MELL proof nets

Part 1

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A brief recap of MLL proof nets

Multiplicative proof structures

Definition

A **multiplicative proof structure** is:

- a (directed, acyclic, multi-) graph built on the nodes

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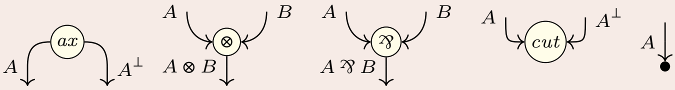

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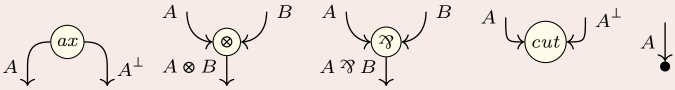
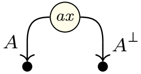


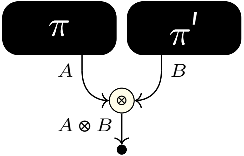
- an ordering of the two input edges of each ⊗ and ⅋ (one left, one right)
- a labelling of edges with MLL formulas, compatible with typing rules
- a linear ordering of the conclusions

Translation of proofs

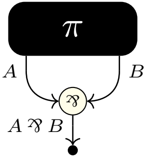
$$\frac{}{A, A^{\perp}}\ (ax) \quad \mapsto$$



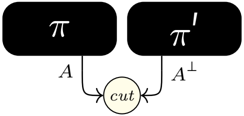
$$\frac{\overset{\pi}{\Gamma, A} \qquad \overset{\pi'}{\Delta, B}}{\Gamma, \Delta, A \otimes B}\ (\otimes) \quad \mapsto$$



$$\frac{\overset{\pi}{\Gamma, A, B}}{\Gamma, A \parr B}\ (\parr) \quad \mapsto$$

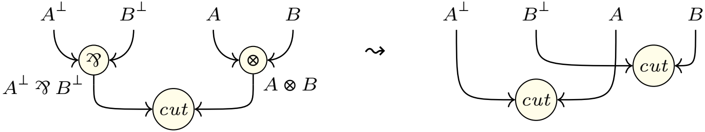


$$\frac{\overset{\pi}{\Gamma, A} \qquad \overset{\pi'}{\Delta, A^{\perp}}}{\Gamma, \Delta}\ (cut) \quad \mapsto$$

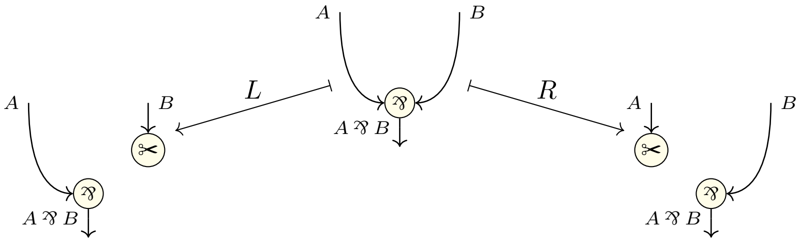


Cut elimination in multiplicative proof structures

provided the in/out edges do not coincide



Switchings and proof nets



Definition

- A **switching** σ of a structure S, is the choice of L or R for each ⅋ of S.
- The **switching graph** S_σ is obtained by replacing each ⅋ accordingly.
- A **proof net** is a proof structure whose switching graphs are all (undirected) acyclic.
- A **connected proof net** is a proof net whose switching graphs have exactly one (undirected) connected component.

Multiplicative proof nets

In proof nets, each cut can be eliminated.

Theorem

Proof nets (resp. connected proof nets) are stable under cut elimination.

Theorem

Cut elimination is confluent and strongly normalizing on proof nets.

Theorem

Connected proof nets are exactly the translations of proofs.

A warm-up: MLL with units

Units

MLL with units:

$$A, B, C, \ldots ::= X \mid X^\perp \mid A \otimes B \mid A \parr B \mid \mathbf{1} \mid \perp$$

$$\mathbf{1}^\perp := \perp \quad \perp^\perp := \mathbf{1}$$

Units

MLL with units:

$$A, B, C, \ldots ::= X \mid X^\perp \mid A \otimes B \mid A \parr B \mid \mathbf{1} \mid \bot$$

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$$\frac{}{\mathbf{1}}\ (\mathbf{1}) \qquad\qquad \frac{\Gamma}{\Gamma, \bot}\ (\bot)$$

Units

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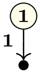
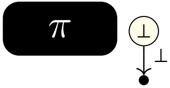


$$\frac{\begin{matrix}\pi \\ \Gamma\end{matrix}}{\Gamma, \perp}\ (\perp) \quad \mapsto$$



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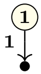

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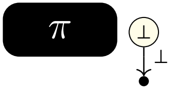

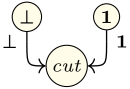

$$\rightsquigarrow \quad \varnothing$$

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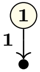

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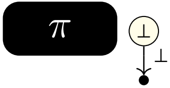
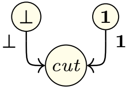




$$\rightsquigarrow \quad \varnothing$$

Breaks connectedness!

Option 1: restoring connectedness

Definition

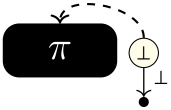
- A **multiplicative proof structure with jumps** is:
  - a proof structure with units
  - a jump from each $\bot$ to some node:

- Switching graphs include jumps as edges.
- A **connected proof net with jumps** is a proof structure with jumps, *whose switching graphs are acyclic and connected.*

Translation of proof trees

In the translation of a proof, we can jump to an arbitrary node:

$$\dfrac{\overset{\pi}{\Gamma}}{\Gamma, \bot}\ (\bot) \quad \mapsto$$



Lemma

Any translation of a proof tree is a connected proof net with jumps.

Proof. Same as without units: attaching a new node with a jump does not create cycles, and does not change the number of connected components. □

What about the converse?

