MELL Proof-Nets (Part 2)

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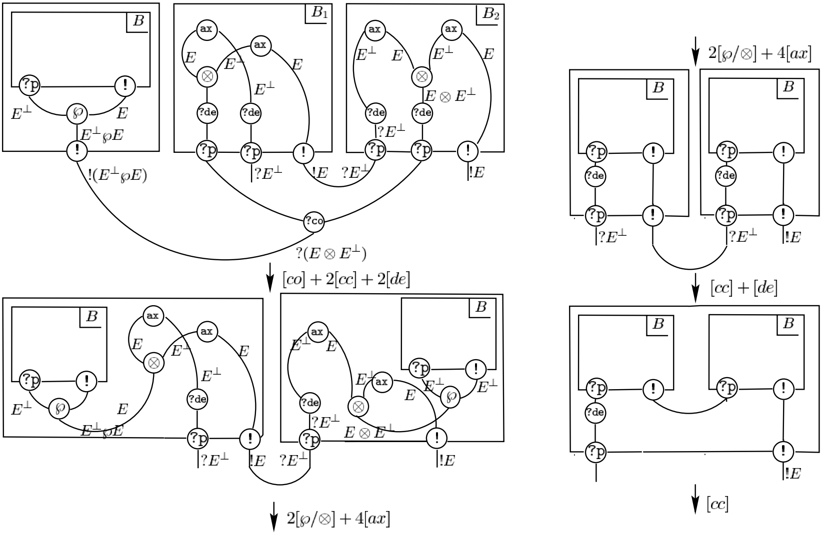
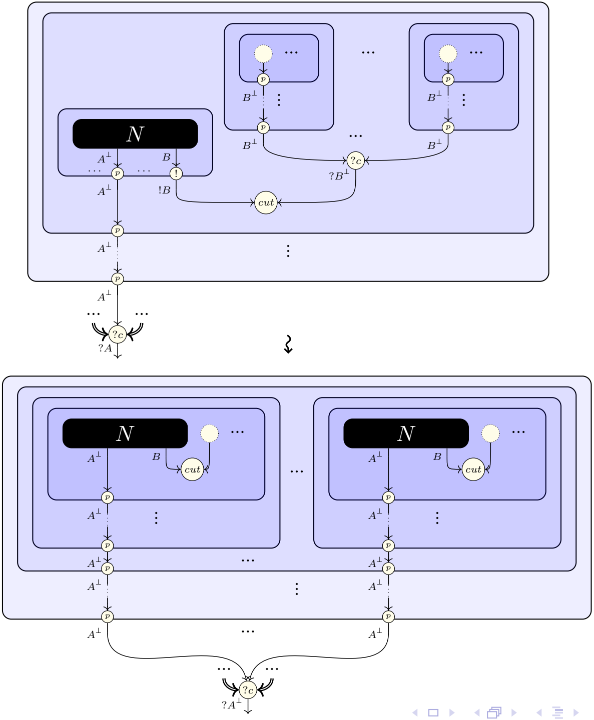
Logic and Interactions 2022
Linear Logic Winter School
24-28 Jan 2022


B
B_1
B_2
$!(E^\perp \wp E)$
$?(E \otimes E^\perp)$
$[co] + 2[cc] + 2[de]$
$2[\wp/\otimes] + 4[ax]$
$2[\wp/\otimes] + 4[ax]$
$[cc] + [de]$
$[cc]$



N
$A^\perp$
B
$!B$
$B^\perp$
$?B^\perp$
$?c$
cut
$?A$
N
N
cut
cut
$?A^\perp$

Strong Normalization for *MELL*

Some notations:

- a cut of type (!/?) where the ?-node has no premise (a weakening) is erasing. All other cuts are non erasing. Associated reduction steps are erasing/non erasing. There are e-reductions and $\neg e$-reductions;
- **WN** =set of proof-nets π s.t. there exists a reduction from π to some normal proof-net;
- $\mathbf{WN}^{\neg e}$ =set of proof-nets π s.t. there exists a $\neg e$-reduction from π to some $\neg e$-normal proof-net;
- **SN** =set of proof-nets π s.t. every reduction from π is finite;
- $\mathbf{SN}^{\neg e}$ =set of proof-nets π s.t. every $\neg e$-reduction from π is finite.

