Cut-elimination in Linear Logic

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Outline of the lecture and preliminary remarks

In this lecture, we will prove the cut-elimination theorem for linear logic and some of its consequences.

Theorem (Hauptsatz)

If $\vdash \Gamma$ is provable in LL, *there exists a proof of conclusion $\vdash \Gamma$ which does not contain the cut inference.*

The lecture is followed by a Click & CoLLect exercise session during which you will practice the proof transformations we will describe here.

- As Laurent said: the most important rule of logic / sequent calculus;
- Allows to invoke extra concepts, as when relying on a lemma, but does not allow to prove more;
- In Gentzen's terms: A "roundabout" reasoning;
- Computationaly, the rule that brings computation into the picture;
- Structurally, the only inference of sequent calculus which breaks the subformula property

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- In Gentzen's terms: A "roundabout" reasoning;
- Computationaly, the rule that brings computation into the picture;
- Structurally, the only inference of sequent calculus which breaks the subformula property

But first, some complements!

LL sequent calculus

LL Inference Rules

$$\frac{}{\vdash F, F^{\perp}}\ (ax) \qquad \frac{\vdash \Gamma, F \qquad \vdash F^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (cut) \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (ex)$$

$$\frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ (\parr) \qquad \frac{\vdash F, \Gamma \qquad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{}{\vdash 1}\ (1)$$

$$\frac{\vdash F, \Gamma \qquad \vdash G, \Gamma}{\vdash F \& G, \Gamma}\ (\&) \qquad \frac{\vdash A_i, \Gamma}{\vdash A_1 \oplus A_2, \Gamma}\ (\oplus_i) \qquad \frac{}{\vdash \top, \Gamma}\ (\top) \qquad \text{(no rule for 0)}$$

$$\frac{\vdash F, \Gamma}{\vdash ?F, \Gamma}\ (?) \qquad \frac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma}\ (!) \qquad \frac{\vdash \Gamma}{\vdash ?F, \Gamma}\ (w) \qquad \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\ (c)$$

Exponential isomorphisms

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^{\perp}, A}\,ax}{\vdash A^{\perp}, ?A}\,?d}{\vdash A^{\perp}, ?A, ?B}\,?w \qquad \dfrac{\dfrac{\dfrac{}{\vdash B^{\perp}, B}\,ax}{\vdash B^{\perp}, ?B}\,?d}{\vdash B^{\perp}, ?A, ?B}\,?w}{\vdash A^{\perp} \mathbin{\&} B^{\perp}, ?A, ?B}\,\&}{\vdash !(A^{\perp} \mathbin{\&} B^{\perp}), ?A, ?B}\,!}{\vdash !(A^{\perp} \mathbin{\&} B^{\perp}), ?A \parr ?B}\,\parr$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,ax}{\vdash A \oplus B, A^{\perp}}\,\oplus_1}{\vdash ?(A \oplus B), A^{\perp}}\,?d}{\vdash ?(A \oplus B), !A^{\perp}}\,! \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B^{\perp}, B}\,ax}{\vdash B^{\perp}, A \oplus B}\,\oplus_2}{\vdash B^{\perp}, ?(A \oplus B)}\,?d}{\vdash !B^{\perp}, ?(A \oplus B)}\,!}{\vdash !A^{\perp} \otimes !B^{\perp}, ?(A \oplus B), ?(A \oplus B)}\,\otimes}{\vdash !A^{\perp} \otimes !B^{\perp}, ?(A \oplus B)}\,?c$$

Positive / Negative LL Formulas

LL Formulas:

$$F ::= a \mid F \otimes F \mid F \oplus F \mid \mathbf{1} \mid \mathbf{0} \mid \exists x.F \mid !F$$
$$a^{\perp} \mid F \parr F \mid F \& F \mid \perp \mid \top \mid \forall x.F \mid ?F$$

Positive formulas:

$$P, Q ::= F \otimes F \mid F \oplus F \mid \mathbf{1} \mid \mathbf{0} \mid \exists x.F \mid !F$$

Negative formulas:

$$N, M ::= F \parr F \mid F \& F \mid \perp \mid \top \mid \forall x.F \mid ?F$$

Definition (Reversible inference)

An inference is reversible if one can derive each of its premises from its conclusion.

Proposition

$\parr, \&, \perp, \top, \forall$, *have reversible inferences. (details about the exponentials to come in the next lecture.)*

Reversibility

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Proposition

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$$\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}(ax) \qquad \dfrac{}{\vdash B, B^\perp}(ax)}{\vdash A, B, A^\perp \otimes B^\perp}(\otimes) \qquad \vdash A \parr B, \Gamma}{\vdash A, B, \Gamma}(cut)$$

$$\dfrac{\dfrac{}{\vdash \mathbf{1}}(\mathbf{1}) \qquad \vdash \bot, \Gamma}{\vdash \Gamma}(cut)$$

$$\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}(ax)}{\vdash A, A^\perp \oplus B^\perp}(\oplus_1) \qquad \vdash A \& B, \Gamma}{\vdash A, \Gamma}(cut)$$

$$\dfrac{\dfrac{}{\vdash B, B^\perp}(ax) \qquad \dfrac{\vdash A \& B, \Gamma}{\vdash B, A^\perp \oplus B^\perp}(\oplus_2)}{\vdash B, \Gamma}(cut)$$

Cut elimination and cut admissibility

In many logical frameworks, one can prove cut-admissibility via a semantical argument, composing a general soundness and a cut-free completeness theorem:

- assume $\mathcal{S}$ is a derivable sequent: by soundness every derivable sequent is valid, so is $\mathcal{S}$;
- by cut-free completeness, every valid sequent has a cut-free proof, so has $\mathcal{S}$.

In LK, such a cut-free completeness result is easily obtained by considering a reversible inference for every connective and applying a contraction together with any use of an existential (right) rule. By applying a fair proof-search strategy, one can build a derivation which is either an LK proof or from which one can define a model falsifying the conclusion sequent.

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In this lecture we shall instead:

- define a reduction relation $\longmapsto_{\mathsf{c}}$ on proofs that transforms proofs with cuts into proofs with "simpler" cuts, such that any proof with cut reduces to proof(s) of the same sequent.
- prove weak-normalization of $\longmapsto_{\mathsf{c}}$: there is a sequence of reductions reaching a cut-free proof.