Translation of proof trees

In the translation of a proof, we can jump to an arbitrary node:

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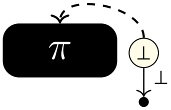


Lemma

Any translation of a proof tree is a connected proof net with jumps.

Proof. Same as without units: attaching a new node with a jump does not create cycles, and does not change the number of connected components. □

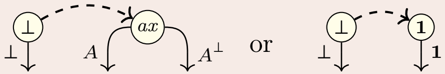
What about the converse? Easier with restricted jumps.

Initial jumps

Definition

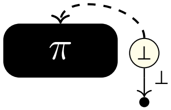
A *multiplicative proof structure with initial jumps* is:

- a proof structure with units
- a jump from each $\bot$ (node) to some initial node:



- In the translation:

$$\frac{\overset{\pi}{\Gamma}}{\Gamma, \bot}\ (\bot) \quad \mapsto$$



  we can always jump to an initial node.

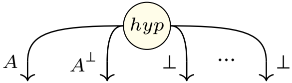
Sequentialization with initial jumps

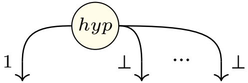
Theorem

A connected proof net with initial jumps is always the translation of a proof.

Proof. Geometrically, this is the same as sequentialization without units, but with generalized "axioms":

$$\dfrac{}{\vdash A, A^{\perp}, \perp, \ldots, \perp}\ (ax) \qquad \dfrac{}{\vdash \mathbf{1}, \perp, \ldots, \perp}\ (\mathbf{1})$$





□

Sequentialization with jumps

Lemma

For each connected proof net with jumps, there is a connected proof net with initial jumps with the same underlying proof structure.

Proof. Move the target of jumps up. □

Theorem

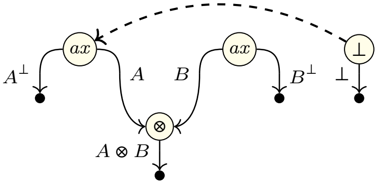
A proof structure is the translation of a proof tree iff it can be equipped with jumps to form a connected proof net with jumps.

(And we can require jumps to be initial.)

Jumps are not canonical

- Even initial ones:

$$\dfrac{\dfrac{}{A^\perp, A}\,(ax) \qquad \dfrac{}{B, B^\perp}\,(ax)}{\dfrac{A^\perp, A \otimes B, B^\perp}{A^\perp, A \otimes B, B^\perp, \perp}\,(\perp)}\,(\otimes) \quad \mapsto$$



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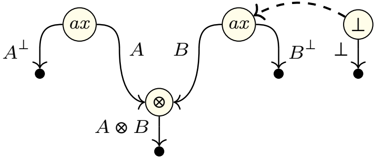

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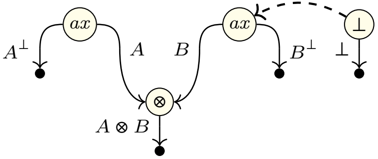
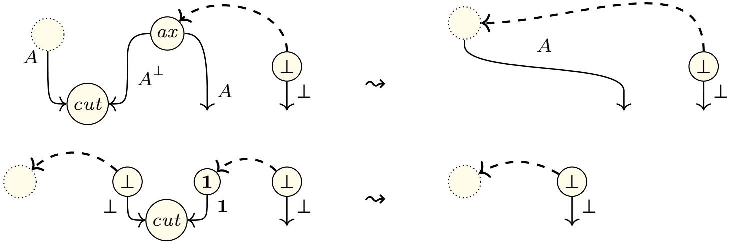


- Cut elimination requires re-routing jumps:



Option 2: acyclicity only

The mix rules:

$$\frac{}{\varepsilon}\,(mix_0) \;\mapsto\; \varnothing \qquad\qquad \frac{\begin{matrix}\pi & \pi' \\ \Gamma & \Delta\end{matrix}}{\Gamma, \Delta}\,(mix) \;\mapsto\; \boxed{\pi}\;\boxed{\pi'}$$

Theorem

A proof structure is a proof net iff it is the translation of a proof of $\mathsf{MLL} + (mix_0) + (mix)$.

Two words about the proof. The main structure (reason by induction, finding a splitting $\otimes$) is essentially the same. But the existence of a splitting tensor *does not* really follow from the connected case. □

- Pro:
  - no extra data
  - same switching graphs, same cut elimination with or without units.

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- Pro:
  - no extra data
  - same switching graphs, same cut elimination with or without units.
- Con:
  - strictly more provable sequents (including the empty one, hence $\vdash \bot$)
  - may identify some proofs that are not equal up to rule permutations

No perfect solution

Stumbling block

Identifying proofs of MLL with units up to rule permutations is hard: PSPACE-complete (Heijltjes–Houston, 2014).

We stick to option 2: forget about connectedness and accept the mix-rules.

MELL proof nets

Introducing exponentials

$$A, B, C, \ldots ::= X \mid X^{\perp} \mid A \otimes B \mid A \parr B \mid !A \mid ?A$$

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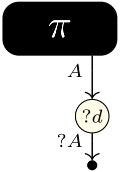
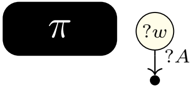
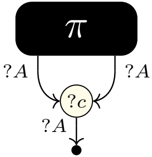


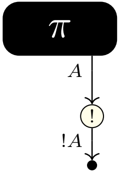
$$\dfrac{\overset{\pi}{\Gamma}}{\Gamma, ?A}\ (?w) \quad \mapsto$$



$$\dfrac{\overset{\pi}{\Gamma, ?A, ?A}}{\Gamma, ?A}\ (?c) \quad \mapsto$$



$$\dfrac{\overset{\pi}{?\Gamma, A}}{?\Gamma, !A}\ (!) \quad \mapsto$$



$$A, B, C, \ldots ::= X \mid X^{\perp} \mid A \otimes B \mid A \parr B \mid !A \mid ?A$$

