Multiplicative Proof Nets

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What is a proof?

When are two proofs equal?

Unit-Free Multiplicative Linear Logic

The paradize of Linear Logic

MLL Sequent Calculus

$$\frac{}{\vdash A, A^{\perp}}\ (ax) \qquad \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\ (ex) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ (\parr) \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ (\otimes)$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\ (cut)$$

Unit-Free Multiplicative Linear Logic

The paradize of Linear Logic

MLL Sequent Calculus

$$\frac{}{\vdash A, A^\perp}\ (ax) \qquad \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\ (ex) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ (\parr) \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ (\otimes)$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\ (cut)$$

Remarks

- SPOILER: **cut is admissible**
- **implicit exchange**: can be reintroduced / follow occurrences / assume all atoms are distinct
- **linearity**: at most one logical rule per occurrence of connective

Example

$$\dfrac{}{\vdash D, D^\perp}\,(ax) \qquad \dfrac{}{\vdash C^\perp, C}\,(ax)$$

$$\dfrac{\vdash D, D^\perp \qquad \vdash C^\perp, C}{\vdash C, D, D^\perp \otimes C^\perp}\,(\otimes)$$

$$\dfrac{\vdash C, D, D^\perp \otimes C^\perp}{\vdash C \parr D, D^\perp \otimes C^\perp}\,(\parr)$$

$$\dfrac{}{\vdash A, A^\perp}\,(ax) \qquad \dfrac{}{\vdash E^\perp, E}\,(ax) \qquad \dfrac{}{\vdash B, B^\perp}\,(ax)$$

$$\dfrac{\vdash A, A^\perp \qquad \vdash E^\perp, E}{\vdash A, A^\perp \otimes E^\perp, E}\,(\otimes)$$

$$\dfrac{\vdash B, B^\perp \qquad \vdash C \parr D, D^\perp \otimes C^\perp}{\vdash B, B^\perp \otimes (C \parr D), D^\perp \otimes C^\perp}\,(\otimes)$$

$$\dfrac{\vdash A, A^\perp \otimes E^\perp, E \qquad \vdash B, B^\perp \otimes (C \parr D), D^\perp \otimes C^\perp}{\vdash A \otimes B, B^\perp \otimes (C \parr D), D^\perp \otimes C^\perp, A^\perp \otimes E^\perp, E}\,(\otimes)$$

$$\dfrac{\vdash A \otimes B, B^\perp \otimes (C \parr D), D^\perp \otimes C^\perp, A^\perp \otimes E^\perp, E}{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), D^\perp \otimes C^\perp, A^\perp \otimes E^\perp, E}\,(\parr)$$

$$\dfrac{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), D^\perp \otimes C^\perp, A^\perp \otimes E^\perp, E}{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), (D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp), E}\,(\parr)$$

$$\dfrac{}{\vdash F^\perp, F}\,(ax)$$

$$\dfrac{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), (D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp), E \qquad \vdash F^\perp, F}{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), ((D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp)) \otimes F^\perp, F, E}\,(\otimes)$$

$$\dfrac{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), ((D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp)) \otimes F^\perp, F, E}{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), ((D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp)) \otimes F^\perp, F \parr E}\,(\parr)$$

Loosing Information

Lacunar Sequent Calculus

Lacunar Rules

$$\frac{}{\vdash A, A^{\perp}}\ (ax) \qquad \frac{\vdash \Gamma}{\vdash A \parr B}\ (\parr) \qquad \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash A \otimes B}\ (\otimes)$$

Example

$$\frac{}{\vdash D, D^\perp}\,(ax) \qquad \frac{}{\vdash C^\perp, C}\,(ax)$$

$$\frac{\vdash D, D^\perp \qquad \vdash C^\perp, C}{\vdash C, D, D^\perp \otimes C^\perp}\,(\otimes)$$

$$\frac{\vdash C, D, D^\perp \otimes C^\perp}{\vdash C \parr D, D^\perp \otimes C^\perp}\,(\parr)$$

$$\frac{}{\vdash A, A^\perp}\,(ax) \qquad \frac{}{\vdash E^\perp, E}\,(ax) \qquad \frac{}{\vdash B, B^\perp}\,(ax)$$