Outline of the proof:

- very easy proof of weak normalization for *MELL*: more precisely, if π is an *MELL* proof-net, then there exists a $\neg e$-reduction sequence to some $\neg e$-normal proof-net ($\pi \in \mathbf{WN}^{\neg e}$);
- "semantic" proof of the conservation theorem: $\mathbf{WN}^{\neg e} = \mathbf{SN}$.
  We present the proof with types but it is actually indipendent from types: it holds for untyped nets (proof-nets without formulas).

Weak Normalization for *MELL* (1)

- $\mathcal{M}_{fin}(\mathbb{N})$ =set of finite multisets of natural numbers: formally, $m \in \mathcal{M}_{fin}(\mathbb{N})$ is a function from $\mathbb{N}$ to $\mathbb{N}$ with value 0, except for a finite set of integers;
- $m \in \mathcal{M}_{fin}(\mathbb{N})$ is defined by its support (its underlying set) and the multipicities of the elements of its support: for $\mu = [4, 5, 6, 6, 6]$, one has $Supp(\mu) = \{4, 5, 6\}$ and $\mu(4) = \mu(5) = 1$, $\mu(6) = 3$, $\mu(7) = \mu(1024) = 0$;
- $\mathcal{M}_{fin}(\mathbb{N})$ is well-ordered: $m + m' < m + [k]$ where $k > q'$ for every $q' \in m'$. Example: $m = [2, 2, 2, 2, 3, 3, 4, 5, 6, 7] < [4, 4, 5, 6, 7] = m'$.
- For a *MELL* proof-net π, let $Cut(\pi) \in \mathcal{M}_{fin}(\mathbb{N})$ such that:
  - $Supp(Cut(\pi)) = \{\sharp A : A/A^{\perp} \text{ is the type of a cut-node of } \pi\}$;
  - if $n \in Supp(Cut(\pi))$, then $Cut(\pi)(n)$ is the number of cut-nodes of π whose premises have types with complexity n.

Weak Normalization for *MELL* (2)

- linear/non linear cuts: A non linear cut has type (!/?) and the ?-node has $n \geq 2$ premises. All other cuts are linear.
- lemma: π MELL proof-net, $\pi \rightsquigarrow t(\pi)$ and t non-erasing cut of π such that:
  - t is linear or
  - t is non linear and π^o is cut-free, where o is the !-link whose main conclusion is a premise of t and π^o is the box of o.

  Then $Cut(t(\pi)) < Cut(\pi)$ (w.r.t. the ordering of $\mathcal{M}_{fin}(\mathbb{N})$).
- theorem: If π is an *MELL* proof-net, then $\pi \in \mathbf{WN}^{\neg e}$ (thus $\pi \in \mathbf{WN}$).
- proof of the theorem: if π contains a non-erasing cut, then there is a cut t in π satisfying the hypothesis of the lemma.

Relational semantics for MELL

Relational model: a (simple and completely degenerate) semantics for LL in the category **Rel** of sets and relations

$$\text{formula } A \longmapsto |A| \text{ set}$$

$$\text{MELL proof-structure } \pi \text{ with conclusions } A_1, \ldots, A_n \longmapsto [\![\pi]\!] \subseteq |A_1| \times \ldots \times |A_n|$$

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We fix an infinite set $\mathcal{A}t$ of atoms.

Let $|\cdot|$ be the function associating with any MELL formula A the set $|A|$ defined by induction on A as follows:

$$|X| = |X^\perp| = \mathcal{A}t, \text{ for any variable } X; \qquad |1| = |\bot| = \{*\};$$

$$|A \otimes B| = |A \parr B| = |A| \times |B|; \qquad |!A| = |?A| = \mathscr{M}_{\text{fin}}(|A|).$$

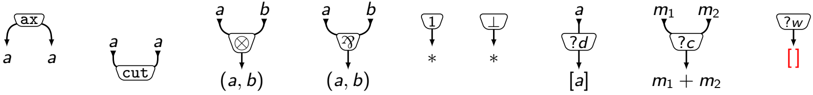
Remark: For any MELL formula A: $|A^\perp| = |A|$.