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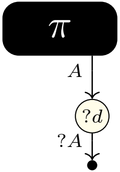

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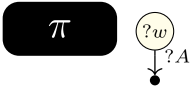

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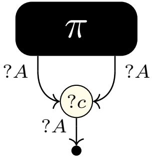

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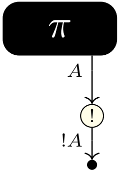
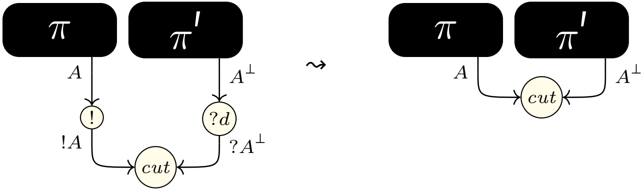


Units are derivable: $\perp := ?(X^{\perp} \otimes X)$ $\quad \mathbf{1} := !(X \parr X^{\perp})$

Cut elimination in MELL

$$\dfrac{\dfrac{\overset{\pi}{?\Gamma, A}}{?\Gamma, !A}\,(!) \qquad \dfrac{\overset{\pi'}{\Delta, A^\perp}}{\Delta, ?A^\perp}\,(?d)}{?\Gamma, \Delta}\,(cut) \quad \rightsquigarrow \quad \dfrac{\overset{\pi}{?\Gamma, A} \qquad \overset{\pi'}{\Delta, A^\perp}}{?\Gamma, \Delta}\,(cut)$$



Cut elimination in MELL

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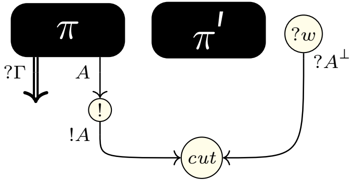
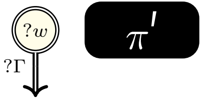
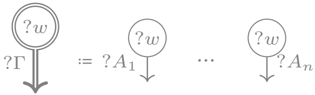


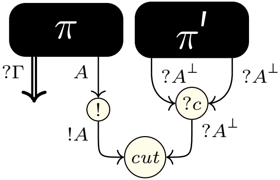
$\rightsquigarrow$





Cut elimination in MELL

$$\dfrac{\dfrac{\overset{\pi}{?\Gamma, A}}{?\Gamma, !A}\,(!) \qquad \dfrac{\overset{\pi'}{\Delta, ?A^\perp, ?A^\perp}}{?A^\perp, \Delta}\,(?c)}{?\Gamma, \Delta}\,(cut) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{\pi}{?\Gamma, A}}{?\Gamma, !A}\,(!) \qquad \dfrac{\dfrac{\overset{\pi}{?\Gamma, A}}{?\Gamma, !A}\,(!) \qquad \overset{\pi'}{\Delta, ?A^\perp, ?A^\perp}}{?\Gamma, \Delta, ?A^\perp}\,(cut)}{?\Gamma, ?\Gamma, \Delta}\,(cut)}{?\Gamma, \Delta}\,(?c)$$



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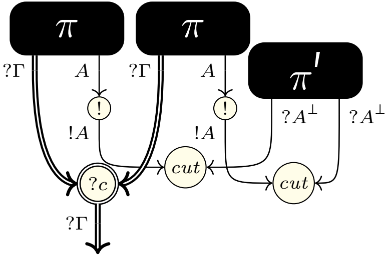
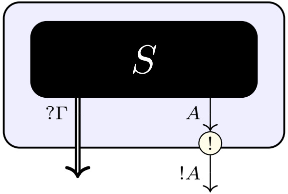
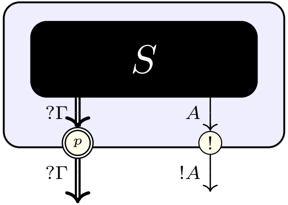


Promotion boxes



- allows to treat the box as one initial node
- amounts to every internal node of S jumping to (!)
- boxes must be either disjoint or nested

Promotion boxes

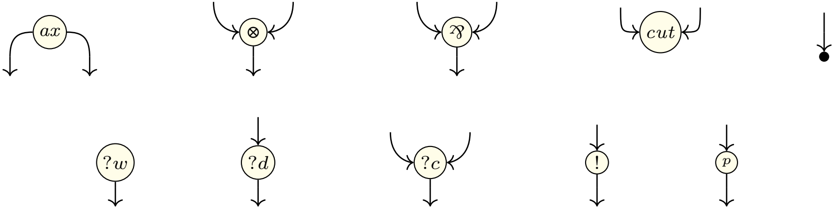


- allows to treat the box as one initial node
- amounts to every internal node of S jumping to (!)
- boxes must be either disjoint or nested
- we add extra nodes as a convenience: box ports = conclusions of the boxed net

MELL proof structures

An **MELL proof structure** is:

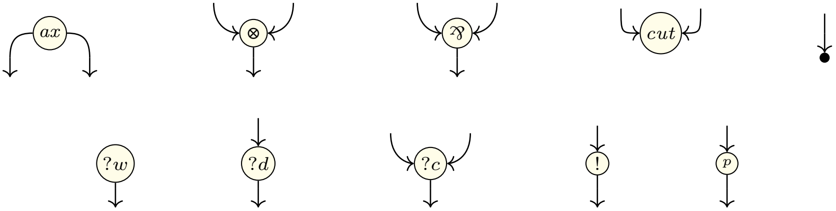
- a (directed, acyclic, multi-) graph built on the nodes



MELL proof structures

An **MELL proof structure** is:

- a (directed, acyclic, multi-) graph built on the nodes



- an ordering of the two input edges of each $\otimes$ and $⅋$ (not $?c$)