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Cut-elimination: proof transformation **Cut-admissibility: proof construction**

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$$\frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ (\parr) \qquad \frac{\vdash F, \Gamma \quad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{}{\vdash 1}\ (1)$$

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$$\frac{\vdash F, \Gamma}{\vdash ?F, \Gamma}\ (?) \qquad \frac{\vdash F, ?\Gamma}{\vdash !F, ?\Gamma}\ (!) \qquad \frac{\vdash \Gamma}{\vdash ?F, \Gamma}\ (w) \qquad \frac{\vdash ?F, ?F, \Gamma}{\vdash ?F, \Gamma}\ (c)$$

LL Inference Rules (with ancestor relation)

$$\frac{}{\vdash F, F^\perp}\ (ax) \qquad \frac{\vdash \Gamma, F \qquad \vdash F^\perp, \Delta}{\vdash \Gamma, \Delta}\ (cut) \qquad \frac{\vdash \Gamma, G, F, \Delta}{\vdash \Gamma, F, G, \Delta}\ (ex)$$

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Cut-Axiom case

$$\dfrac{\begin{array}{c}\pi_1 \\ \vdash \Gamma, C\end{array} \qquad \dfrac{}{\vdash C^{\perp}, C}\,(ax)}{\vdash \Gamma, C}\,(cut) \quad \xrightarrow[\mathsf{ax}]{} \quad \begin{array}{c}\pi_1 \\ \vdash C, \Gamma\end{array}$$

Multiplicative case

$$\dfrac{\dfrac{\begin{matrix}\pi_1 \\ \vdash \Gamma_1, B\end{matrix} \qquad \begin{matrix}\pi_2 \\ \vdash \Gamma_2, C\end{matrix}}{\vdash \Gamma_1, \Gamma_2, B \otimes C}\,(\otimes) \qquad \dfrac{\begin{matrix}\pi_3 \\ \vdash B^{\perp}, C^{\perp}, \Delta\end{matrix}}{\vdash B^{\perp} \parr C^{\perp}, \Delta}\,(\parr)}{\vdash \Gamma_1, \Gamma_2, \Delta}\,(cut^{\alpha})$$

$$\xrightarrow[\otimes/\parr]{} \quad \dfrac{\begin{matrix}\pi_2 \\ \vdash \Gamma_2, C\end{matrix} \qquad \dfrac{\begin{matrix}\pi_1 \\ \vdash \Gamma_1, B\end{matrix} \qquad \begin{matrix}\pi_3 \\ \vdash B^{\perp}, C^{\perp}, \Delta\end{matrix}}{\vdash C^{\perp}, \Gamma_1, \Delta}\,(cut^{\gamma})}{\vdash \Gamma_1, \Gamma_2, \Delta}\,(cut^{\beta})$$

Additive case

$$\dfrac{\dfrac{\begin{matrix}\pi_0\\ \vdash \Gamma, C_i\end{matrix}}{\vdash \Gamma, C_1 \oplus C_2}\ (\oplus_i) \qquad \dfrac{\begin{matrix}\pi_1\\ \vdash C_1^{\perp}, \Delta\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdash C_2^{\perp}, \Delta\end{matrix}}{\vdash, C_1^{\perp} \& C_2^{\perp}, \Delta}\ (\&)}{\vdash \Gamma, \Delta}\ (cut^{\alpha})$$

$$\xrightarrow[\oplus/\&]{} \qquad \dfrac{\begin{matrix}\pi_0\\ \vdash \Gamma, C_i\end{matrix} \qquad \begin{matrix}\pi_i\\ \vdash C_i^{\perp}, \Delta\end{matrix}}{\vdash \Gamma, \Delta}\ (cut^{\beta}) \qquad i \in \{1, 2\}$$

Exponential case (1)

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\;(!) \qquad \dfrac{\overset{\pi_2}{\vdash B^{\perp}, \Delta}}{\vdash ?B^{\perp}, \Delta}\;(?)}{\vdash ?\Gamma, \Delta}\;(cut^{\alpha}) \quad \underset{!/?}{\longrightarrow} \quad \dfrac{\overset{\pi_1}{\vdash ?\Gamma, B} \qquad \overset{\pi_2}{\vdash B^{\perp}, \Delta}}{\vdash ?\Gamma, \Delta}\;(cut^{\beta})$$

Exponential case (2)

$$
\dfrac{\dfrac{\overset{\pi_1}{\vdots}}{\vdash ?\Gamma, B}\ (!)\quad\text{gives}\quad \vdash ?\Gamma, !B \qquad \dfrac{\overset{\pi_2}{\vdots}}{\vdash \Delta}\ (w)\quad\text{gives}\quad \vdash ?B^{\perp}, \Delta}{\vdash ?\Gamma, \Delta}\ (cut^{\alpha})
$$

$$
\dfrac{\dfrac{\overset{\pi_1}{\vdots}\atop \vdash ?\Gamma, B}{\vdash ?\Gamma, !B}\ (!) \qquad \dfrac{\overset{\pi_2}{\vdots}\atop \vdash \Delta}{\vdash ?B^{\perp}, \Delta}\ (w)}{\vdash ?\Gamma, \Delta}\ (cut^{\alpha}) \quad \xrightarrow[!/?]{} \quad \dfrac{\overset{\pi_2}{\vdots}\atop \vdash \Delta}{\vdash ?\Gamma, \Delta}\ (w^{\star})
$$

Exponential case (3)

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash ?\Gamma, B\end{array}}{\vdash ?\Gamma, !B}\,(!) \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdash ?B^\perp, ?B^\perp, \Delta\end{array}}{\vdash ?B^\perp, \Delta}\,(c)}{\vdash ?\Gamma, \Delta}\,(cut^\alpha) \quad \xrightarrow[!/?]{}$$

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Commutation cases

$$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A, B, \Gamma, C}}{\vdash A \parr B, \Gamma, C}\ (\parr) \qquad \dfrac{\pi_2}{\vdash C^{\perp}, \Delta}}{\vdash A \parr B, \Gamma, \Delta}\ (cut^{\alpha})$$

$$\underset{\mathsf{comm}(\parr)}{\longrightarrow} \qquad \dfrac{\dfrac{\dfrac{\pi_1}{\vdash A, B, \Gamma, C} \qquad \dfrac{\pi_2}{\vdash C^{\perp}, \Delta}}{\vdash A, B, \Gamma, \Delta}\ (cut^{\beta})}{\vdash A \parr B, \Gamma, \Delta}\ (\parr)$$

Are those cut transformations really cut simplifications?

Definition (Cut rank, proof rank, size of a cut occurrence)

Let $\pi : \vdash \Gamma$ be a proof and r an occurrence of a cut inference of π.
The **cut rank** of r, $\mathsf{rk}(r)$ is the complexity of its cut-formula (ie. the number of connectives of the cut formula). The **proof rank** of π, $\mathsf{rk}(\pi)$ is the supremum of its cut ranks.
The **size** of r, $\mathsf{sz}(r)$, is the size of the proof tree rooted in r *neglecting the exchange rules*.