Experiment of a MELL proof-structure (Girard, 1987)

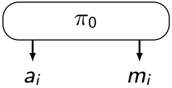
A $[\![\,]\!]$-experiment of a MELL proof-structure π is a function (= labelling) e s.t.

$$\begin{cases} p \mapsto e(p) \in |A| & \text{for any edge } p\colon A \text{ of } \pi \text{ with depth } 0 \\ \mathtt{B} \mapsto e(\mathtt{B}) = [e_1, \dots, e_n] & \text{for any box } \mathtt{B} \text{ of } \pi \text{ with depth } 0, \text{ where } n \geq 0 \text{ and} \\ & e_1, \dots, e_n \text{ are } [\![\,]\!]\text{-experiments of the content of } \mathtt{B} \end{cases}$$

(definition by induction on the depth of π) and the following conditions hold:



if $\mathtt{B}$ is a box of π with depth 0 (π_0 is its content), if $e(\mathtt{B}) = [e_1, \dots, e_n]$ and, for any $1 \leq i \leq n$, e_i is such that



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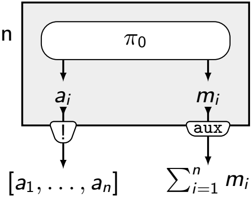

Relational semantics of a MELL proof-structure

Definition (Girard, 1987)

Let π be a MELL proof-structure with conclusions $p_1 : A_1, \ldots, p_n : A_n$.

1. If e is a $[\![]\!]$-experiment of π, the result of e is $|e| = (e(p_1), \ldots, e(p_n))$.
2. $[\![\pi]\!] = \{|e| : e \text{ a } [\![]\!]\text{-exp. of } \pi\}$ is the relational interpretation of π.

Example:

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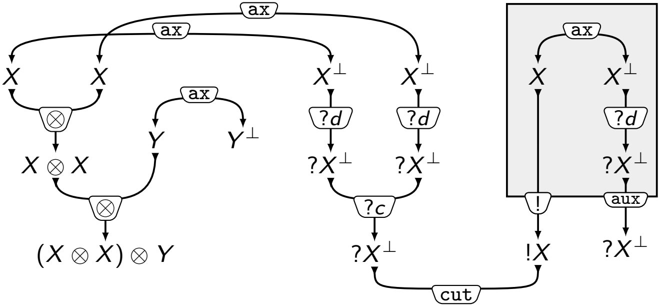
$\pi =$ 

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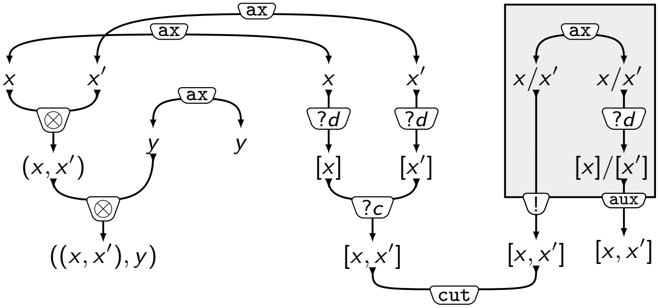
Example: $x, x', y \in \mathcal{A}t = |X| = |X^\perp| = |Y| = |Y^\perp|$



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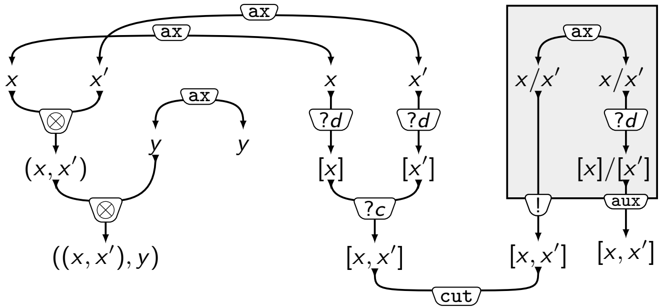
Example:



the result of e: $|e| = (((x, x'), y), y, [x, x']) \in [\![\pi]\!]$

Relational semantics of a MELL proof-structure

Definition (Girard, 1987)

Let π be a MELL proof-structure with conclusions $p_1 : A_1, \ldots, p_n : A_n$.