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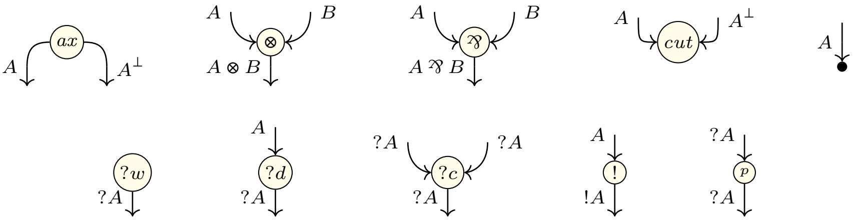

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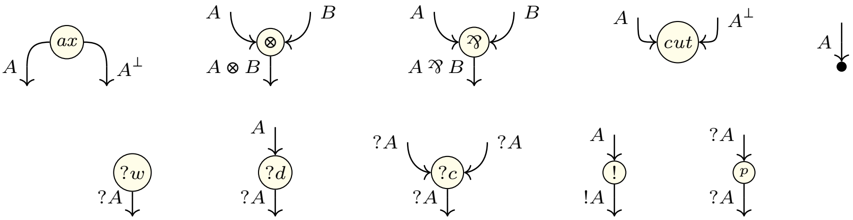
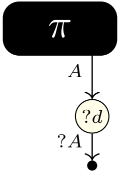


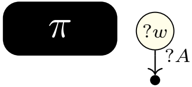
- an ordering of the two input edges of each ⊗ and ⅋ (not ?c)
- a labelling of edges with MELL formulas, compatible with typing rules
- a tree order on ! nodes (+ a root for top level) and a graph morphism to this tree
  - content of a box = preimage of a subtree
  - ! and p nodes mark the border between a box and its parent

Translation of proofs

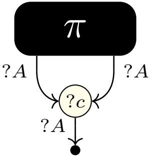
$$\dfrac{\overset{\pi}{\Gamma, A}}{\Gamma, ?A}\ (?d) \quad \mapsto$$



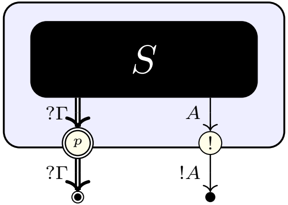
$$\dfrac{\overset{\pi}{\Gamma}}{\Gamma, ?A}\ (?w) \quad \mapsto$$



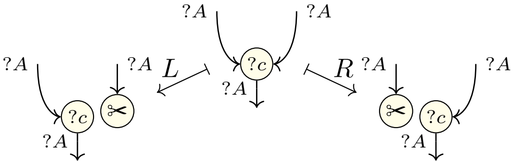
$$\dfrac{\overset{\pi}{\Gamma, ?A, ?A}}{\Gamma, ?A}\ (?c) \quad \mapsto$$

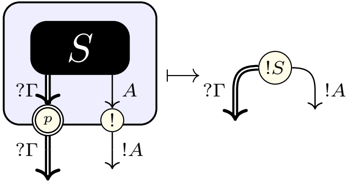


$$\dfrac{\overset{\pi}{?\Gamma, A}}{?\Gamma, !A}\ (!) \quad \mapsto$$



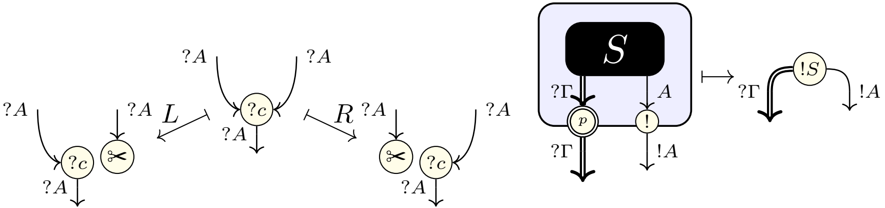
Switchings and proof nets





Definition

- A **switching** σ of a structure S, is the choice of L or R for each ⅋ or $?c$ of S.
- The **switching graph** S_σ is obtained by replacing each ⅋ or $?c$ accordingly, and each box with an initial node.
- A **proof net** is a proof structure whose switching graphs are all (undirected) acyclic,



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- A **proof net** is a proof structure whose switching graphs are all (undirected) acyclic, and such that the content of each box is a proof net, inductively.

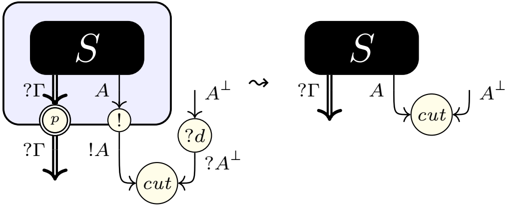
Sequentialization

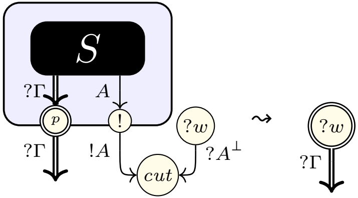
Theorem

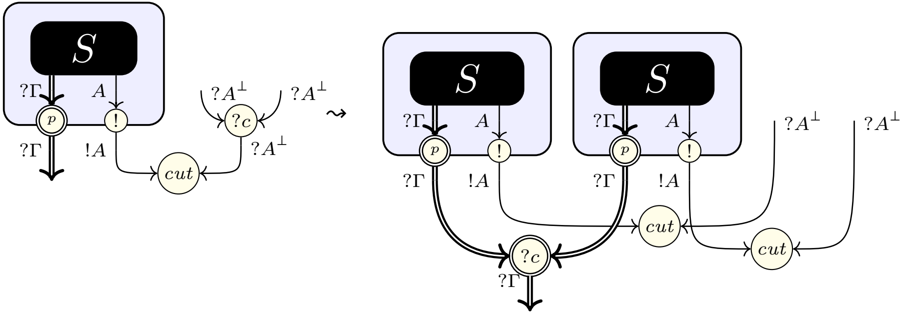
A proof structure is a proof net iff it is the translation of a proof of $\mathsf{MELL} + (mix_0) + (mix)$.

