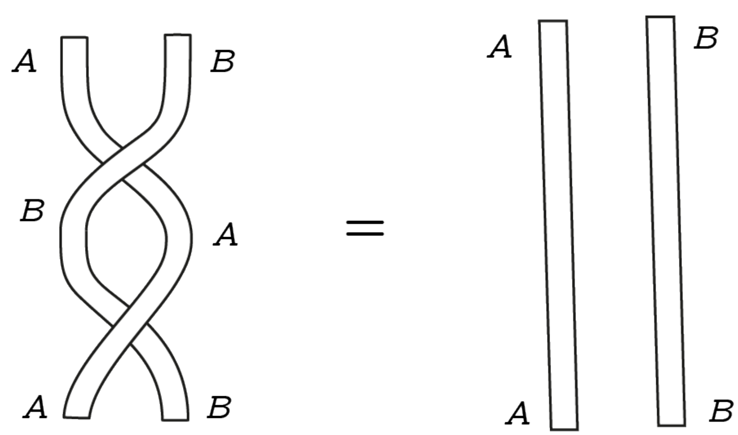
A **symmetry** in a monoidal category is a braiding

$$\gamma_{A,B} \quad : \quad A \otimes B \longrightarrow B \otimes A$$

satisfying the additional equation

$$A \otimes B \xrightarrow{\gamma_{A,B}} B \otimes A \xrightarrow{\gamma_{B,A}} A \otimes B \quad = \quad A \otimes B \xrightarrow{id_{A \otimes B}} A \otimes B$$

The equation may be depicted in string diagrams:



Symmetric monoidal categories

Definition.

A **symmetric monoidal category** is a monoidal category

equipped with a **symmetry**:

$$\gamma_{A,B} \quad : \quad A \otimes B \longrightarrow B \otimes A$$

Observation: a symmetric monoidal category is the same thing as

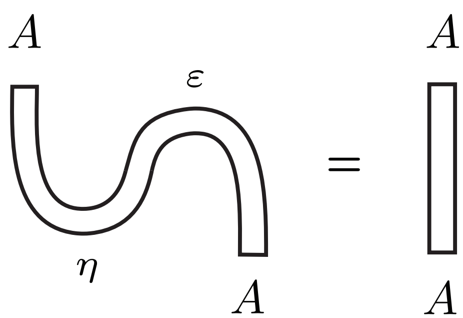
a balanced category whose twist is trivial

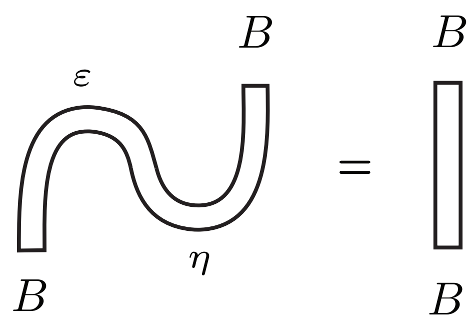
Compact-closed categories

Definition.

A **compact-closed category** is a symmetric monoidal category

where every object A has a right dual B as depicted below:





Observation: a compact-closed category is the same thing as

a ribbon category whose twist is trivial

Proof invariants

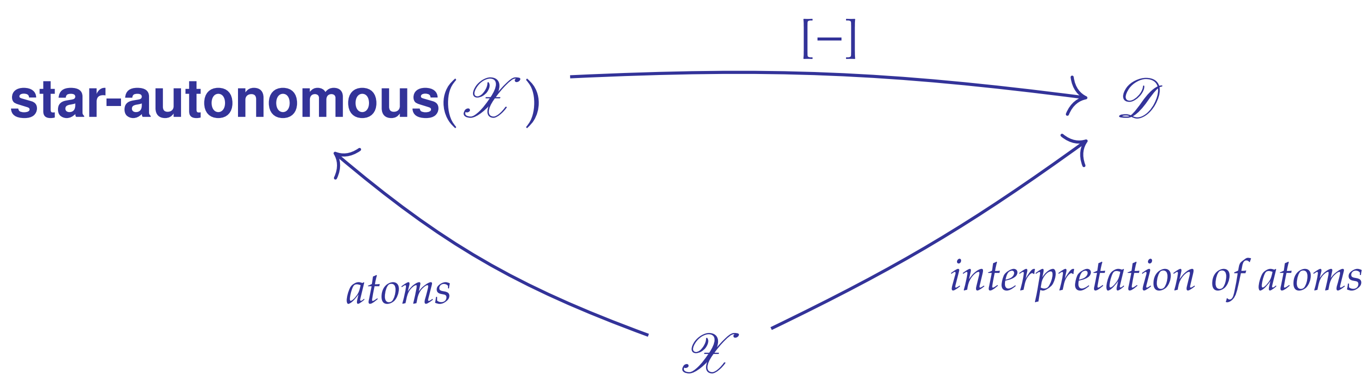
Theorem. Every functor to a star-autonomous category $\mathscr{D}$

$$\mathscr{X} \longrightarrow \mathscr{D}$$

lifts uniquely (⋆) to a functor of star-autonomous categories

$$[-] \quad : \quad \textbf{star-autonomous}(\mathscr{X}) \longrightarrow \mathscr{D}$$

defining a **proof invariant** modulo cut-elimination:



(⋆) up to a unique iso

Symmetric monoidal closed categories

Crossing the boundary between topology and logic

Symmetric monoidal closed categories (smcc)

Definition.

A **symmetric monoidal closed category** is

a symmetric monoidal category

together with, for all objects A and B:

- an object $A \multimap B$
- a map

$$\mathrm{eval}_{A,B} \quad : \quad A \otimes (A \multimap B) \longrightarrow B$$

satisfying a universal property described in the next slide.

Universal property of the linear implication

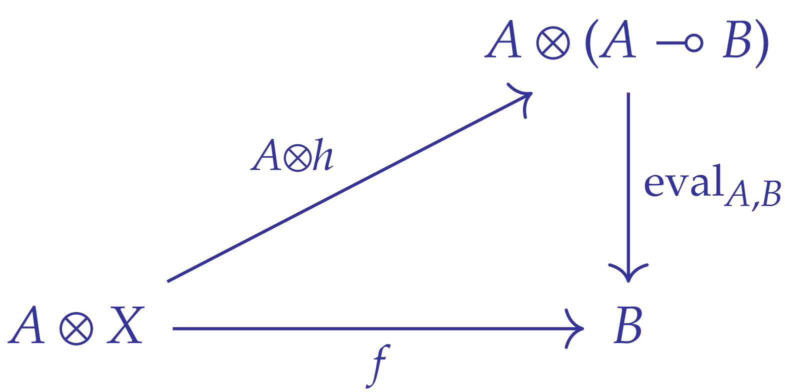
For every object X and for every map

$$f \quad : \quad A \otimes X \longrightarrow B$$

there exists a unique map

$$h \quad : \quad X \longrightarrow A \multimap B$$

making the diagram below commute:



Monoidal exponentiation

Suppose given an object A of a symmetric monoidal category $\mathscr{C}$.