$$\frac{\vdash A, A^\perp \qquad \vdash E^\perp, E}{\vdash A, A^\perp \otimes E^\perp, E}\,(\otimes) \qquad \frac{\vdash B, B^\perp \qquad \vdash C \parr D, D^\perp \otimes C^\perp}{\vdash B, B^\perp \otimes (C \parr D), D^\perp \otimes C^\perp}\,(\otimes)$$

$$\frac{\vdash A, A^\perp \otimes E^\perp, E \qquad \vdash B, B^\perp \otimes (C \parr D), D^\perp \otimes C^\perp}{\vdash A \otimes B, B^\perp \otimes (C \parr D), D^\perp \otimes C^\perp, A^\perp \otimes E^\perp, E}\,(\otimes)$$

$$\frac{\vdash A \otimes B, B^\perp \otimes (C \parr D), D^\perp \otimes C^\perp, A^\perp \otimes E^\perp, E}{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), D^\perp \otimes C^\perp, A^\perp \otimes E^\perp, E}\,(\parr)$$

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$$\frac{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), (D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp), E \qquad \vdash F^\perp, F}{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), ((D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp)) \otimes F^\perp, F, E}\,(\otimes)$$

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Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^{\perp}}(ax) \quad \dfrac{}{\vdash E^{\perp}, E}(ax)}{\vdash \quad A^{\perp} \otimes E^{\perp}}(\otimes) \quad \dfrac{\dfrac{}{\vdash B, B^{\perp}}(ax) \quad \dfrac{\dfrac{\dfrac{}{\vdash D, D^{\perp}}(ax) \quad \dfrac{}{\vdash C^{\perp}, C}(ax)}{\vdash \quad D^{\perp} \otimes C^{\perp}}(\otimes)}{\vdash C \parr D}(\parr)}{\vdash \quad B^{\perp} \otimes (C \parr D)}(\otimes)}{\vdash A \otimes B}(\otimes)}{\vdash (A \otimes B) \parr (B^{\perp} \otimes (C \parr D))}(\parr)}{\vdash \quad (D^{\perp} \otimes C^{\perp}) \parr (A^{\perp} \otimes E^{\perp})}(\parr) \quad \dfrac{}{\vdash F^{\perp}, F}(ax)}{\vdash \quad ((D^{\perp} \otimes C^{\perp}) \parr (A^{\perp} \otimes E^{\perp})) \otimes F^{\perp}}(\otimes)}{\vdash \quad F \parr E}(\parr)
$$

Lacunar Sequent Calculus

Lacunar Rules

$$\frac{}{\vdash A, A^{\perp}}\ (ax) \qquad \frac{\vdash \Gamma}{\vdash A \parr B}\ (\parr) \qquad \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash A \otimes B}\ (\otimes)$$

Lemma

A lacunar proof maps to at most one proof.

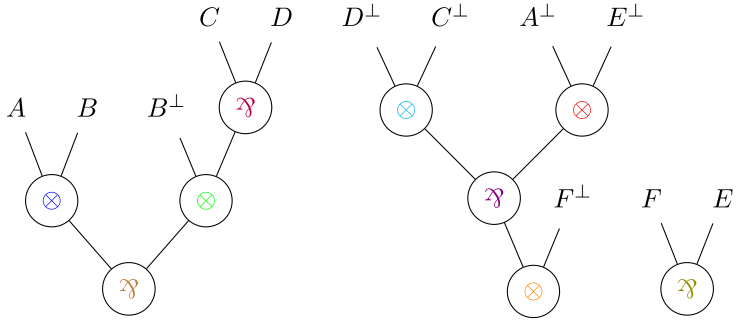
[exactly one if careful about linear use of occurrences]

Proof.