Proof. Same proof as for MLL with units, inductively: treat boxes as general axioms in switching graphs. □

Cut elimination in MELL proof nets

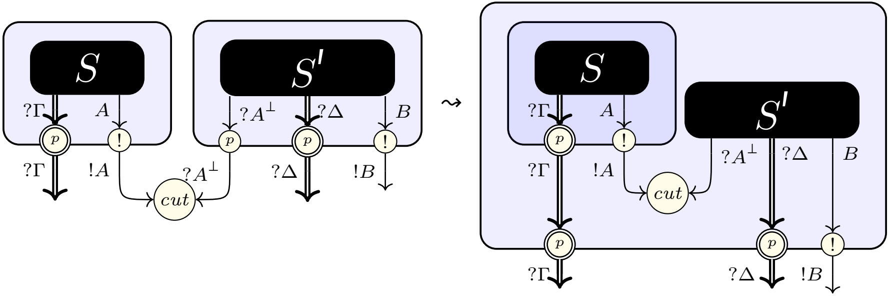






Cut elimination in MELLproof nets

Commuting cuts are back



Properties of cut elimination in MELL proof nets

Theorem

Proof nets are stable under cut elimination

Proof. No cut elimination rule can create a cycle. □

Lemma

Cut elimination is locally confluent

Proof. Inspect critical pairs and solve them. □

Theorem

Cut elimination is confluent and strongly normalizing.

Proof (teaser). By Newman's lemma, it "suffices" to prove strong normalization:

- a general pattern is to first prove weak normalizability of cut elimination without erasing steps ($!$ *vs* $?w$) then deduce SN;
- an example of proof technique in the next lecture.

□

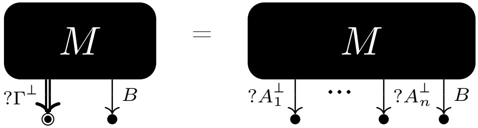
λ-calculus and proof nets

Translating the λ-calculus

One famous translation:

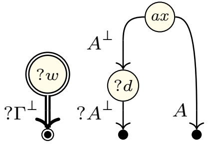
$$(A \Rightarrow B) \mapsto (!A \multimap B)$$

Given $\Gamma \vdash M : B$ with $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ we construct

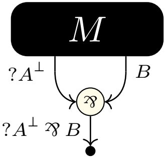


Translating the λ-calculus

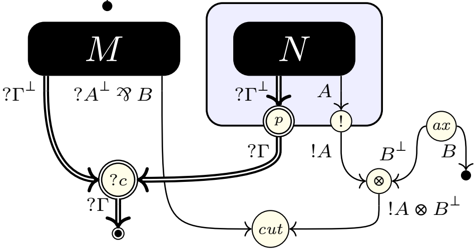
$$\frac{}{\Gamma, x : A \vdash x : A}\ (var) \mapsto$$



$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}\ (abs) \mapsto$$

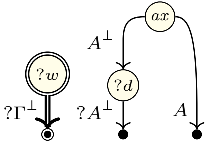


$$\frac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}\ (app) \mapsto$$

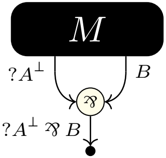


Translating the λ-calculus: $\Lambda \to \mathsf{NJ} \to \mathsf{MELL} \to \mathsf{MELL}$ proof nets

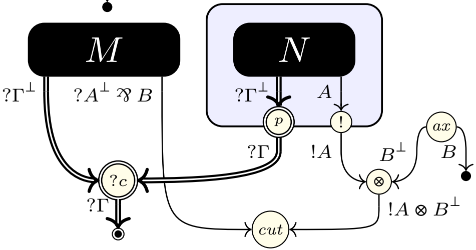
$$\frac{}{\Gamma, x : A \vdash x : A}\ (var) \mapsto$$



$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}\ (abs) \mapsto$$

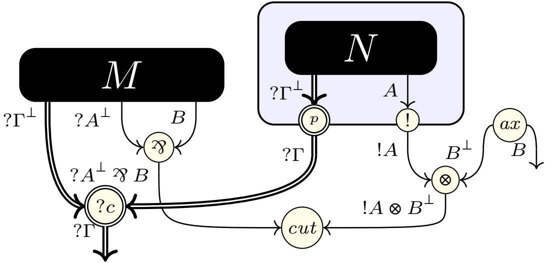


$$\frac{\Gamma \vdash M : A \Rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}\ (app) \mapsto$$



Simulating β-reduction

$$\dfrac{\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \Rightarrow B}\,(abs) \qquad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x.M)N : B}\,(app)$$



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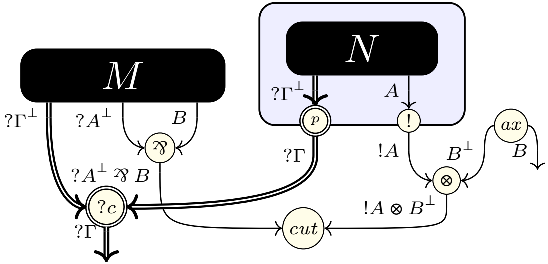

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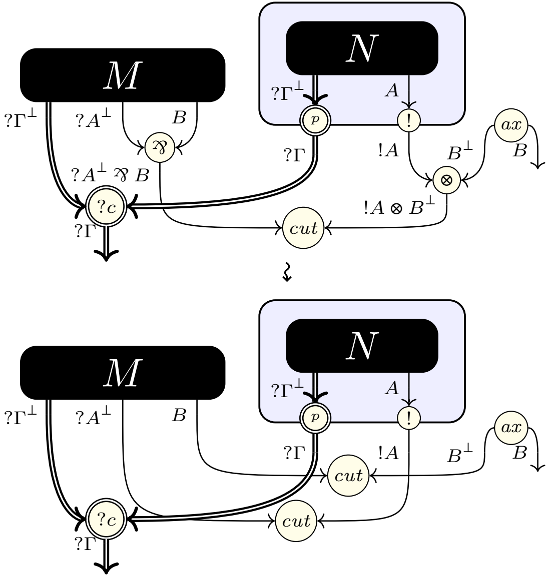