In previous cases, in the proof resulting from the transformation step, the cut which are involved in the proof-transformation rule:

- either have disappeared;
- or concern subformulas of the cut-formula of the original proof;
- or if not, they have the same cut-formula but are applied on subtrees of the original cut-premise (at least one of which is a strict subtree).

Therefore a natural measure to consider is a lexicographic ordering on the complexity of the cut-formula and the size of the proof-tree rooted in the cut: that is a well-founded ordering.

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But does it decrease on each reduction?

Back to the promotion / contraction case

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \vdash ?\Gamma, B\end{matrix}}{\vdash ?\Gamma, !B}\,(!) \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ \vdash ?B^\perp, ?B^\perp, \Delta\end{matrix}}{\vdash ?B^\perp, \Delta}\,(c)}{\vdash ?\Gamma, \Delta}\,(cut^{\alpha}) \quad \xrightarrow[!/?]{}$$

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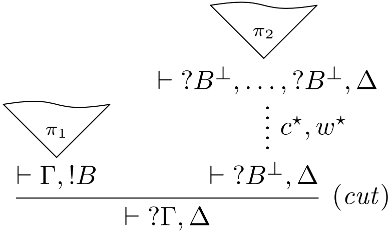
Structural cut (1)

To solve this issue, a natural workaround is to postpone the promotion/contraction case as much as possible by considering, in the case of an exponential cut, not the last rule of the subtree containing the ?-cut formula, but a branch leaving the cut up and containing only structural rules on the cut-formula:



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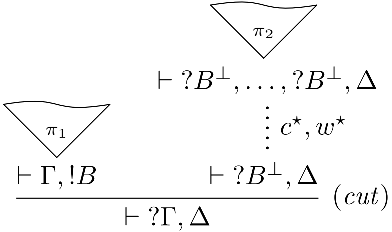

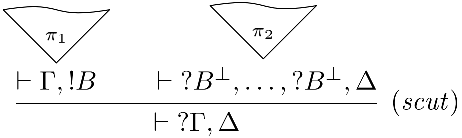

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$$\dfrac{\begin{array}{c}\pi_1\\ \vdash \Gamma, !B\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash ?B^\perp,\dots,?B^\perp,\Delta\\ \vdots\ c^\star, w^\star\\ \vdash ?B^\perp,\Delta\end{array}}{\vdash ?\Gamma,\Delta}\ (cut) \qquad\qquad \dfrac{\begin{array}{c}\pi_1\\ \vdash \Gamma, !B\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash ?B^\perp,\dots,?B^\perp,\Delta\end{array}}{\vdash ?\Gamma,\Delta}\ (scut)$$

This abstraction is a specialization of Gentzen's proof for LK, that he named the *mix* rule.

The promotion / contract case is simpler now:

$$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\,(!) \qquad \dfrac{\dfrac{\pi_2}{\vdash ?{B^{\perp}}^{l+1}, \Delta}}{\vdash ?{B^{\perp}}^{l}, \Delta}\,(c)}{\vdash ?\Gamma, \Delta}\,(scut^{\alpha}) \quad \underset{!/?}{\longrightarrow} \quad \dfrac{\dfrac{\dfrac{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\,(!) \qquad \dfrac{\pi_2}{\vdash ?{B^{\perp}}^{l+1}, \Delta}}{\vdash ?\Gamma, \Delta}\,(scut^{\beta})$$

Structural cut (2)

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$$
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\quad \xrightarrow[!/?]{} \quad
\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots \\ \vdash ?\Gamma, B\end{matrix}}{\vdash ?\Gamma, !B}\,(!) \qquad \begin{matrix}\pi_2\\ \vdots \\ \vdash ?{B^{\perp}}^{l+1}, \Delta\end{matrix}}{\vdash ?\Gamma, \Delta}\,(scut^{\beta})
$$

There will be some price to pay though...

Structural cut (3)

In the following, we shall therefore consider LL extended with the following (derivable) rule:

$$\dfrac{\begin{array}{c}\pi_1 \\ \vdots \\ \vdash \Gamma, B^k\end{array} \qquad \begin{array}{c}\pi_2 \\ \vdots \\ \vdash {B^{\perp}}^{l}, \Delta\end{array}}{\vdash \Gamma, \Delta}\ (scut)$$

with $k, l \in \mathbb{N}$ such that k (resp. l) can differ from 1 only if it labels a ?-formula.

Theorem

There exists a (contextual) binary relation, $\longmapsto_{\mathsf{c}}$*, on* LL *proofs satisfying:*

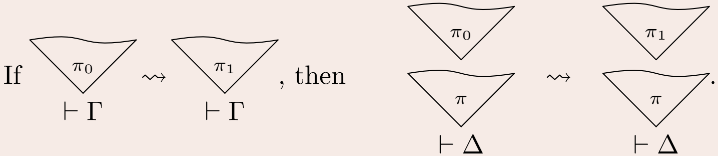
- *if* $\begin{array}{c}\pi \\ \vdots \\ \vdash \Gamma\end{array}$ *and* $\pi \longmapsto_{\mathsf{c}} \pi'$*, then* $\begin{array}{c}\pi' \\ \vdots \\ \vdash \Gamma\end{array}$ *;*
- *for any* LL *proof* π*, there exists some scut-free* π^{cf} *such that* $\pi \longmapsto_{\mathsf{c}}^{\star} \pi^{\mathsf{cf}}$*.*

Contextual proof reductions

In the following, all relations we shall consider will be assumed to have the property that they relate two proofs only if they have the same conclusion sequents.

Definition (Contextual reduction)

A binary relation $\rightsquigarrow$ on proof trees is *contextual* if for every proofs π_0, π_1, π'_0 such that $\pi_0 \rightsquigarrow \pi_1$, the proof π'_1 obtained by replacing a subtree of π'_0 equal to π_0 with π_1 is a proof such that $\pi'_0 \rightsquigarrow \pi'_1$:



A sufficient condition for scut-elimination: rank-decreasing reductions

Definition (Rank-decreasing reduction)

A contextual reduction $\rightsquigarrow$ is *rank-decreasing* if for any proof $\pi = \dfrac{\dfrac{\pi_1}{\vdash \Gamma, C^k} \qquad \dfrac{\pi_2}{\vdash {C^\perp}^l, \Delta}}{\vdash \Gamma, \Delta}\ (scut)$

such that $\mathsf{rk}(\pi_1), \mathsf{rk}(\pi_2) < \mathsf{rk}(\pi)$, there exists π' such that $\pi \rightsquigarrow^\star \pi'$ and $\mathsf{rk}(\pi') < \mathsf{rk}(\pi)$.