1. If e is a $\llbracket\rrbracket$-experiment of π, the result of e is $|e| = (e(p_1), \ldots, e(p_n))$.
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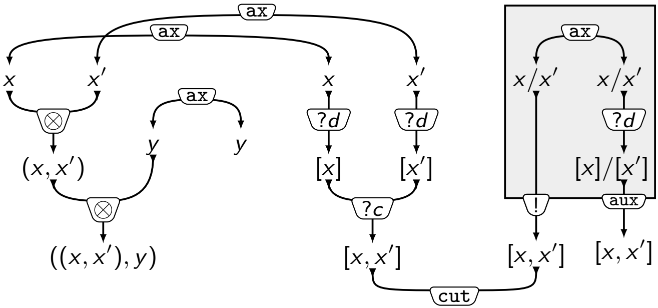
Example:



Intuition: take as many experiments inside a box as copies of the box required from the outside (by means of cuts).

Relational semantics of a MELL proof-net

Theorem (Girard, 1987)

Let π be a MELL proof-net and suppose $\pi \rightsquigarrow^* \pi'$. Then $[\![\pi]\!] = [\![\pi']\!]$.

Proof:

1) For e experiment of π, there exists e' experiment of π' s.t. $|e| = |e'|$;
2) For e' experiment of π', there exists e experiment of π s.t. $|e| = |e'|$.

A semantic characterization of **SN**

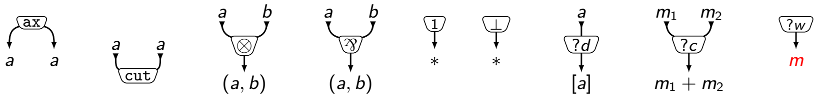
- We look for a result of the shape: $\pi \in \mathbf{SN} \iff [\![\pi]\!] \neq \emptyset$.
- If there exist infinite computations, one can easily find $\pi \notin \mathbf{SN}$ and $\pi' \in \mathbf{SN}$ s.t. $\pi \rightsquigarrow^* \pi'$. And since $[\![\pi]\!] = [\![\pi']\!]$, the previous characterization cannot hold.
- But in the examples of π, π' s.t. $\pi \notin \mathbf{SN}$, $\pi \rightsquigarrow^* \pi'$ and $\pi' \in \mathbf{SN}$, one can notice that actually $\pi \rightsquigarrow_e \pi'$: erasing cuts can hide infinite computations.

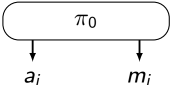
Non erasing experiment of a MELL proof-structure

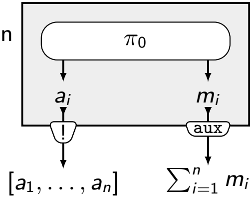
A ⦇⦈-experiment of a MELL proof-structure π is a function (= labelling) e s.t.

$$\begin{cases} p \mapsto e(p) \in |A| & \text{for any edge } p\colon A \text{ of } \pi \text{ with depth } 0 \\ \mathtt{B} \mapsto e(\mathtt{B}) = [e_1, \ldots, e_n] & \text{for any box } \mathtt{B} \text{ of } \pi \text{ with depth } 0, \text{ where } n > 0 \text{ and} \\ & e_1, \ldots, e_n \text{ are ⦇⦈-experiments of the content of } \mathtt{B} \end{cases}$$

(definition by induction on the depth of π) and the following conditions hold:



if $\mathtt{B}$ is a box of π with depth 0 (π_0 is its content), if $e(\mathtt{B}) = [e_1, \ldots, e_n]$ and, for any $1 \leq i \leq n$, e_i is such that 

, then 

Non erasing interpretation of a MELL proof-structure

π MELL proof-structure with conclusions $p_1 : A_1, \ldots, p_n : A_n$:

Definition

1. If e is a $(\!|\,|\!)$-experiment of π, the result of e is $|e| = (e(p_1), \ldots, e(p_n))$
2. If e is a $(\!|\,|\!)$-experiment of π, $\mathcal{W}(e)$ is the finite multiset of all the labels of the weakeing of π.
3. The non erasing relational interpretation of π is $(\!|\pi|\!) = \{(|e|, \mathcal{W}(e)) : e \text{ is a } (\!|\,|\!)\text{-exp, of } \pi\}$.