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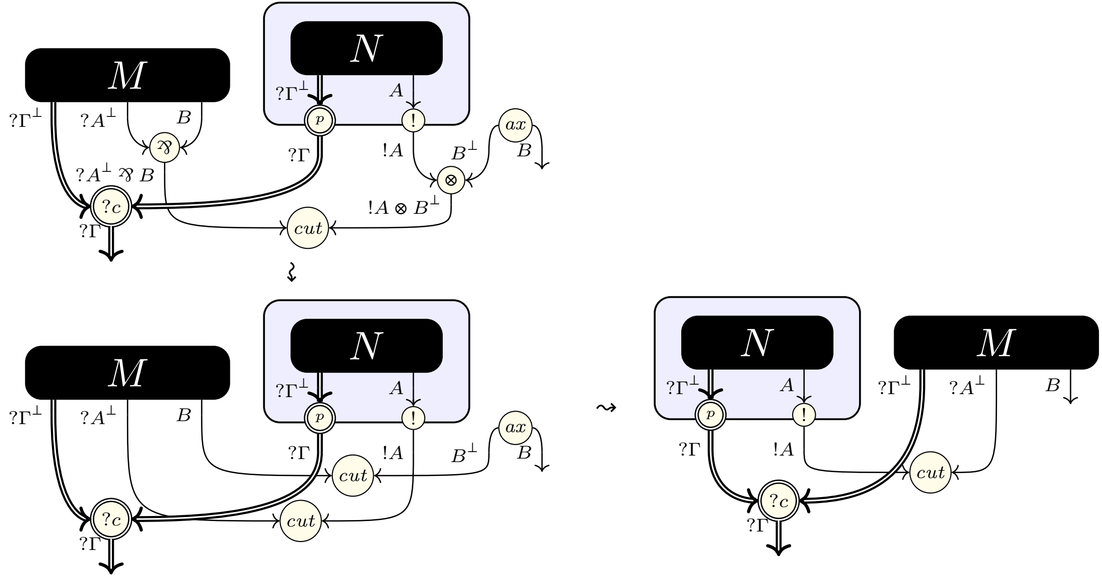
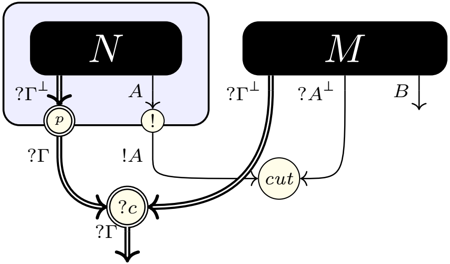


Simulating substitution

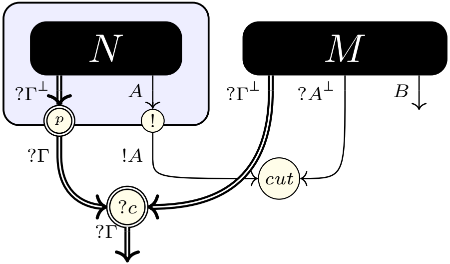


Lemma

Iterating exponential cut elimination in the above net yields the translation of $M[N/x]$.

Write $M\langle N/x\rangle$ for the above proof net.

Simulating substitution



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In fact, not quite…
Write $M\langle N/x\rangle$ for the above proof net.

Substitutions. . .

Let $M = x(x\,y)$ and $N = x$.

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Let $M = x_1(x_2\, y)$ and $N = x_3$.

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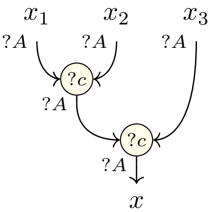
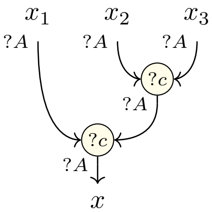
$M\langle N/y\rangle$ $\qquad$ $M[N/y] = x_1(x_2\, x_3)$





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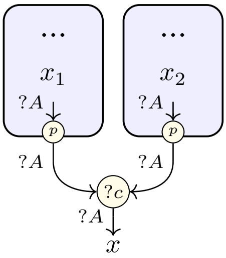
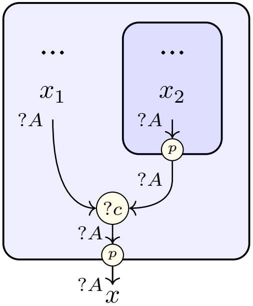
$M\langle N/y\rangle$



$M[N/y] = z(x_1\, x_2)$



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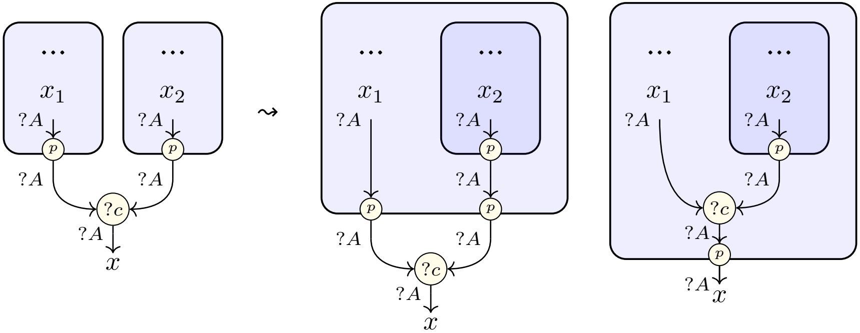
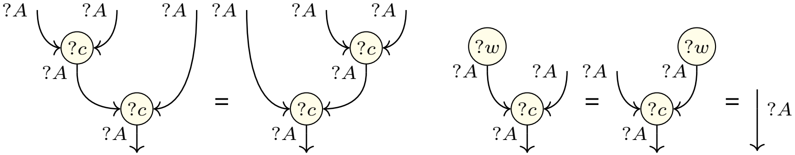
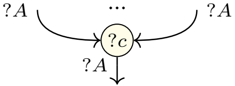