Theorem

If $\rightsquigarrow$ is rank-decreasing, then for any sequent Γ and for any proof $\pi : \vdash \Gamma$, there exists a cut-free proof π' of $\vdash \Gamma$ such that $\pi \rightsquigarrow^\star \pi'$.

Proof.

By lexicographic induction on $\mathsf{w}(\pi) = \begin{cases} (0,0) & \text{if } \pi \text{ is scut-free} \\ (r,n) & \text{otherwise, with } r = \mathsf{rk}(\pi) \text{ and } n \\ & \text{the number of scuts of rank } r \text{ in } \pi. \end{cases}$

applying the rank-decreasing property on a topmost scut of maximal rank. □

Definition of a rank decreasing cut-reduction

We define a rank-decreasing cut-reduction for proofs having a structural cut as a last rule, by case analysis the cut-formula and on the last rules of π_1 and π_2.
Without loss of generality, assume that the proofs ends with (P being a positive formula):

$$\pi = \cfrac{\begin{array}{c}\pi_1\\ \vdash \Gamma, P\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash {P^{\perp}}^{l}, \Delta\end{array}}{\vdash \Gamma, \Delta}\ (scut)$$

1. If both P and $P^{\perp}$ are principal in logical rules concluding π_1, π_2, one defines a *key logical-step*.
2. Otherwise, if P or $P^{\perp}$ is principal in an axiom rule, one defines an *key axiom-step*;
3. Otherwise, if P is not principal in the last rule of π_1, one defines a *commutation rule* of the last rule of π_1 below the *scut*;
4. If P is principal in a logical rule concluding π_1 and $P^{\perp}$ is not principal in the last rule of π_2, one defines a *commutation rule* of the last rule of π_2 below the *scut*.

1. Key logical-step

For multiplicative and additive connectives, scut is simply a regular cut: this case is as in the introduction, that we call $\underset{\otimes/\parr}{\longrightarrow}$, $\underset{\mathbf{1}/\bot}{\longrightarrow}$, $\underset{\oplus/\&}{\longrightarrow}$.

For the exponential connectives, we have the following cases:

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\,(!) \qquad \dfrac{\overset{\pi_2}{\vdash B^\perp, ?{B^\perp}^l, \Delta}}{\vdash ?{B^\perp}^{l+1}, \Delta}\,(?)}{\vdash ?\Gamma, \Delta}\,(scut) \qquad \dfrac{\dfrac{\overset{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\,(!) \qquad \dfrac{\overset{\pi_2}{\vdash ?{B^\perp}^l, \Delta}}{\vdash ?{B^\perp}^{l+1}, \Delta}\,(w)}{\vdash ?\Gamma, \Delta}\,(scut) \qquad \dfrac{\dfrac{\overset{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\,(!) \qquad \dfrac{\overset{\pi_2}{\vdash ?{B^\perp}^{l+1}, \Delta}}{\vdash ?{B^\perp}^{l}, \Delta}\,(c)}{\vdash ?\Gamma, \Delta}\,(scut)$$

We define reduction $\underset{!/?}{\longrightarrow}$ in the following slides

1. Key logical-step (2)

Promotion versus dereliction

- Promotion versus dereliction ($l = 0$):

$$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\,(!) \qquad \dfrac{\dfrac{\pi_2}{\vdash B^\perp, \Delta}}{\vdash ?B^\perp, \Delta}\,(?)}{\vdash ?\Gamma, \Delta}\,(scut^\alpha) \quad \xrightarrow[!/?]{} \quad \dfrac{\dfrac{\pi_1}{\vdash ?\Gamma, B} \qquad \dfrac{\pi_2}{\vdash B^\perp, \Delta}}{\vdash ?\Gamma, \Delta}\,(scut^\beta)$$

- Promotion versus dereliction ($l \geq 1$):

$$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\,(!) \qquad \dfrac{\dfrac{\pi_2}{\vdash B^\perp, ?{B^\perp}^l, \Delta}}{\vdash ?{B^\perp}^{l+1}, \Delta}\,(?)}{\vdash ?\Gamma, \Delta}\,(scut^\alpha) \quad \xrightarrow[!/?]{} \quad \dfrac{\dfrac{\dfrac{\pi_1}{\vdash ?\Gamma, B} \qquad \dfrac{\dfrac{\dfrac{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\,(!) \qquad \dfrac{\pi_2}{\vdash B^\perp, ?{B^\perp}^l, \Delta}}{\vdash ?\Gamma, B^\perp, \Delta}\,(scut^\gamma)}{\vdash ?\Gamma, ?\Gamma, \Delta}\,(scut^\beta)}{\vdash ?\Gamma, \Delta}\,(c^\star)$$

$$
\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdash ?\Gamma, B\end{matrix}}{\vdash ?\Gamma, !B}\ (!) \qquad \dfrac{\begin{matrix}\pi_2\\ \vdash ?{B^{\perp}}^{l}, \Delta\end{matrix}}{\vdash ?{B^{\perp}}^{l+1}, \Delta}\ (w)}{\vdash ?\Gamma, \Delta}\ (scut^{\alpha})
\quad \xrightarrow[!/?]{} \quad
\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdash ?\Gamma, B\end{matrix}}{\vdash ?\Gamma, !B}\ (!) \qquad \begin{matrix}\pi_2\\ \vdash ?{B^{\perp}}^{l}, \Delta\end{matrix}}{\vdash ?\Gamma, \Delta}\ (scut^{\beta})
$$

1. Key logical-step (3)

Promotion versus weakening

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\,(!) \qquad \dfrac{\overset{\pi_2}{\vdash ?{B^{\perp}}^{l}, \Delta}}{\vdash ?{B^{\perp}}^{l+1}, \Delta}\,(w)}{\vdash ?\Gamma, \Delta}\,(scut^{\alpha}) \quad \underset{!/?}{\longrightarrow} \quad \dfrac{\dfrac{\overset{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\,(!) \qquad \overset{\pi_2}{\vdash ?{B^{\perp}}^{l}, \Delta}}{\vdash ?\Gamma, \Delta}\,(scut^{\beta})$$

$$\left(\text{optional reduction:} \quad \dfrac{\dfrac{\overset{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\,(!) \qquad \overset{\pi_2}{\vdash \Delta}}{\vdash ?\Gamma, \Delta}\,(scut^{\alpha}) \quad \underset{!/?}{\longrightarrow} \quad \dfrac{\overset{\pi_2}{\vdash \Delta}}{\vdash ?\Gamma, \Delta}\,(w^{\star}) \right)$$