Definition.

A **monoidal exponentiation** of A is a pair consisting of a functor

$$A \multimap - \quad : \quad \mathscr{C} \longrightarrow \mathscr{C}$$

and of a family of bijections

$$\phi_{A,B,C} \quad : \quad \mathbf{Hom}(A \otimes B, C) \xrightarrow{\cong} \mathbf{Hom}(B, A \multimap C)$$

natural in the parameters B and C.

Alternative definition

Definition.

A **symmetric monoidal closed category** is

a symmetric monoidal category

together with a monoidal exponentiation

$$\frac{A \otimes B \longrightarrow C}{B \longrightarrow A \multimap C} \quad \phi_{A,B,C}$$

for all objects A of the category.

The evaluation map

In that formulation, the map

$$\mathrm{eval}_{A,B} : A \otimes (A \multimap B) \longrightarrow B$$

is defined in the following way:

$$\frac{A \multimap B \xrightarrow{id} A \multimap B}{A \otimes (A \multimap B) \longrightarrow B} \quad \phi^{-1}_{A \multimap B, A, B}$$

Multiplicative intuitionistic linear logic

$$A, B \quad ::= \quad \mathbf{1} \mid A \otimes B \mid A \multimap B \mid \alpha$$

Axiom $$\frac{}{A \vdash A}$$

$\multimap$ left $$\frac{\Delta \vdash A \qquad \Gamma, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}$$

$\multimap$ right $$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

$\otimes$ left $$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$\otimes$ right $$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

1 left $$\frac{\Gamma, 1 \vdash A}{\Gamma \vdash A}$$

1 right $$\frac{}{\vdash \mathbf{1}}$$

Cut $$\frac{\Delta \vdash A \qquad \Gamma, A \vdash B}{\Gamma, \Delta \vdash B}$$

Exchange $$\frac{\Gamma, A_1, A_2, \Delta \vdash B}{\Gamma, A_2, A_1, \Delta \vdash B}$$

From symmetric monoidal closed categories

to star-autonomous categories

The joys and marvels of classical linear duality

A general observation

Every pair of objects $A, \bot$ in a smcc comes with an identity

$$id_{A \multimap \bot} \quad : \quad A \multimap \bot \longrightarrow A \multimap \bot$$

which is transported by the bijection $\phi^{-1}_{A \multimap \bot, A, \bot}$ to the map

$$\mathrm{eval}_{A,\bot} \quad : \quad A \otimes (A \multimap \bot) \longrightarrow \bot$$

then becomes by precomposing with symmetry:

$$(A \multimap \bot) \otimes A \longrightarrow \bot$$

and is finally transported by the bijection $\phi_{A \multimap \bot, A, \bot}$ to the map

$$A \longrightarrow (A \multimap \bot) \multimap \bot$$

Star-autonomous categories

Definition

An object $\bot$ is called **dualizing** when the canonical map

$$\partial_A \quad : \quad A \longrightarrow (A \multimap \bot) \multimap \bot$$

is an isomorphism for every object A.

Definition

A **star-autonomous category** is a smcc with a dualizing object.

The category $\mathbf{Coh}$ is star-autonomous

The dualizing object $\bot = 1^*$ is the **singleton** coherence space.

$$
\begin{array}{rcccl}
e = id_{A\multimap\bot} & : & A \multimap \bot \longrightarrow A \multimap \bot & = & \{ ((a,*),(a,*)) \mid a \in |A| \} \\
f = \phi^{-1}_{A\multimap\bot,A,\bot}(e) & : & A \otimes (A \multimap \bot) \longrightarrow \bot & = & \{ ((a,(a,*)),*) \mid a \in |A| \} \\
g = f \circ \gamma_{A,A\multimap\bot} & : & (A \multimap \bot) \otimes A \longrightarrow \bot & = & \{ ((((a,*),a),*) \mid a \in |A| \} \\
\partial_A = \phi_{A\multimap\bot,A,\bot}(g) & : & A \longrightarrow (A \multimap \bot) \multimap \bot & = & \{ (a,((a,*),*)) \mid a \in |A| \}
\end{array}
$$

The resulting map is an isomorphism

$$\partial_A \quad : \quad A \longrightarrow (A \multimap \bot) \multimap \bot$$

with inverse defined as

$$\partial_A^{-1} \quad = \quad \{ (((a,*),*),a) \mid a \in |A| \}$$

Multiplicative linear logic (MLL)

$$A, B ::= A \otimes B \mid \mathbf{1} \mid A ⅋ B \mid \bot \mid \alpha$$

Axiom	$\dfrac{}{\vdash A^{\perp}, A}$
$\otimes$	$\dfrac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$
⅋	$\dfrac{\vdash \Gamma, A, B}{\vdash \Gamma, A ⅋ B}$
1	$\dfrac{}{\vdash 1}$
$\bot$	$\dfrac{\vdash \Gamma}{\vdash \Gamma, \bot}$

- MLL can be interpreted in every **star-autonomous** category.

Multiplicative additive linear logic (MALL)

$$A, B ::= A \oplus B \mid A \otimes B \mid 0 \mid \mathbf{1} \mid A \& B \mid A ⅋ B \mid \top \mid \bot \mid \alpha$$

$\oplus$ left	$\dfrac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}$
$\oplus$ right	$\dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}$
$\&$	$\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \& B}$
0	no rule
$\top$	$\dfrac{}{\vdash \Gamma, \top}$

- MALL can be interpreted in every $\left\{ \begin{array}{c} \textbf{star-autonomous} \\ \textbf{and cartesian} \end{array} \right\}$ category.

The exponential modality

The alchimy of combining additives and multiplicatives

A new ingredient: the exponential

The **exponential modality**

$$A \mapsto\ !A$$

transports a coherence space A to the coherence space $!\,A$

- $\triangleright$ whose web $|\,!\,A\,|$ is the set of finite cliques of A,
- $\triangleright$ $u \frown_{!\,A} v$ iff the union $u \cup v$ is a finite clique of A.

The coherence space $?\,A$ is defined by de Morgan duality:

$$?\,A \quad = \quad (\,!\,A^{\perp}\,)^{\perp}$$

The exponential alchimy

The **exponential modality** transmutes the **additives** into **multiplicatives**

The terminology « exponential » is justified by the isomorphisms:

$$!(A \& B) \cong !A \otimes !B \qquad !\top \cong 1$$

which are reminiscent of the set-theoretic bijections:

$$\wp(A + B) \cong \wp(A) \times \wp(B)$$

The exponential alchimy

We will study the formal properties of the exponential required by

a Seely category

in order to define a model of linear logic.