The non erasing interpretation can see even through erasing cuts!

Non erasing interpretation of a MELL proof-net

Theorem

Let π be a MELL proof-net and suppose $\pi \leadsto_{\neg e} \pi_1$. Then $(\!|\pi|\!) = (\!|\pi_1|\!)$.

Proof:

1) For e $(\!|\,|\!)$-experiment of π, there exists e_1 $(\!|\,|\!)$-experiment of π_1 s.t. $(|e|, \mathcal{W}(e)) = (|e_1|, \mathcal{W}(e_1))$;
2) For e_1 $(\!|\,|\!)$-experiment of π_1, there exists e $(\!|\,|\!)$-experiment of π s.t. $(|e|, \mathcal{W}(e)) = (|e_1|, \mathcal{W}(e_1))$.

And one can define a size on $(\!|\,|\!)$-experiments that shrinks during normalization: for every proof-net π, for every $(\!|\,|\!)$-experiment e of π, we define, by induction on $\mathrm{depth}(\pi)$, the *size of e*, $s(e)$ for short, as follows:

$$s(e) = \|ground(\pi)\| + \sum_{o \in !(ground(\pi))} \sum_{e^o \in e(o)} s(e^o) \ .$$

- $ground(\pi)$=the graph obtained by substituting every box with depth 0 by a single node (with the appropriate conclusions);
- $\|ground(\pi)\|$=number of logical+axiom (no cut, no structural) nodes of $ground(\pi)$.

Non erasing interpretation of a MELL proof-net

Improving the proof of the theorem...

Key lemma: Let π and π_1 be two proof-nets such that $\pi \rightsquigarrow_{\neg e} \pi_1$. Then

1. for every $(\!|\,|\!)$-experiment e of π, there exists an $(\!|\,|\!)$-experiment e_1 of π_1 such that $(|e|, \mathcal{W}(e)) = (|e_1|, \mathcal{W}(e_1))$ and $s(e_1) < s(e)$;
2. for every $(\!|\,|\!)$-experiment e_1 of π_1, there exists an $(\!|\,|\!)$-experiment e of π such that $(|e|, \mathcal{W}(e)) = (|e_1|, \mathcal{W}(e_1))$ and $s(e_1) < s(e)$.

...we obtain another one (the kind of result we looked for):

Theorem: $\pi \in \mathbf{SN} \iff (\!|\pi|\!) \neq \emptyset$.

Non erasing interpretation of a MELL proof-net

Proposition: $\pi \in \mathbf{WN}^{\neg e} \Rightarrow (\!|\pi|\!) \neq \emptyset$.

Proof of the proposition: if $\pi \in \mathbf{WN}^{\neg e}$, then let π_0 be a $\neg e$-normal form of π. There exists an experiment e_0 of π_0 (easy check). Then (by invariance of the non erasing interpretation w.r.t. $\neg e$-steps) $(|e_0|, \mathcal{W}(e_0)) \in (\!|\pi|\!)$.

Proof of $\pi \in \mathbf{SN} \iff (\!|\pi|\!) \neq \emptyset$: It is enough to prove that $\pi \in \mathbf{SN}^{\neg e} \iff (\!|\pi|\!) \neq \emptyset$ (because $\mathbf{SN} = \mathbf{SN}^{\neg e}$). If $\pi \in \mathbf{SN}^{\neg e}$, then $\pi \in \mathbf{WN}^{\neg e}$ and the proposition allows to conclude. Conversely, since $(\!|\pi|\!) \neq \emptyset$, by induction on $\min\{s(e)\,;\ e \text{ is an } (\!|\,|\!)\text{-experiment of } \pi\}$, one proves (using the Key lemma) that $\pi \in \mathbf{SN}^{\neg e}$.