Simple induction: propagate information down. □

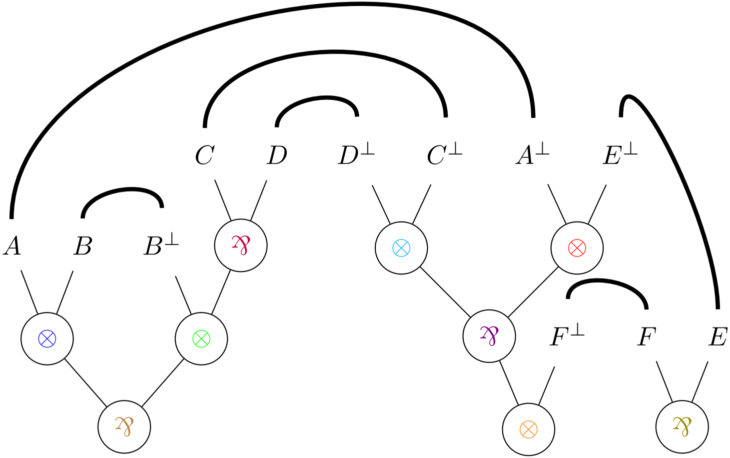
Formula Trees

Rely on: **logical rules** $\longleftrightarrow$ **connectives**



Formula Trees

Rely on: **logical rules** $\longleftrightarrow$ **connectives**



Formula Trees

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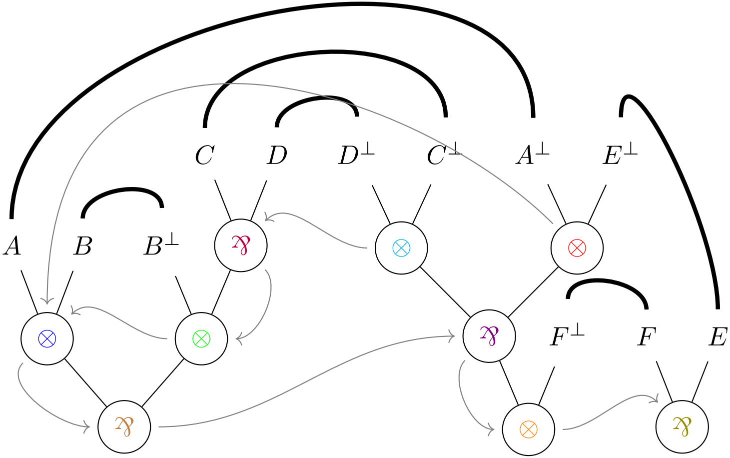