1. Key logical-step (4)

Promotion versus contraction

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\,(!) \qquad \dfrac{\overset{\pi_2}{\vdash ?{B^{\perp}}^{l+1}, \Delta}}{\vdash ?{B^{\perp}}^{l}, \Delta}\,(c)}{\vdash ?\Gamma, \Delta}\,(scut^{\alpha}) \quad \xrightarrow[!/?]{} \quad \dfrac{\dfrac{\overset{\pi_1}{\vdash ?\Gamma, B}}{\vdash ?\Gamma, !B}\,(!) \qquad \overset{\pi_2}{\vdash ?{B^{\perp}}^{l+1}, \Delta}}{\vdash ?\Gamma, \Delta}\,(scut^{\beta})$$

We notice the following relations about the ranks and levels of cuts:

- (!) versus (?) ($l = 0$): $\mathsf{rk}(\beta) < \mathsf{rk}(\alpha)$ and $\mathsf{sz}(\beta) = \mathsf{sz}(\alpha) - 2$.
- (!) versus (?) ($l \geq 1$): $\mathsf{rk}(\beta) < \mathsf{rk}(\alpha)$, $\mathsf{rk}(\gamma) = \mathsf{rk}(\alpha)$, $\mathsf{sz}(\gamma) = \mathsf{sz}(\alpha) - 1$ (but $\mathsf{sz}(\beta)$ may be larger than the $\mathsf{sz}(\alpha)$).
- (!) versus (w): $\mathsf{rk}(\beta) = \mathsf{rk}(\alpha)$, $\mathsf{sz}(\beta) = \mathsf{sz}(\alpha) - 1$.
- (!) versus (c): $\mathsf{rk}(\beta) = \mathsf{rk}(\alpha)$, $\mathsf{sz}(\beta) = \mathsf{sz}(\alpha) - 1$.

2. Key axiom-step

- k=1

$$\dfrac{\overset{\pi_1}{\vdash \Gamma, C} \qquad \dfrac{}{\vdash C^{\perp}, C}\,(ax)}{\vdash \Gamma, C}\,(scut) \quad \xrightarrow[\text{ax}]{} \quad \overset{\pi_1}{\vdash C, \Gamma}$$

- k=0

$$\dfrac{\overset{\pi_1}{\vdash \Gamma} \qquad \dfrac{}{\vdash !C^{\perp}, ?C}\,(ax)}{\vdash \Gamma, ?C}\,(scut) \quad \xrightarrow[\text{ax}]{} \quad \dfrac{\overset{\pi_1}{\vdash \Gamma}}{\vdash ?C, \Gamma}\,(w)$$

- k>1

$$\dfrac{\overset{\pi_1}{\vdash ?C^k, \Gamma} \qquad \dfrac{}{\vdash !C^{\perp}, ?C}\,(ax)}{\vdash \Gamma, ?C}\,(scut) \quad \xrightarrow[\text{ax}]{} \quad \dfrac{\overset{\pi_1}{\vdash ?C^k, \Gamma}}{\vdash ?C, \Gamma}\,(c^{k-1})$$

Notice that the *scut* disappeared.

3. Commuting the last rule of π_1 with *scut*

$\parr$ commutation step

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash A, B, \Gamma, P\end{array}}{\vdash A \parr B, \Gamma, P}(\parr) \qquad \begin{array}{c}\pi_2\\ \vdash {P^{\perp}}^{l}, \Delta\end{array}}{\vdash A \parr B, \Gamma, \Delta}(scut^{\alpha}) \quad \underset{\mathsf{comm}(\parr)}{\longrightarrow} \quad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash A, B, \Gamma, P\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash {P^{\perp}}^{l}, \Delta\end{array}}{\vdash A, B, \Gamma, \Delta}(scut^{\beta})}{\vdash A \parr B, \Gamma, \Delta}(\parr)$$

We notice that $\mathsf{rk}(\beta) = \mathsf{rk}(\alpha)$ and $\mathsf{sz}(\beta) < \mathsf{sz}(\alpha)$.

$\perp$ commutation step

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash \Gamma, P\end{array}}{\vdash \perp, \Gamma, P}(\perp) \qquad \begin{array}{c}\pi_2\\ \vdash {P^{\perp}}^{l}, \Delta\end{array}}{\vdash \perp, \Gamma, \Delta}(scut^{\alpha}) \quad \underset{\mathsf{comm}(\perp)}{\longrightarrow} \quad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash \Gamma, P\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash {P^{\perp}}^{l}, \Delta\end{array}}{\vdash \Gamma, \Delta}(scut^{\beta})}{\vdash \perp, \Gamma, \Delta}(\perp)$$

We notice that $\mathsf{rk}(\beta) = \mathsf{rk}(\alpha)$ and $\mathsf{sz}(\beta) < \mathsf{sz}(\alpha)$.

3. Commuting the last rule of π_1 with *scut*

$\otimes$ commutation step Without loss of generality, assume that the (only occurrence of the) cut-formula P goes to the left premise of the $\otimes$, we consider the following commutation relation (the other case is similar):

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdash A,\Gamma\end{matrix}\qquad \begin{matrix}\pi_2\\ \vdash B,\Gamma',P\end{matrix}}{\vdash A\otimes B,\Gamma,\Gamma',P}\,(\otimes) \qquad \begin{matrix}\pi_3\\ \vdash {P^{\perp}}^{l},\Delta\end{matrix}}{\vdash A\otimes B,\Gamma,\Gamma',\Delta}\,(scut^{\alpha})$$

$$\xrightarrow[\mathsf{comm}(\otimes)]{} \quad \dfrac{\begin{matrix}\pi_1\\ \vdash A,\Gamma\end{matrix}\qquad \dfrac{\begin{matrix}\pi_2\\ \vdash B,\Gamma',P\end{matrix}\qquad \begin{matrix}\pi_3\\ \vdash {P^{\perp}}^{l},\Delta\end{matrix}}{\vdash B,\Gamma',\Delta}\,(scut^{\beta})}{\vdash A\otimes B,\Gamma,\Gamma',\Delta}\,(\otimes)$$

We notice that $\mathsf{rk}(\beta)=\mathsf{rk}(\alpha)$ and $\mathsf{sz}(\beta)<\mathsf{sz}(\alpha)$.

$\mathbf{1}$ commutation step Impossible as there is (exactly one occurrence of) P in the premise.