The conservation theorem for *MELL*: a semantic proof

- Proposition: $\pi \in \mathbf{WN}^{\neg e} \Rightarrow (\!|\pi|\!) \neq \emptyset$.
- Theorem: $\pi \in \mathbf{SN} \iff (\!|\pi|\!) \neq \emptyset$.
- Conservation theorem: $\mathbf{WN}^{\neg e} = \mathbf{SN}$.

Strong Normalization is an immediate consequence of weak normalization (for every *MELL* proof-net π one has $\pi \in \mathbf{WN}^{\neg e}$) and the conservation theorem ($\mathbf{WN}^{\neg e} = \mathbf{SN}$).

Bounded (elementary) complexity and Linear Logic: hints

Fundamental remark (Girard): By restricting in a suitable (geometric) way *MELL* proof-nets the complexity of cut-elimination does not depend on the complexity of cut-formulas, but only on the graph representation of proofs as proof-nets. One can then bound the complexity of cut-elimination (and of the representable functions).

Bounded (elementary) complexity and Linear Logic: hints

For elementary time:

- untyped *LL*: *LL* proof-nets without formulas (just the nodes);
- (untyped) *ELL*: every exponential branch (top-down path from dereliction to contraction) crosses exactly one box;
- In sequent calculus, this corresponds to the functorial promotion:

$$\frac{\vdash \Gamma, A}{\vdash ?\Gamma, !A}$$

  (But it is not enough for all elementary functions: it becomes enough adding second order quantifiers)
- stability of cut-elimination for *ELL*: if $\pi \in ELL$ and (with the big-step procedure) $\pi \rightsquigarrow^* \pi'$, then $\pi' \in ELL$.
- cut-elimination works by layer: only nodes at the same depth can interact. In particular, the depth of a proof-net cannot increase during (big step) cut-elimination.

Bounded (elementary) complexity and Linear Logic: hints

Definition (representable function): Every integer n can be represented by a cut-free proof-net ω_n. A function $f : \mathbb{N} \to \mathbb{N}$ is representable by a proof-net π_f when for every integer n, the proof-net $cut(\pi_f, \omega_n)$ reduces to $\omega_{f(n)}$.

ELL proof-nets are a computational model of elementary time:

- Adequacy: For π proof-net of *ELL* with h nodes and depth d, if π_0 is π's normal form, then the number of cut elimination steps leading from π to π_0 is less than $\underbrace{2^{2^{\cdot^{\cdot^{2^h}}}}}_{d \text{ times } 2}$. When d is fixed the number of steps is thus elementary: as a consequence, in case the proof-net π_g represents a function g that is not elementary, $\pi_g \notin ELL$.
- Completeness: If the values of f can be computed in elementary time (by a Turing Machine), then it is representable in *ELL*: there exists a proof-net π_f of *ELL* representing f.

(!/?) :

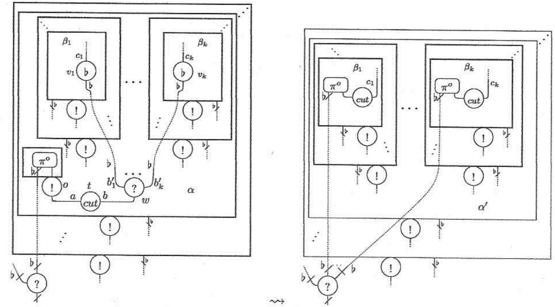


Figure 4: Cut elimination for nets. In the (!/?) case, since every switching of every g-structure of π is an acyclic graph, we know that no conclusion of o appears above w; what happens is that the !-link o dispatches k copies of π^o ($k \geq 0$ being the arity of the ?-node w premise of the cut) inside the !-boxes (if any) containing the $\flat$-nodes associated with the premises of w; notice also that the reduction duplicates k times the premises of ?-nodes which are associated with the auxiliary conclusions of o.