Information Encoding

$$
\begin{prooftree}
\AxiomC{}\RightLabel{(ax)}\UnaryInfC{$\vdash A, A^\perp$}
\AxiomC{}\RightLabel{(ax)}\UnaryInfC{$\vdash E^\perp, E$}
\RightLabel{$(\otimes)$}\BinaryInfC{$\vdash A, A^\perp \otimes E^\perp, E$}
\AxiomC{}\RightLabel{(ax)}\UnaryInfC{$\vdash B, B^\perp$}
\AxiomC{}\RightLabel{(ax)}\UnaryInfC{$\vdash D, D^\perp$}
\AxiomC{}\RightLabel{(ax)}\UnaryInfC{$\vdash C^\perp, C$}
\RightLabel{$(\otimes)$}\BinaryInfC{$\vdash C, D, D^\perp \otimes C^\perp$}
\RightLabel{$(\parr)$}\UnaryInfC{$\vdash C \parr D, D^\perp \otimes C^\perp$}
\RightLabel{$(\otimes)$}\BinaryInfC{$\vdash B, B^\perp \otimes (C \parr D), D^\perp \otimes C^\perp$}
\RightLabel{$(\otimes)$}\BinaryInfC{$\vdash A \otimes B, B^\perp \otimes (C \parr D), D^\perp \otimes C^\perp, A^\perp \otimes E^\perp, E$}
\RightLabel{$(\parr)$}\UnaryInfC{$\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), D^\perp \otimes C^\perp, A^\perp \otimes E^\perp, E$}
\RightLabel{$(\parr)$}\UnaryInfC{$\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), (D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp), E$}
\AxiomC{}\RightLabel{(ax)}\UnaryInfC{$\vdash F^\perp, F$}
\RightLabel{$(\otimes)$}\BinaryInfC{$\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), ((D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp)) \otimes F^\perp, F, E$}
\RightLabel{$(\parr)$}\UnaryInfC{$\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), ((D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp)) \otimes F^\perp, F \parr E$}
\end{prooftree}
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\begin{prooftree}
\AxiomC{}\RightLabel{(ax)}\UnaryInfC{$\vdash A, A^\perp$}
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\RightLabel{$(\otimes)$}\BinaryInfC{$\vdash \quad A^\perp \otimes E^\perp$}
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\AxiomC{}\RightLabel{(ax)}\UnaryInfC{$\vdash F^\perp, F$}
\RightLabel{$(\otimes)$}\BinaryInfC{$\vdash \quad ((D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp)) \otimes F^\perp$}
\RightLabel{$(\parr)$}\UnaryInfC{$\vdash \quad F \parr E$}
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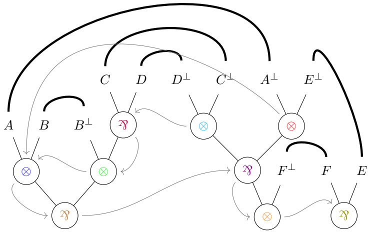
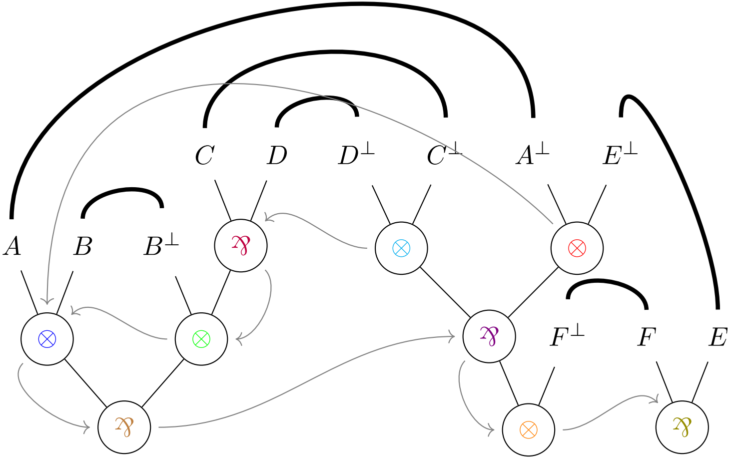
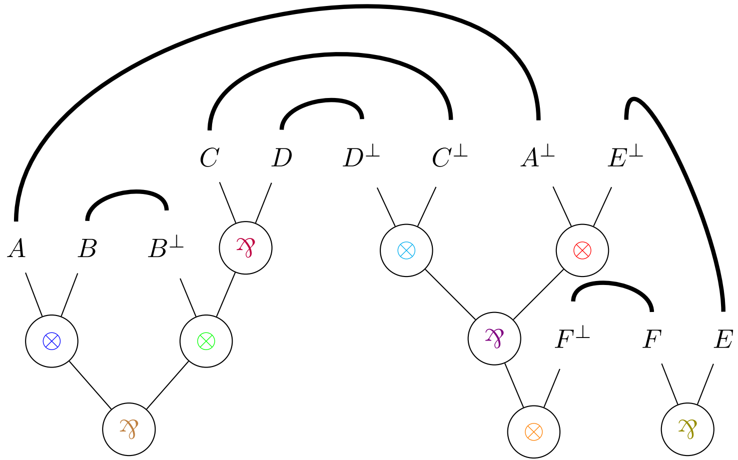


Less Sequentiality



Less Sequentiality



Multiple Proofs

$$\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), ((D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp)) \otimes F^\perp, F \parr E$$

Multiple Proofs

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$$\dfrac{\vdash A, A^\perp \otimes E^\perp, E \qquad \vdash B, B^\perp \otimes (C \parr D), D^\perp \otimes C^\perp}{\vdash A \otimes B, B^\perp \otimes (C \parr D), D^\perp \otimes C^\perp, A^\perp \otimes E^\perp, E}\,(\otimes)$$

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$$\dfrac{}{\vdash F^\perp, F}\,(ax)$$

$$\dfrac{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), (D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp), E \qquad \vdash F^\perp, F}{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), ((D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp)) \otimes F^\perp, F, E}\,(\otimes)$$

$$\dfrac{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), ((D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp)) \otimes F^\perp, F, E}{\vdash (A \otimes B) \parr (B^\perp \otimes (C \parr D)), ((D^\perp \otimes C^\perp) \parr (A^\perp \otimes E^\perp)) \otimes F^\perp, F \parr E}\,(\parr)$$