3. Commuting the last rule of π_1 with *scut*

& commutation step

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash A,\Gamma,P\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash B,\Gamma,P\end{array}}{\vdash A \,\&\, B,\Gamma,P}\,(\&) \qquad \begin{array}{c}\pi_3\\ \vdash {P^{\perp}}^{l},\Delta\end{array}}{\vdash A \,\&\, B,\Gamma,\Delta}\,(scut^{\alpha})$$

$$\xrightarrow[\mathsf{comm}(\&)]{} \quad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash A,\Gamma,P\end{array} \qquad \begin{array}{c}\pi_3\\ \vdash {P^{\perp}}^{l},\Delta\end{array}}{\vdash A,\Gamma,\Delta}\,(scut^{\beta}) \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdash B,\Gamma,P\end{array} \qquad \begin{array}{c}\pi_3\\ \vdash {P^{\perp}}^{l},\Delta\end{array}}{\vdash B,\Gamma,\Delta}\,(scut^{\gamma})}{\vdash A \,\&\, B,\Gamma,\Delta}\,(\&)$$

We notice that $\mathsf{rk}(\beta) = \mathsf{rk}(\gamma) = \mathsf{rk}(\alpha)$ and $\mathsf{sz}(\beta), \mathsf{sz}(\gamma) < \mathsf{sz}(\alpha)$.

$\top$ commutation step

$$\dfrac{\dfrac{}{\vdash \top,\Gamma,P}\,(\top) \qquad \begin{array}{c}\pi\\ \vdash {P^{\perp}}^{l},\Delta\end{array}}{\vdash \top,\Gamma,\Delta}\,(scut^{\alpha}) \quad \xrightarrow[\mathsf{comm}(\top)]{} \quad \dfrac{}{\vdash \top,\Gamma,\Delta}\,(\top)$$

We notice that the resulting proof is cut-free.

3. Commuting the last rule of π_1 with *scut*

$\oplus$ commutation step

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash A_i, \Gamma, P}}{\vdash A_1 \oplus A_2, \Gamma, P}\,(\oplus_i) \qquad \overset{\pi_2}{\vdash {P^{\perp}}^{l}, \Delta}}{\vdash A_1 \oplus A_2, \Gamma, \Delta}\,(scut^{\alpha})$$

$$\underset{\mathsf{comm}(\oplus)}{\longrightarrow} \quad \dfrac{\dfrac{\overset{\pi_1}{\vdash A_i, \Gamma, P} \qquad \overset{\pi_2}{\vdash {P^{\perp}}^{l}, \Delta}}{\vdash A_1 \oplus A_2, \Gamma, \Delta}\,(scut^{\beta})}{\vdash A_1 \oplus A_2, \Gamma, \Delta}\,(\oplus_i)$$

We notice that $\mathsf{rk}(\beta) = \mathsf{rk}(\alpha)$ and $\mathsf{sz}(\beta) < \mathsf{sz}(\alpha)$.

0 commutation step This case cannot occur.

3. Commuting the last rule of π_1 with *scut*

Promotion commutation step This case cannot occur as P is neither a ? formula nor principal in the sequent.

Dereliction commutation step

$$\cfrac{\cfrac{\overset{\pi_1}{\vdots}}{\cfrac{\vdash A, \Gamma, P}{\vdash ?A, \Gamma, P}\;(?)} \qquad \cfrac{\overset{\pi_2}{\vdots}}{\vdash {P^{\perp}}^{l}, \Delta}}{\vdash ?A, \Gamma, \Delta}\;(scut^{\alpha})$$

$$\underset{\mathsf{comm}(?)}{\longrightarrow} \qquad \cfrac{\cfrac{\cfrac{\overset{\pi_1}{\vdots}}{\vdash A, \Gamma, P} \qquad \cfrac{\overset{\pi_2}{\vdots}}{\vdash {P^{\perp}}^{l}, \Delta}}{\vdash A, \Gamma, \Delta}\;(scut^{\beta})}{\vdash ?A, \Gamma, \Delta}\;(?)$$

We notice that $\mathsf{rk}(\beta) = \mathsf{rk}(\alpha)$ and $\mathsf{sz}(\beta) < \mathsf{sz}(\alpha)$.

Structural commutation step

For contraction:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash ?A, ?A, \Gamma, P\end{array}}{\vdash ?A, \Gamma, P}\,(c) \qquad \begin{array}{c}\pi_2\\ \vdash {P^{\perp}}^{l}, \Delta\end{array}}{\vdash ?A, \Gamma, \Delta}\,(scut^{\alpha}) \quad \underset{\mathsf{comm(c)}}{\longrightarrow} \quad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash ?A, ?A, \Gamma, P\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash {P^{\perp}}^{l}, \Delta\end{array}}{\vdash ?A, ?A, \Gamma, \Delta}\,(scut^{\beta})}{\vdash ?A, \Gamma, \Delta}\,(c)$$

For weakening:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash \Gamma, P\end{array}}{\vdash ?A, \Gamma, P}\,(w) \qquad \begin{array}{c}\pi_2\\ \vdash {P^{\perp}}^{l}, \Delta\end{array}}{\vdash ?A, \Gamma, \Delta}\,(scut^{\alpha}) \quad \underset{\mathsf{comm(w)}}{\longrightarrow} \quad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdash \Gamma, P\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash {P^{\perp}}^{l}, \Delta\end{array}}{\vdash \Gamma, \Delta}\,(scut^{\beta})}{\vdash ?A, \Gamma, \Delta}\,(w)$$

For both cases, we notice that $\mathsf{rk}(\beta) = \mathsf{rk}(\alpha)$ and $\mathsf{sz}(\beta) < \mathsf{sz}(\alpha)$.

3. Commuting the last rule of π_1 with *scut*

ax commutation step (similar to the case of **1**) it cannot occur since P does occur in the sequent.

scut commutation step (Similar to the $\otimes$ commutation) Without loss of generality, assume that P goes to the right premise of the topmost scut (the other case is similar), one considers the following commutation relation requiring that the $\mathsf{rk}(\alpha)$ is greater that the $\mathsf{rk}(\beta)$:

$$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash \Gamma, D^m} \quad \dfrac{\pi_2}{\vdash {D^{\perp}}^{m'}, \Gamma', P}}{\vdash \Gamma, \Gamma', P}\,(scut^{\beta}) \quad \dfrac{\pi_3}{\vdash {P^{\perp}}^{l}, \Delta}}{\vdash \Gamma, \Gamma', \Delta}\,(scut^{\alpha}) \quad \underset{\mathsf{comm}(cut)}{\longrightarrow} \quad \dfrac{\dfrac{\pi_1}{\vdash \Gamma, D^m} \quad \dfrac{\dfrac{\pi_2}{\vdash {D^{\perp}}^{m'}, \Gamma', P} \quad \dfrac{\pi_3}{\vdash {P^{\perp}}^{l}, \Delta}}{\vdash {D^{\perp}}^{m'}, \Gamma', \Delta}\,(scut^{\delta})}{\vdash \Gamma, \Gamma', \Delta}\,(scut^{\gamma})$$

We notice that $\mathsf{rk}(\beta) = \mathsf{rk}(\gamma)$, $\mathsf{rk}(\alpha) = \mathsf{rk}(\delta)$ and $\mathsf{sz}(\delta) < \mathsf{sz}(\alpha)$ (while $\mathsf{sz}(\gamma) > \mathsf{sz}(\beta)$).