- ▷ every object $!A$ defines a **commutative comonoid** $(\,!A\,,\,d_A\,,\,e_A\,)$,
- ▷ the exponential modality defines a **comonad** $(\,!\,,\,\delta\,,\,\epsilon\,)$
- ▷ the cartesian diagonal

$$A \longrightarrow A \,\&\, A$$

is transported to the comonoidal diagonal

$$!A \longrightarrow\ !A \otimes !A.$$

Linear logic (LL)

$$A, B ::= A \oplus B \mid A \otimes B \mid !A \mid 0 \mid \mathbf{1} \mid A \& B \mid A \parr B \mid ?A \mid \top \mid \bot \mid \alpha$$

contraction $\dfrac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}$

weakening $\dfrac{\vdash \Gamma}{\vdash \Gamma, ?A}$

dereliction $\dfrac{\vdash \Gamma, A}{\vdash \Gamma, ?A}$

digging $\dfrac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}$

Monoids

A **monoid** in a monoidal category $(\mathscr{C}, \otimes, 1)$ is a triple

$$1 \xrightarrow{\ u\ } A \xleftarrow{\ m\ } A \otimes A$$

consisting of an object A and of two maps making the diagrams commute:

$$\begin{array}{ccccc}
A \otimes (A \otimes A) & \xrightarrow{\ \alpha\ } & (A \otimes A) \otimes A & \xrightarrow{\ m \otimes A\ } & A \otimes A \\
{\scriptstyle A \otimes m}\downarrow & & & & \downarrow{\scriptstyle m} \\
A \otimes A & & \xrightarrow{\qquad\quad m \qquad\quad} & & A
\end{array}$$

$$\begin{array}{ccccc}
1 \otimes A & \xrightarrow{\ u \otimes A\ } & A \otimes A & \xleftarrow{\ A \otimes u\ } & A \otimes 1 \\
{\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \rho} \\
A & = & A & = & A
\end{array}$$

Comonoids

Dually, a **comonoid** in a monoidal category $(\mathscr{C}, \otimes, 1)$ is a triple

$$1 \xleftarrow{\ e\ } A \xrightarrow{\ d\ } A \otimes A$$

consisting of an object A and of two maps making the diagrams commute:

$$\begin{array}{ccccc}
A & & \xrightarrow{\qquad\qquad d \qquad\qquad} & & A \otimes A \\
{\scriptstyle d}\downarrow & & & & \downarrow{\scriptstyle d \otimes A} \\
A \otimes A & \xrightarrow{A \otimes d} & A \otimes (A \otimes A) & \xrightarrow{\ \alpha\ } & (A \otimes A) \otimes A
\end{array}$$

$$\begin{array}{ccccc}
A & = & A & = & A \\
{\scriptstyle \lambda}\uparrow & & \downarrow{\scriptstyle d} & & \uparrow{\scriptstyle \rho} \\
1 \otimes A & \xleftarrow{e \otimes A} & A \otimes A & \xrightarrow{A \otimes e} & A \otimes 1
\end{array}$$

Commutative comonoid

A comonoid in a symmetric monoidal category

$$1 \xleftarrow{\;e\;} A \xrightarrow{\;d\;} A \otimes A$$

is **commutative** when the diagram below commutes:

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{d} & & \searrow^{d} & \\ A \otimes A & & \xrightarrow{\quad \gamma_{A,A} \quad} & & A \otimes A \end{array}$$

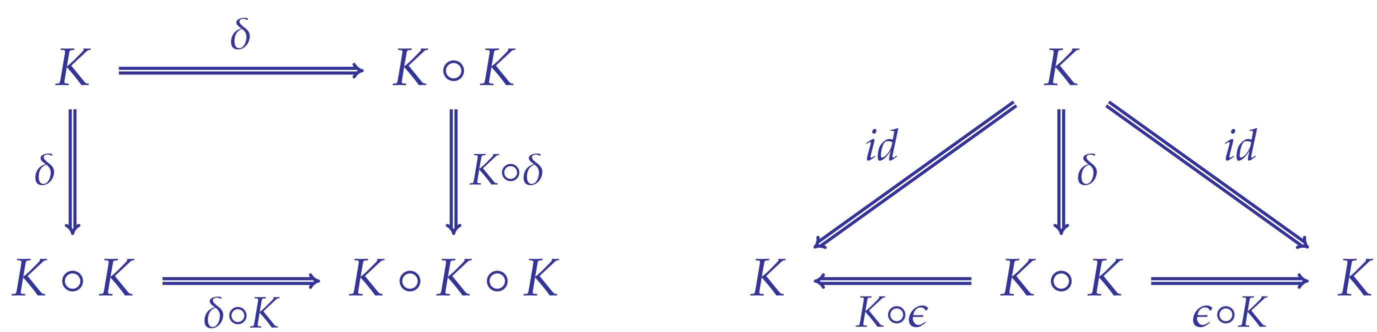
Comonad

A comonad (K, δ, ϵ) in a category $\mathscr{C}$ is the data of

- ▷ a functor $K : \mathscr{C} \longrightarrow \mathscr{C}$
- ▷ two natural transformations

$$\delta : K \Longrightarrow K \circ K \qquad\qquad \epsilon : K \Longrightarrow Id_{\mathscr{C}}$$

such that the following diagrams commute:



Seely categories

Definition. A **Seely category** is

a **star-autonomous** and **cartesian** category $(\mathscr{L}, \otimes, 1)$

equipped with a comonad

$$(\,!\,,\,\delta\,,\,\epsilon\,) \quad : \quad \mathscr{L} \longrightarrow \mathscr{L}$$

and two natural isomorphisms

$$m_{A,B} \quad : \quad !\,A \otimes !\,B \;\cong\; !\,(A \,\&\, B) \qquad\qquad m \quad : \quad 1 \;\cong\; !\,\top$$

defining a **symmetric monoidal functor**

$$(\,!\,,\,m\,) \quad : \quad (\mathscr{L}, \&, \top) \longrightarrow (\mathscr{L}, \otimes, \mathbf{1})$$

from the cartesian structure of $\mathscr{L}$ to its symmetric monoidal structure.

Seely categories

One asks in addition that the diagram

$$\begin{array}{ccc}
!A \otimes !B & \xrightarrow{\;m\;} & !(A \,\&\, B) \\
 & & \downarrow{\scriptstyle \delta_{A \,\&\, B}} \\
\downarrow{\scriptstyle \delta_A \otimes \delta_B} & & !!(A \,\&\, B) \\
 & & \downarrow{\scriptstyle !\langle !\pi_1, !\pi_2 \rangle} \\
!!A \otimes !!B & \xrightarrow{\;m\;} & !(!A \,\&\, !B)
\end{array}$$

commutes in the category $\mathscr{L}$ for all objects A and B.

The polychromatic interpretation of linear logic

Definition.