Rétoré conversions

Contraction and weakening form a commutative monoid:

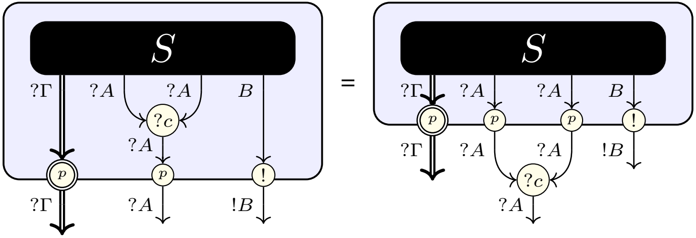


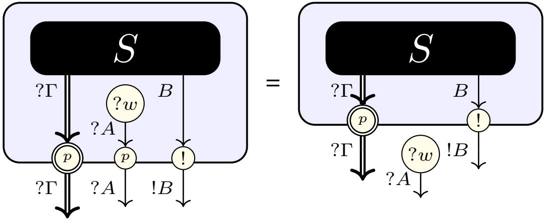
We should thus consider n-ary contractions:



Rétoré conversions

Contraction and weakening cross the border of boxes:





Rétoré conversions and ?-canonical nets

Theorem (Retore, 1987)

Up to Retoré conversions, cut elimination refines β-reduction.

Rétoré conversions and ?-canonical nets

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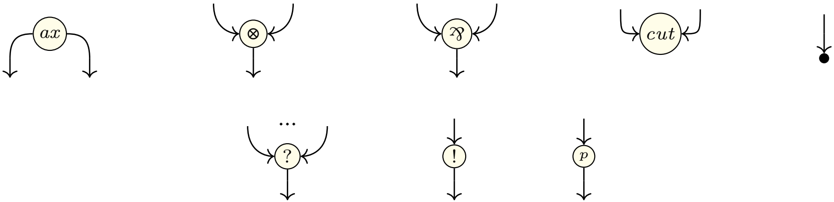
Up to Retoré conversions, cut elimination refines β-reduction.

With n-ary contractions, normal forms for (the obvious orientation of) these conversions are **?-canonical nets** (the old name for this is *nouvelle syntaxe*).
And cut elimination can be defined directly on ?-canonical nets (Danos & Regnier, 1993).

?-canonical proof structures

An **?-canonical proof structure** is:

- a (directed, acyclic, multi-) graph built on the nodes



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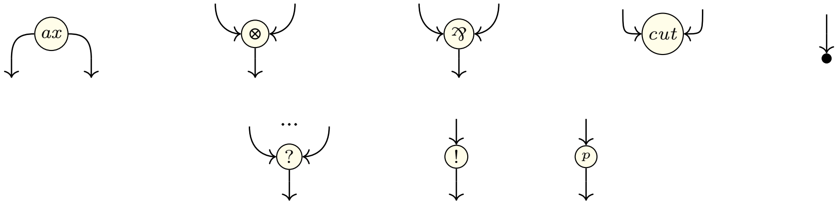

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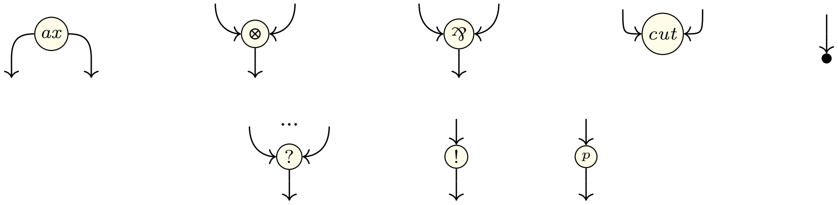

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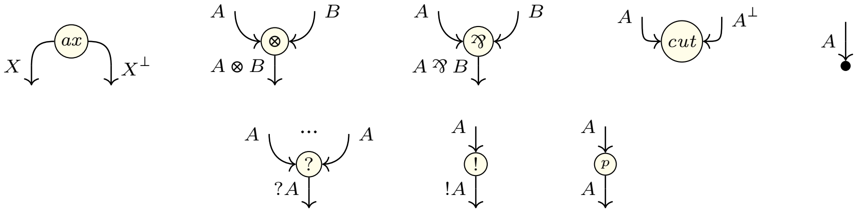

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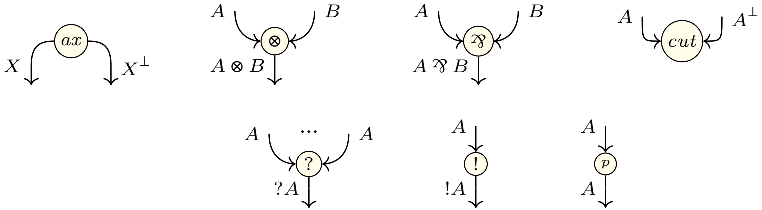
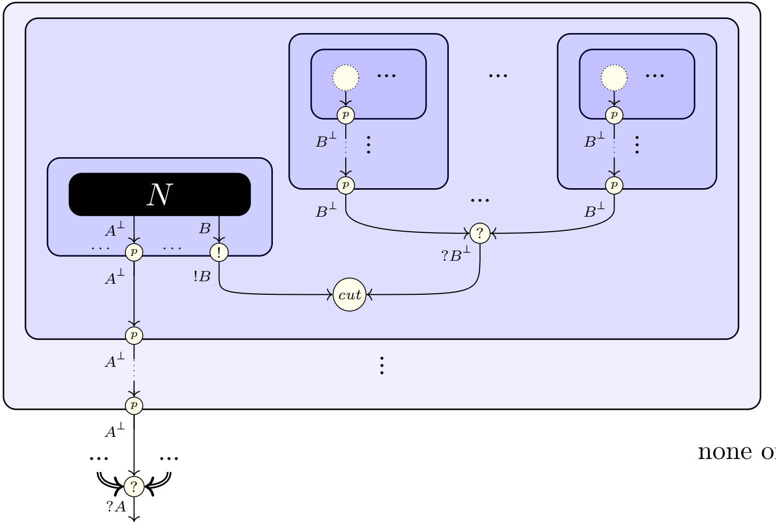