4. Commuting the last rule of π_2 with *scut*

In this last case, we know that the last rule of π_1 is a logical rule on P. In particular, in the case of an exponential cut, that means π_1 ends with a promotion rule... and the context is of the form $?\Gamma$, that is, **it can be contracted and weakened at will!**

For the multiplicative and additive cuts, the commutation rules are similar as in the previous section: one can apply them freely (the other premise does not need to have its cut formula principal) as the only property that we use on the positive cut formula was the fact that it had **a unique occurrence in the premise of the cut rule** which holds for non-? formulas.

However, a new situation arises for exponential cuts: indeed, the cut formula being a ?-formula, it may have $l \neq 1$ occurrences in the conclusion sequent of π_2, which may interfere with the **non-unary mutiplicative rules and the identity rules** (the commutation rules are unchanged for the other).

New cases arise when the last rule of π_2 is

- $(\mathbf{1}), (ax)$ for the nullary rules;
- $(\otimes), (scut)$ for the binary rules.

4. Commuting the last rule of π_2 with *scut* – new cases for $?C$

⊗ **commutation step** $P^\perp = ?C$, one considers the following commutation relation:

$$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A, \Gamma, ?C^k} \qquad \dfrac{\pi_2}{\vdash B, \Gamma', ?C^{k'}}}{\vdash A \otimes B, \Gamma, \Gamma', ?C^{k+k'}}(\otimes) \qquad \dfrac{\pi_3}{\vdash !C^\perp, ?\Delta'}}{\vdash A \otimes B, \Gamma, \Gamma', ?\Delta'}(scut^\alpha)$$

$$\underset{\mathsf{comm}(\otimes)}{\longrightarrow} \quad \dfrac{\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A, \Gamma, ?C^k} \qquad \dfrac{\pi_3}{\vdash !C^\perp, ?\Delta'}}{\vdash A, \Gamma, ?\Delta'}(scut^\beta) \qquad \dfrac{\dfrac{\pi_2}{\vdash B, \Gamma', ?C^{k'}} \qquad \dfrac{\pi_3}{\vdash !C^\perp, ?\Delta'}}{\vdash B, \Gamma', ?\Delta'}(scut^\gamma)}{\vdash A \otimes B, \Gamma, \Gamma', ?\Delta', ?\Delta'}(\otimes)}{\vdash A \otimes B, \Gamma, \Gamma', ?\Delta'}(c^\star)$$

We notice that $\mathsf{rk}(\beta) = \mathsf{rk}(\gamma) = \mathsf{rk}(\alpha)$ and $\mathsf{sz}(\beta), \mathsf{sz}(\gamma) < \mathsf{sz}(\alpha)$.

***scut* commutation step** Similar (possibly restricting to the commutation of smaller-rank cuts only).

4. Commuting the last rule of π_2 with *scut* – new cases

1 commutation step This case can only occur in the following case:

$$\dfrac{\dfrac{\vdots\ \pi_1}{\vdash ?\Delta, !C^{\perp}} \qquad \dfrac{}{\vdash \mathbf{1}}\ (\mathbf{1})}{\vdash ?\Delta, \mathbf{1}}\ (scut^{\alpha}) \quad \xrightarrow{\mathsf{comm}(\mathbf{1})} \quad \dfrac{\dfrac{}{\vdash \mathbf{1}}\ (\mathbf{1})}{\vdash ?\Delta, \mathbf{1}}\ (w^{\star})$$

We notice that the resulting proof is cut-free.

***ax* commutation step** Similar.

$$\dfrac{\dfrac{\vdots\ \pi_1}{\vdash ?\Delta, !C^{\perp}} \qquad \dfrac{}{\vdash A, A^{\perp}}\ (ax)}{\vdash ?\Delta A, A^{\perp}}\ (scut^{\alpha}) \quad \xrightarrow{\mathsf{comm}(ax)} \quad \dfrac{\dfrac{}{\vdash A, A^{\perp}}\ (ax)}{\vdash ?\Delta, A, A^{\perp}}\ (w^{\star})$$

We notice that the resulting proof is cut-free.

Definition ($\longmapsto_{\mathsf{c}}$)

Consider

$$\begin{aligned}
\longrightarrow \quad = \quad & \underset{\mathsf{ax}}{\longrightarrow} \\
& \cup \underset{\otimes/\parr}{\longrightarrow} \cup \underset{\parr/\otimes}{\longrightarrow} \cup \underset{\mathbf{1}/\bot}{\longrightarrow} \cup \underset{\bot/\mathbf{1}}{\longrightarrow} \cup \underset{\oplus/\&}{\longrightarrow} \cup \underset{\&/\oplus}{\longrightarrow} \\
& \cup \underset{!/?}{\longrightarrow} \cup \underset{?/!}{\longrightarrow} \\
& \cup \underset{\mathsf{comm}(\parr)}{\longrightarrow} \cup \underset{\mathsf{comm}(\bot)}{\longrightarrow} \cup \underset{\mathsf{comm}(\otimes)}{\longrightarrow} \cup \underset{\mathsf{comm}(\mathbf{1})}{\longrightarrow} \cup \underset{\mathsf{comm}(\&)}{\longrightarrow} \cup \underset{\mathsf{comm}(\top)}{\longrightarrow} \cup \underset{\mathsf{comm}(\oplus)}{\longrightarrow} \\
& \cup \underset{\mathsf{comm}(?)}{\longrightarrow} \cup \underset{\mathsf{comm}(\mathsf{c})}{\longrightarrow} \cup \underset{\mathsf{comm}(\mathsf{w})}{\longrightarrow} \cup \underset{\mathsf{comm}(!)}{\longrightarrow} \\
& \cup \underset{\mathsf{comm}(ax)}{\longrightarrow} \cup \underset{\mathsf{comm}(cut)}{\longrightarrow}
\end{aligned}$$

$\longmapsto_{\mathsf{c}}$ is the contextual closure of $\longrightarrow$.