A model of linear logic is a **symmetric monoidal adjunction**

$$\mathscr{M} \underset{Lin}{\overset{Mult}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathscr{L}$$

$\mathscr{M}$ cartesian

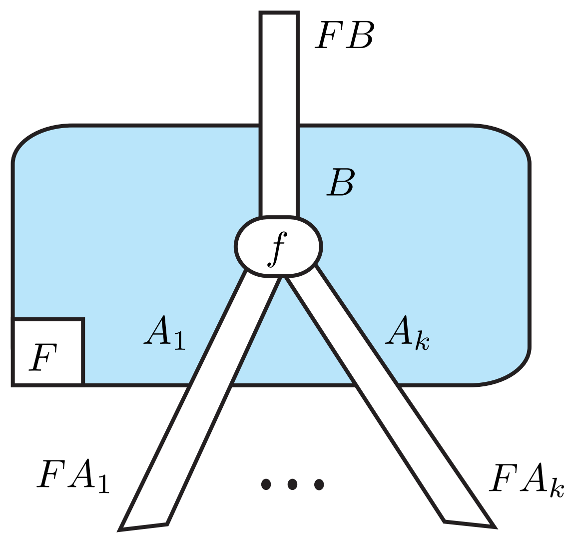
$\mathscr{L}$ star-autonomous

$$\boxed{\; ! \quad = \quad Lin \circ Mult \;}$$

Equivalently: an adjunction whose left adjoint Lin is strong monoidal

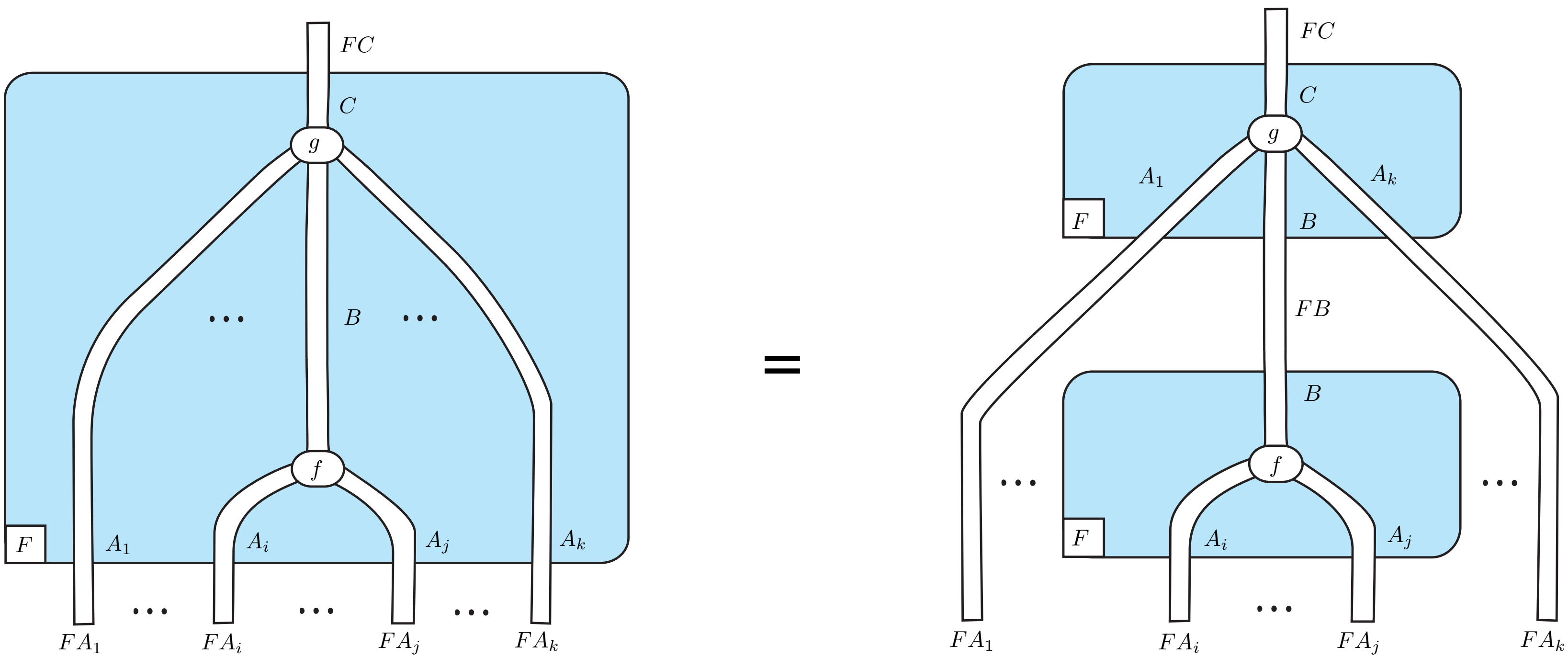
Lax monoidal functor

A **lax monoidal functor** is a box with **many inputs - one output**.



$$F(f) \circ m_{[A_1,\cdots,A_k]} \quad : \quad FA_1 \otimes \cdots \otimes FA_k \longrightarrow FB$$

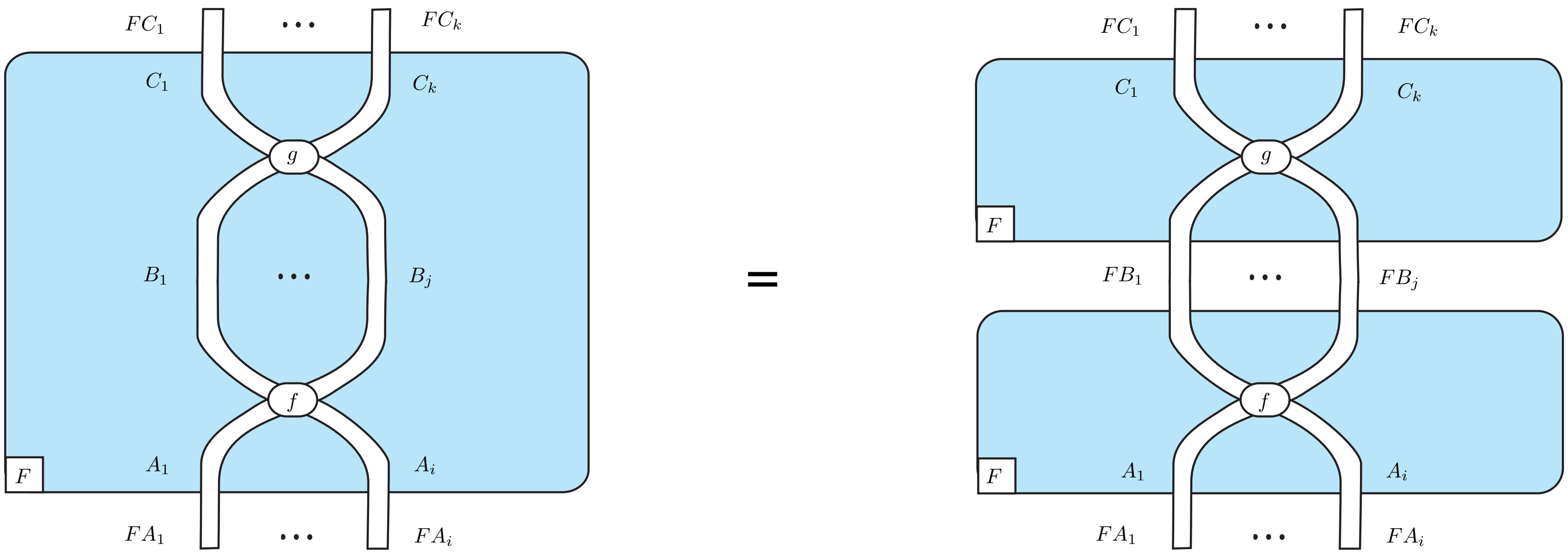
Functorial equalities (on lax functors)