Multiple Proofs

$$\dfrac{\dfrac{}{\vdash A^\perp, A}\,(ax) \qquad \dfrac{}{\vdash B, B^\perp}\,(ax)}{\vdash A \otimes B, B^\perp, A^\perp}\,(\otimes)$$

$$\dfrac{\vdash A \otimes B, B^\perp, A^\perp \qquad \dfrac{}{\vdash E^\perp, E}\,(ax)}{\vdash A \otimes B, B^\perp, A^\perp \otimes E^\perp, E}\,(\otimes)$$

$$\dfrac{\dfrac{}{\vdash D, D^\perp}\,(ax) \qquad \dfrac{}{\vdash C^\perp, C}\,(ax)}{\vdash C, D, D^\perp \otimes C^\perp}\,(\otimes)$$

$$\dfrac{\vdash C, D, D^\perp \otimes C^\perp}{\vdash C \parr D, D^\perp \otimes C^\perp}\,(\parr)$$

$$\dfrac{\vdash A \otimes B, B^\perp, A^\perp \otimes E^\perp, E \qquad \vdash C \parr D, D^\perp \otimes C^\perp}{\vdash A \otimes B, B^\perp \otimes (C \parr D), D^\perp \otimes C^\perp, A^\perp \otimes E^\perp, E}\,(\otimes)$$

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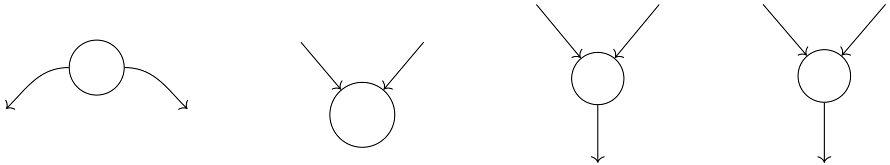
A Graphical Syntax

Proof Structures

Definition

- Directed acyclic graph (dag)

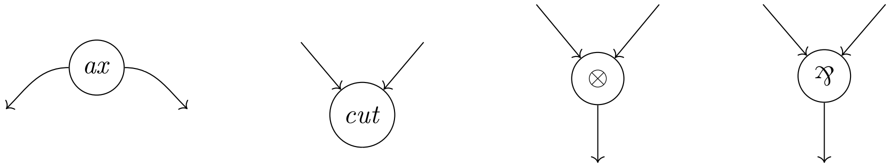


Proof Structures

Rely on: **rule $\leftrightarrow$ node** rather than connective $\leftrightarrow$ node

Definition

- Directed acyclic graph (dag)
- Nodes labelled with rules names: ax, cut, $\otimes$, $⅋$

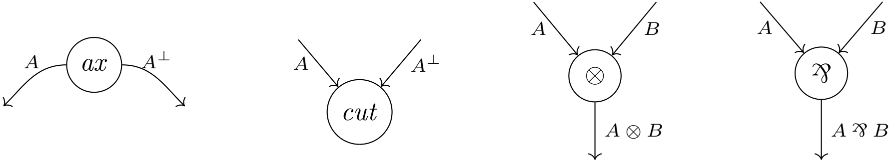


Proof Structures

Rely on: **rule ↔ node** rather than connective ↔ node

Definition

- Directed acyclic graph (dag)
- Nodes labelled with rules names: ax, cut, $\otimes$, $\parr$
- Edges labelled with formulas

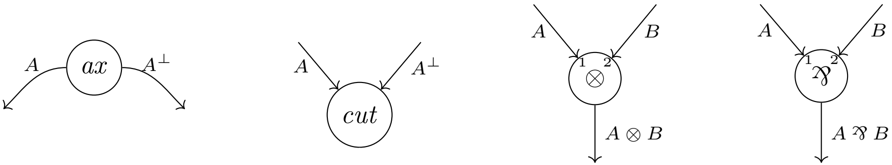


Proof Structures

Rely on: **rule ↔ node** rather than connective ↔ node

Definition

- Directed acyclic graph (dag)
- Nodes labelled with rules names: ax, cut, $\otimes$, $⅋$
- Edges labelled with formulas
- Order on premisses of $\otimes$ and $⅋$ nodes