Lemma

Let π be an LL *structural proof of the form:*

$$\dfrac{\overset{\pi_1}{\vdash \Gamma, P} \qquad \overset{\pi_2}{\vdash P^{\perp^l}, \Delta}}{\vdash \Gamma, \Delta}\ (scut)$$

such that $\mathsf{rk}(\pi_1), \mathsf{rk}(\pi_2) < \mathsf{rk}(\pi)$*, then there exists* π' *such that* $\pi \longmapsto_{\mathsf{c}}{}^{\star} \pi'$ *and* $\mathsf{rk}(\pi') < \mathsf{rk}(\pi)$.

Proof.

We prove the lemma by induction on the size of π.

Base case: Assume that both premises of π are or size 1: they are either an axiom, a **1** or a $\top$. Then π reduces to some scut-free π', therefore of rank 0.

□

$\longmapsto_c$ is rank-decreasing

Proof.

Inductive case: We reason by case analysis on the cut-formula and the last inferences of π_1 and π_2:

1. If both P and $P^\perp$ are principal in logical rules concluding π_1, π_2.
2. Otherwise, if P or $P^\perp$ is principal in an axiom rule.
3. Otherwise, if P is not principal in the last rule of π_1.
4. If P is principal in a logical rule concluding π_1 and $P^\perp$ is not principal in the last rule of π_2.

1. If both P and $P^\perp$ are principal in logical rules concluding π_1, π_2, one applied a *key logical-step*:
(i) in the multiplicative or additive cases, the rank is smaller;
(ii) in the exponential case, either the rank is smaller, or it is equal and the size of the proof is smaller: by applying the induction hypothesis one expected result. *(Notice the order in which one reduces the promotion/dereliction case which creates two cuts: the bottomost has its rank reduced, so the inductive hypothesis can indeed be applied on the topmost proof.)*

□

$\longmapsto_{\mathsf{c}}$ is rank-decreasing

Proof.

2. Otherwise, if P or $P^\perp$ is principal in an axiom rule, the scut disappears with a *key axiom-step*.
3. Otherwise, if P is not principal in the last rule of π_1, one applies *commutation rule* of the last rule of π_1 below the *scut*:
   (i) in the logical/structural cases, if cuts of the maximal rank remain, they are on different branches and are conclusions of smaller proofs: the induction hypothesis applies.
   (ii) in the scut commutation, notice that by hypothesis of the lemma, the scut that is commuted down is necessarily of a lower rank (by the maximality condition). The cut of maximal rank now concludes a smaller proof and the induction hypothesis allows to conclude.
4. If P is principal in a logical rule concluding π_1 and $P^\perp$ is not principal in the last rule of π_2, one defines a *commutation rule* of the last rule of π_2 below the *scut*:
   (i) most case are treated as in the previous case;
   (ii) the new cases allow the invoke the induction hypothesis and conclude.

□

$\longmapsto_{\mathsf{c}}$ is rank-decreasing

Proof.

2. Otherwise, if P or $P^\perp$ is principal in an axiom rule, the scut disappears with a *key axiom-step*.
3. Otherwise, if P is not principal in the last rule of π_1, one applies *commutation rule* of the last rule of π_1 below the *scut*:
   (i) in the logical/structural cases, if cuts of the maximal rank remain, they are on different branches and are conclusions of smaller proofs: the induction hypothesis applies.
   (ii) in the scut commutation, notice that by hypothesis of the lemma, the scut that is commuted down is necessarily of a lower rank (by the maximality condition). The cut of maximal rank now concludes a smaller proof and the induction hypothesis allows to conclude.
4. If P is principal in a logical rule concluding π_1 and $P^\perp$ is not principal in the last rule of π_2, one defines a *commutation rule* of the last rule of π_2 below the *scut*:
   (i) most case are treated as in the previous case;
   (ii) the new cases allow the invoke the induction hypothesis and conclude.

□

This concludes the cut-elimination proof!

Additional slides
(not covered during the lecture)

Consequences of cut-elimination

All LL inferences but the *cut* rule are such that the premise sequents are made of subformulas of the conclusion sequent. This extends from inferences to cut-free proofs:

Proposition (Subformula property)

If $\vdash \Gamma$ is provable, it can be proved using only subformulas of the conclusion sequent.

Theorem

$\vdash$ *, the empty sequent, is not derivable in* LL.

Theorem (Consistency of LL)

Neither $\vdash \mathbf{0}$, nor $\vdash \bot$ is not provable in LL.
For no formula A are both A and $A^\perp$ provable.

Consequences of cut-elimination

Proposition (Disjunction and witness property)

- *If $A \oplus B$ is provable, then at least one of A or B is provable;*
- *(in the FO case) If $\exists x.A$ is provable, there exists some term t such that $A[t/x]$ is provable.*

Theorem

(First-order) MALL is decidable.

Some more remarks on cut-elimination

- Cut-elimination holds for both first and second-order quantifiers:
  - First-order: 1) take care of genericity / eigenvariable conditions on $\forall$ quantifier to ensure that cut-reductions and cut-commutations do not break the side-conditions on the $\forall$-inference. 2)weak-normalization follows from embedding the first-order part as a unary-additive connective for instance.
  - Second-order: Beyond the scope of these lectures...
- Contrarily to LK cut-elimination, the weak-normalization can be strengthened into a strong-normalization result. (see next slide.)
- Despite cut-elimination, even in the propositional case, provability in LL is undecidable.

$\longmapsto_{\mathsf{c}}$ is strongly normalizing

The weak-normalization proof can be strengthened into a strong normalization proof as an immediate corollary of the following lemma:

Lemma

Let π be an LL *structural proof of the form:*

$$\dfrac{\begin{matrix}\pi_1 \\ \vdash \Gamma, C^k\end{matrix} \qquad \begin{matrix}\pi_2 \\ \vdash {C^{\perp}}^l, \Delta\end{matrix}}{\vdash \Gamma, \Delta}\ (scut^{\alpha})$$

such that π_1 and π_2 are strongly normalizing, then π is strongly normalizing.

Proof.

Since π_1 and π_2 are SN by hypothesis, there is (by König's lemma) a bound on the lengths of the reduction sequences from π which never reduce the bottommost cut (call them root-passive). Let $\mathsf{l}(\pi)$ be the supremum of these lengths. The proof is by induction on $(\mathsf{rk}(\alpha), \mathsf{l}(\pi), \mathsf{sz}(\pi))$.
Indeed, every reduction considered on *scut* reduces this measure: (i) commutative reductions reduce $\mathsf{sz}(\pi)$, (ii) key cases reduce $\mathsf{rk}(\pi)$ or $\mathsf{sz}(\pi)$ (iii) inner applications of a cut-elimination step reduce $\mathsf{l}(\pi)$. In the case when two scuts are created one above the other, the inductive step applied on the topmost scut ensures that the bottommost indeed has premises which are both SN. □