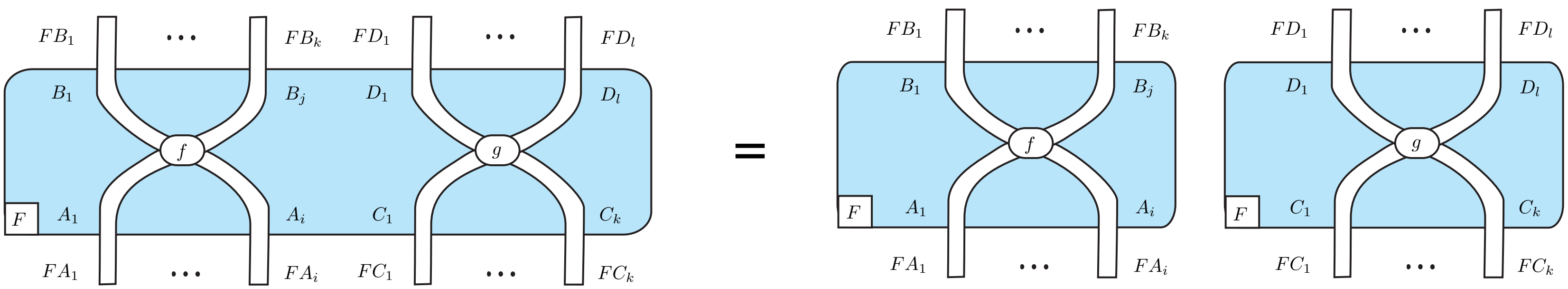
Strong monoidal functors

A **strong monoidal functor** is a box with **many inputs - many outputs**

Functorial equalities (on strong functors)



Functorial equalities (on strong functors)

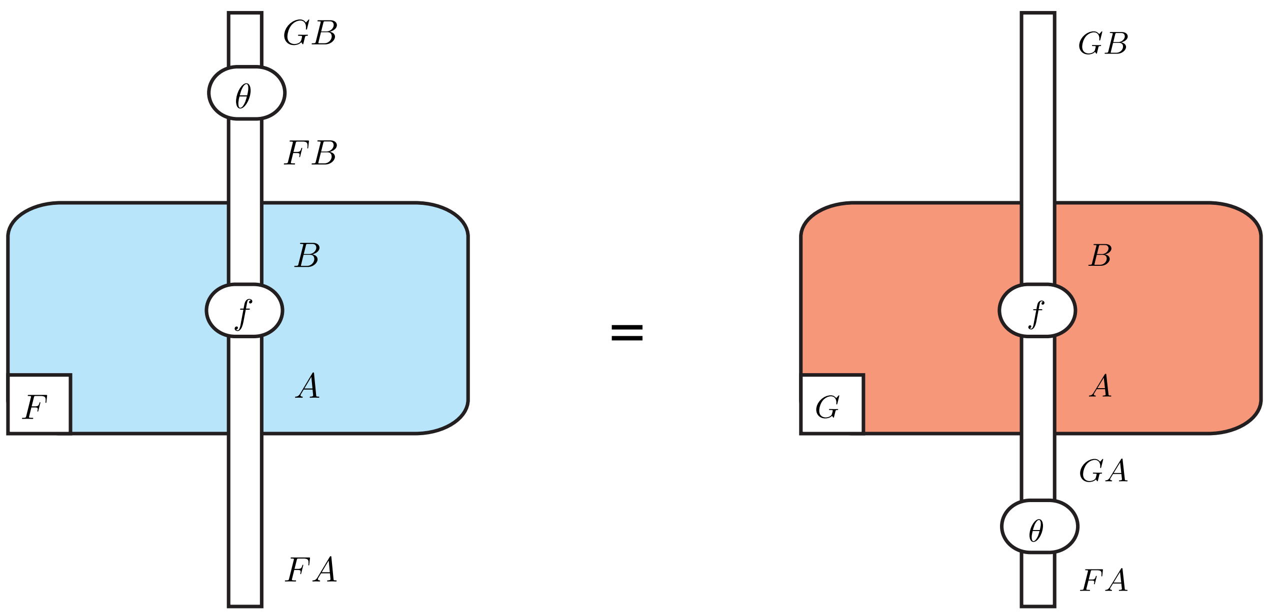


Natural transformations

About one hour ago, we have seen that a natural transformation

$$\theta \quad : \quad F \Longrightarrow G \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

satisfies the pictorial equation in string diagrams:

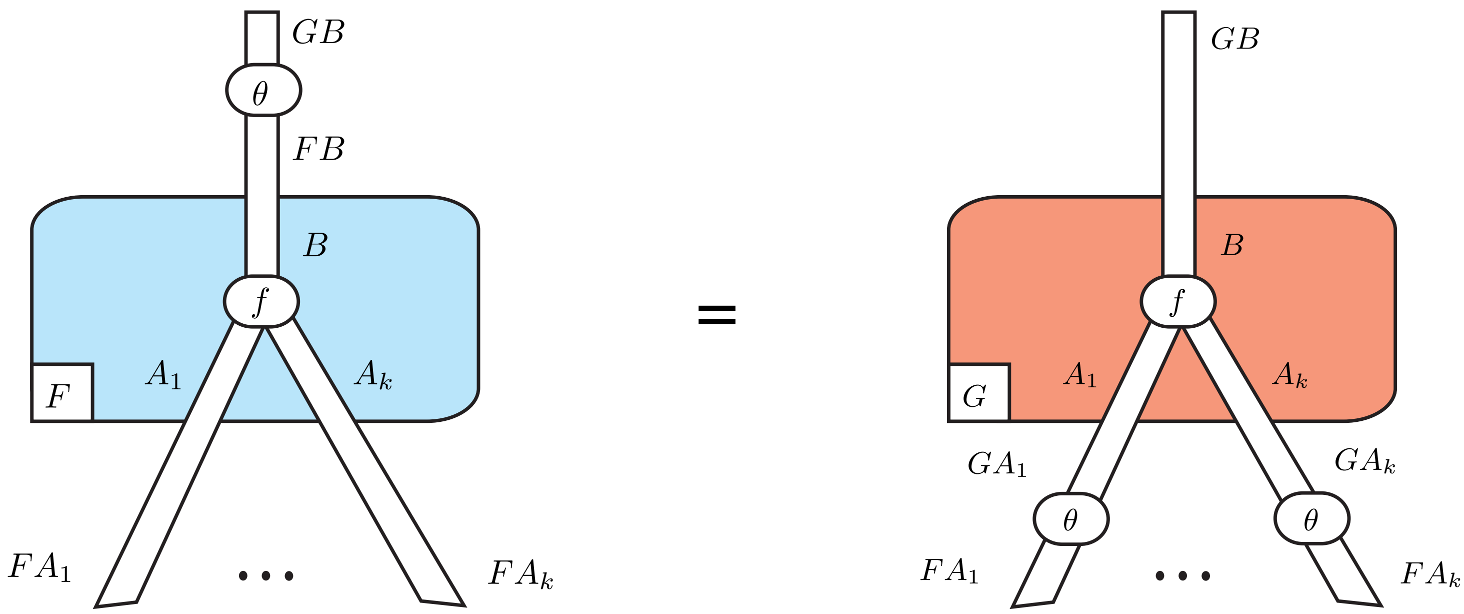


Monoidal natural transformations

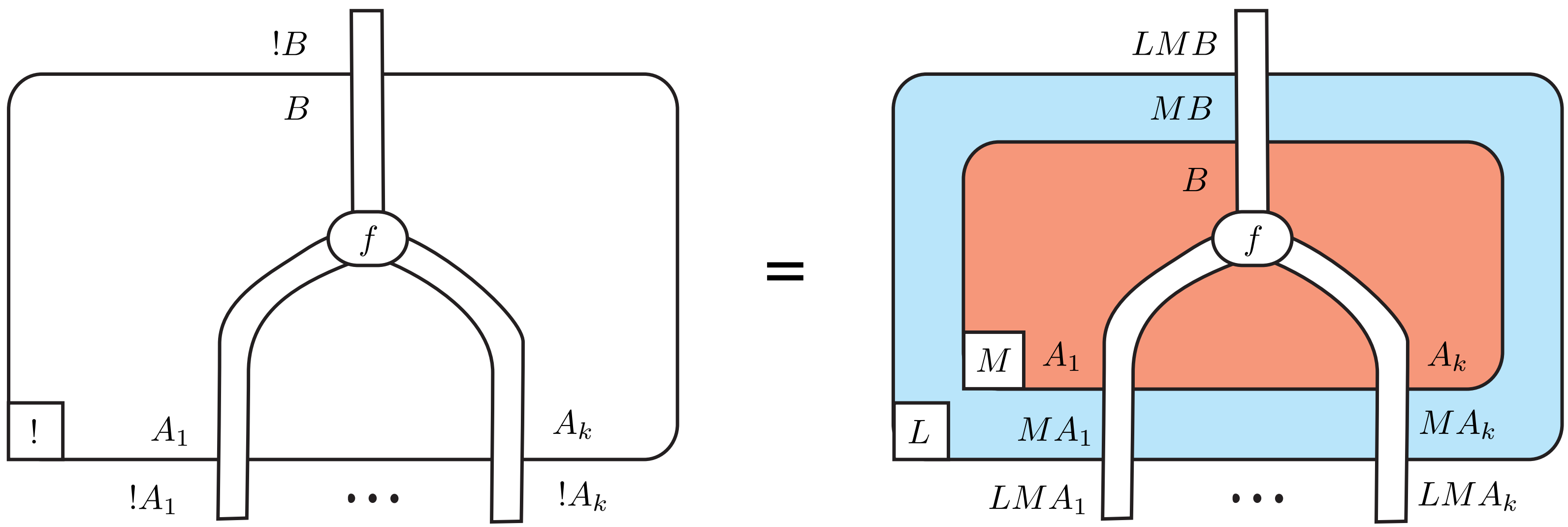
Similarly, a **monoidal** natural transformation

$$\theta \quad : \quad F \Longrightarrow G \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

satisfies the pictorial equation:



Decomposition of the exponential box



Decomposition of the contraction node

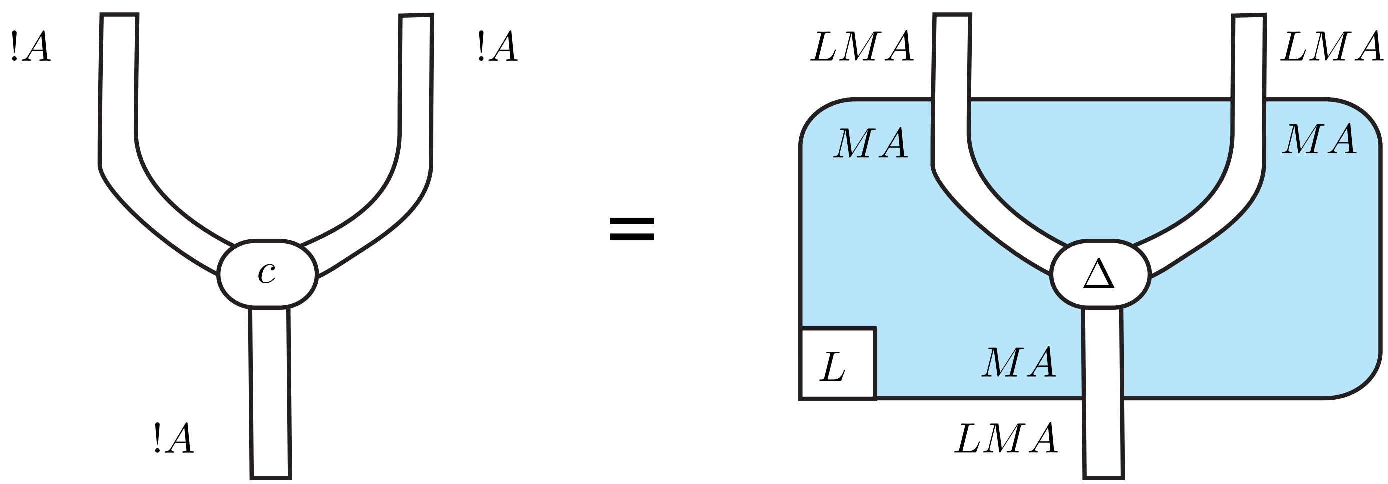
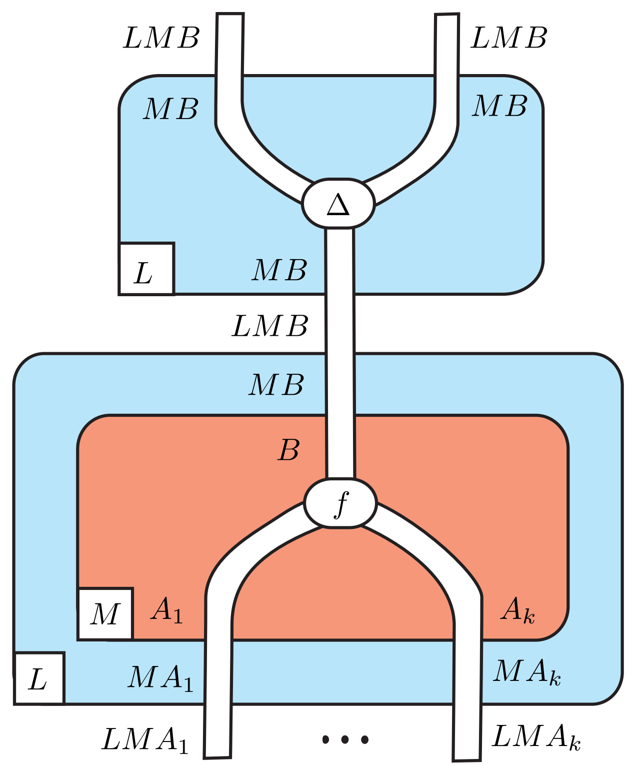
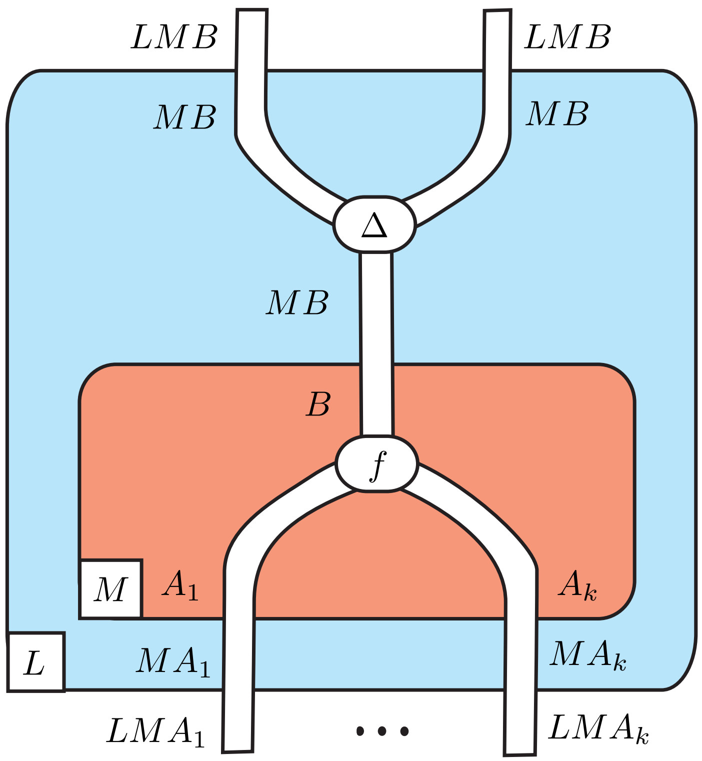


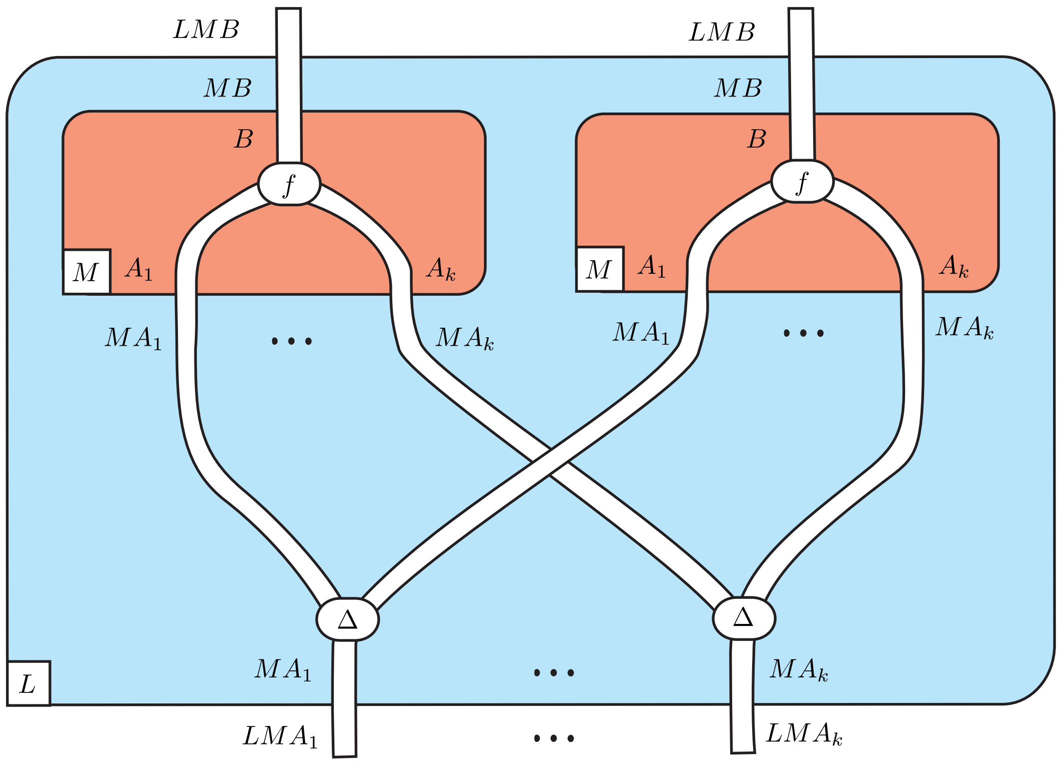
Illustration: duplication of the exponential box