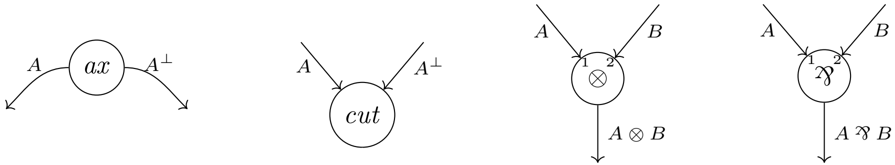


Proof Structures

Rely on: **rule ↔ node** rather than connective ↔ node

Definition

- Directed acyclic graph (dag)
- Nodes labelled with rules names: ax, cut, $\otimes$, $⅋$
- Edges labelled with formulas
- Order on premisses of $\otimes$ and $⅋$ nodes
- Local conditions on arities and typing

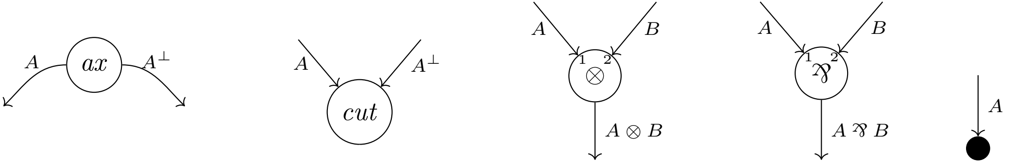


Proof Structures

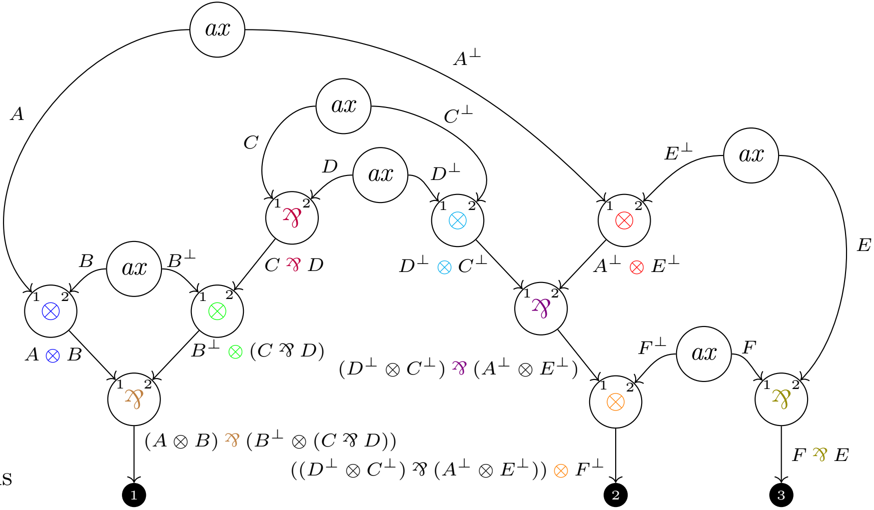
Rely on: **rule $\leftrightarrow$ node** rather than connective $\leftrightarrow$ node

Definition

- Directed acyclic graph (dag)
- Nodes labelled with rules names: ax, cut, $\otimes$, $\parr$
  + conclusion nodes
- Edges labelled with formulas
- Order on premisses of $\otimes$ and $\parr$ nodes
- Local conditions on arities and typing
- Total order on conclusion nodes