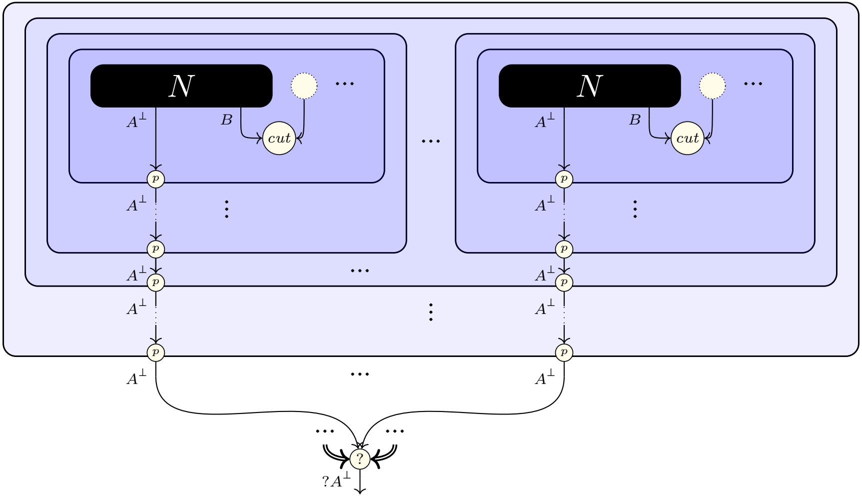
- an ordering of the two input edges of each (⊗) and (⅋)
- a tree order on (!) nodes (+ a root for top level) and a graph morphism to this tree, respecting box conditions
- a labelling of edges with MELL formulas, compatible with typing rules
- moreover such that the conclusion of a (p) must target a (p) or (?)

The exponential cut elimination rule in ?-canonical nets



none of the is a p

The exponential cut elimination rule in ?-canonical nets



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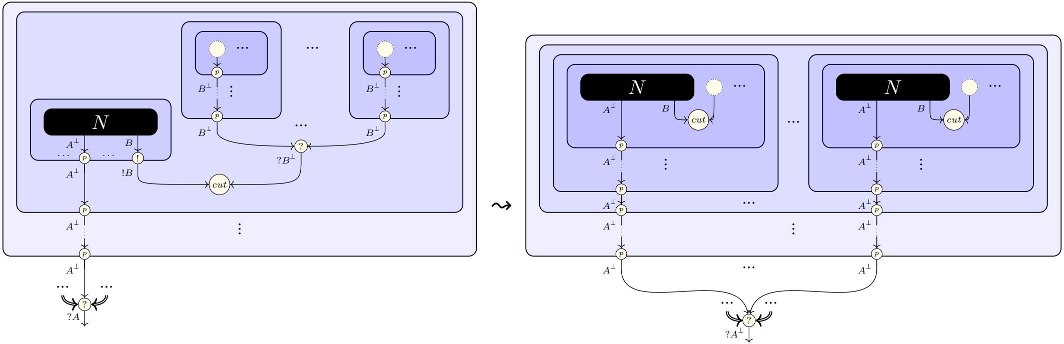

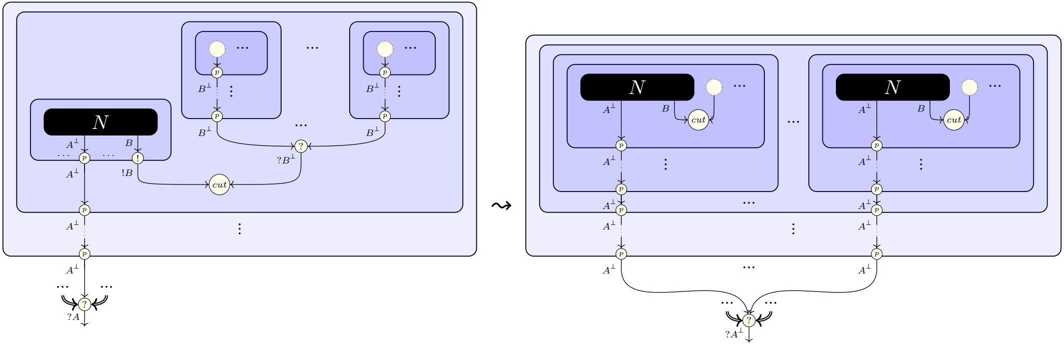

Theorem

Cut elimination in ?-canonical nets simulates β-reduction.

Analysing β-reduction with proof nets

- Conversely, one can look for a counterpart of small-step exponential cut elimination: *explicit substitutions* (Di Cosmo & Kesner, 1997).
- Untyped λ-calculus: type proof nets with $o = (o \Rightarrow o) = (?o^\perp \parr o)$, $!o$, $o^\perp$ and $?o^\perp$.
- A versatile tool to find better rewriting theories (e.g., for CBV *via* another translation of $\Rightarrow$, Carraro & Guerrieri, 2014).

Conclusion: what was not in this tutorial?

- explicit substitutions
- pure types for simulating the untyped λ-calculus
- variants for call-by-value β-reduction and other strategies

Conclusion: what was not in this tutorial?

- explicit substitutions
- pure types for simulating the untyped λ-calculus
- variants for call-by-value β-reduction and other strategies
- a proof of strong normalization (next lecture)
- restrictions for implicit complexity (a hint in the next lecture)
- denotational semantics (tomorrow)
- geometry of interaction (tomorrow)
- additives, quantifiers
- polarized / intuitionistic variants
- differential nets and Taylor expansion
- *etc.*