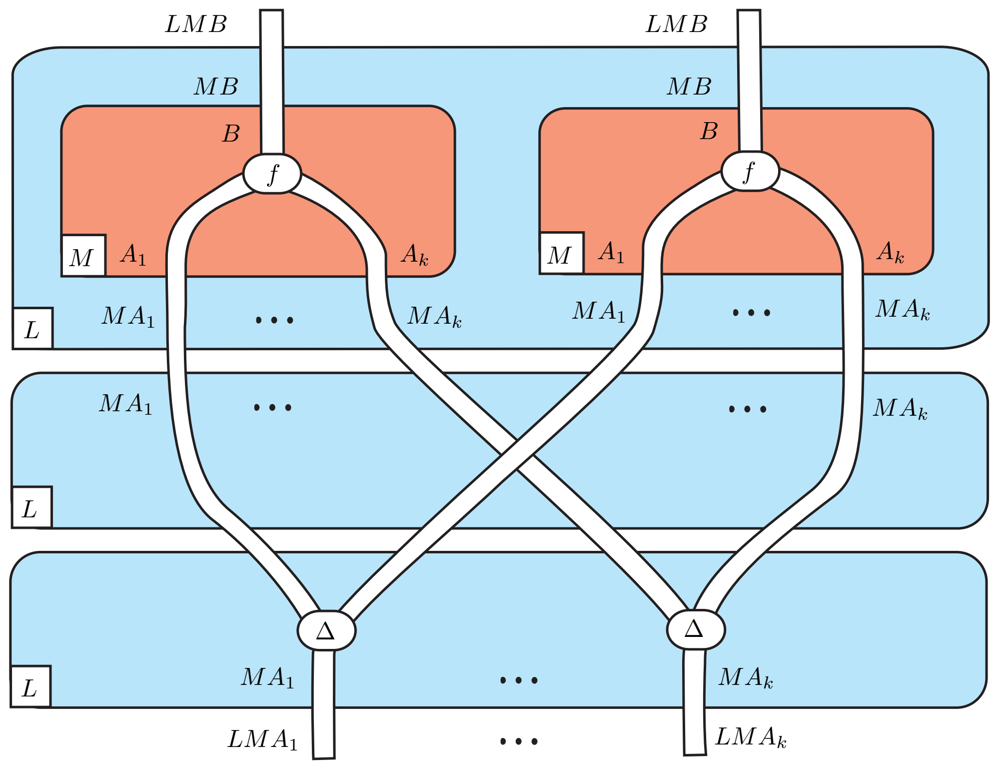
Duplication (step 1)



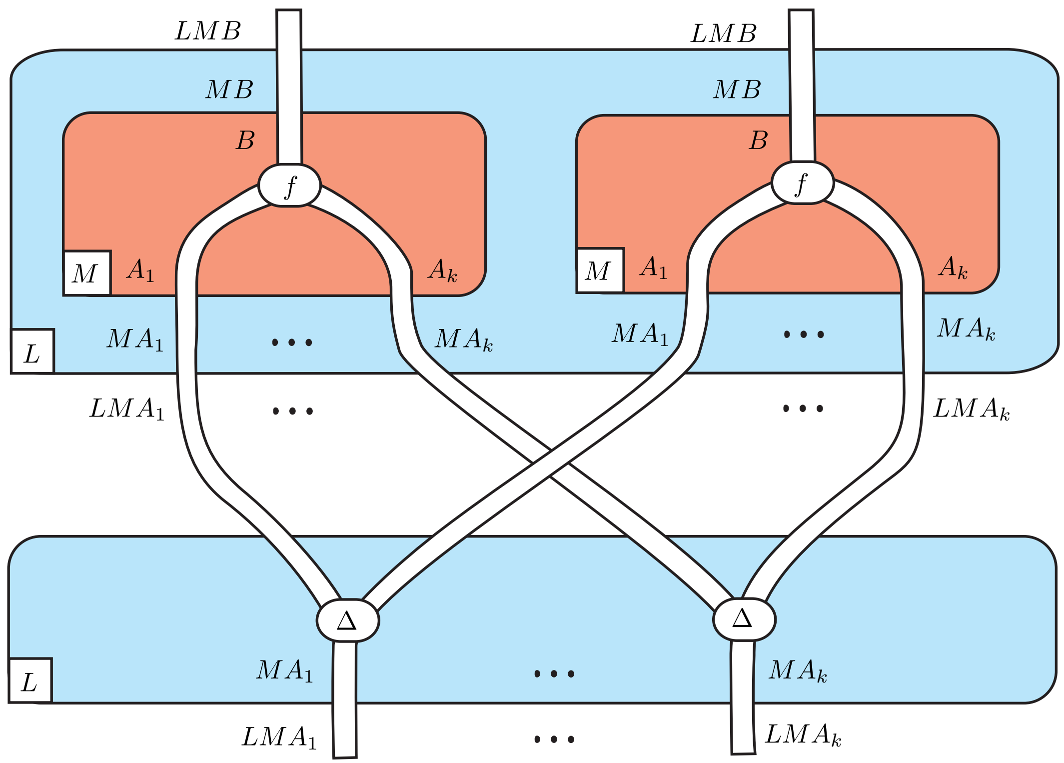
Duplication (step 2)



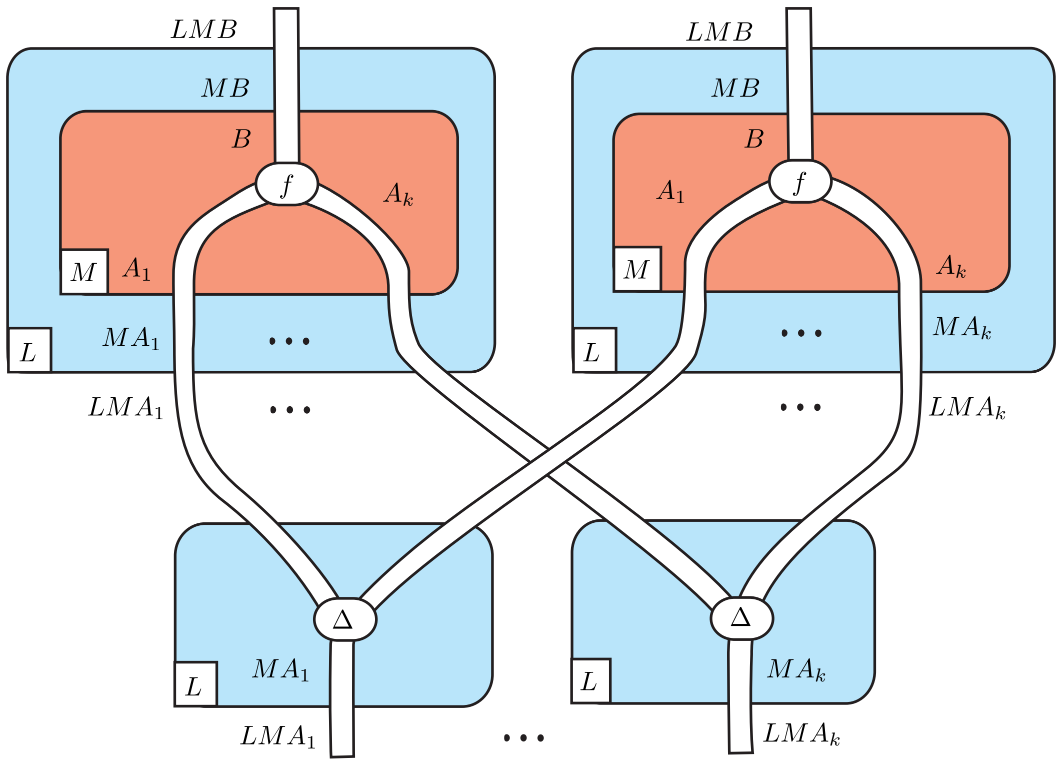
Duplication (step 3)



Duplication (step 4)



Duplication (step 5)



Five polychromatic steps!

The five diagrammatic steps follow very carefully

the categorical proof of soundness

for linear-non-linear models of linear logic.

Thank you !