Example



dag

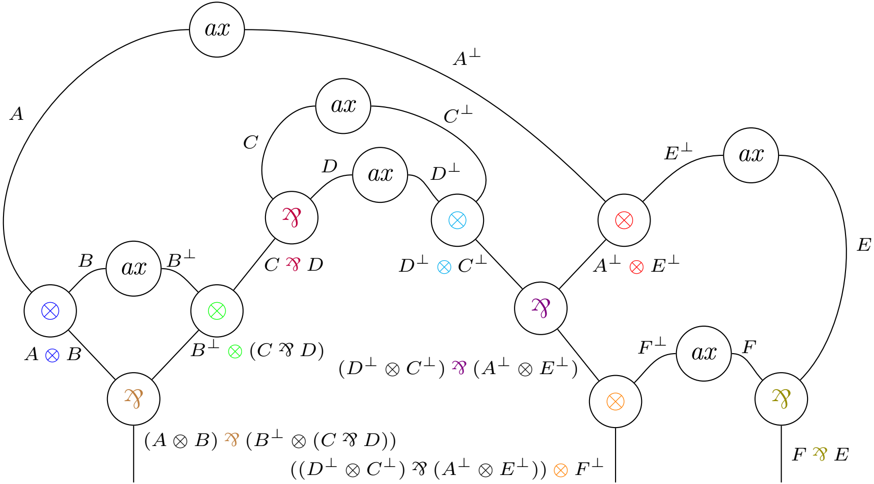
edges: formulas

nodes: ax, cut, $\otimes$, $⅋$

order tag on $\otimes$, $⅋$

ordered conclusions

Example



dag

edges: formulas

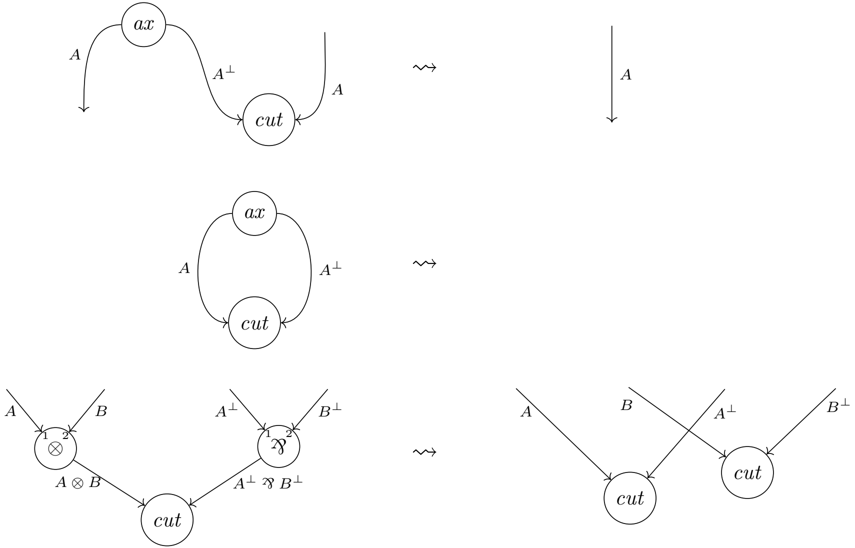
nodes: ax, cut, $\otimes$, $\parr$

Axiom Expansion

Properties

- **Termination**: size of *ax* formulas decreases
- **Confluence**: no critical pair
- Admissibility of axiom rule from atomic axiom

Cut Reduction



Cut Reduction

Properties

- **Exhaustiveness**: any *cut* is reducible
- **Termination**: number of nodes decreases
- **Confluence**: local confluence + Newman's lemma

Cut Reduction

Properties

- **Exhaustiveness**: any *cut* is reducible
- **Termination**: number of nodes decreases
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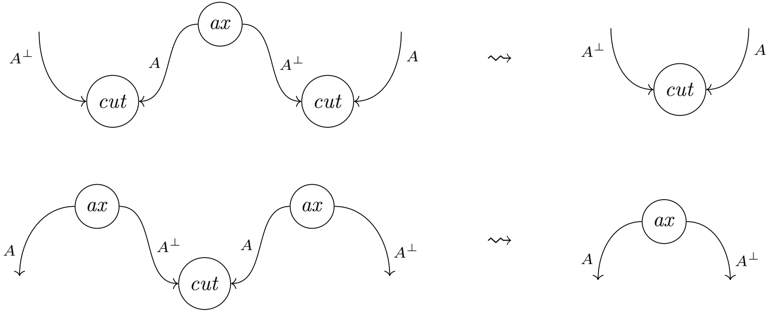

Reducing Cuts and Expanding Axioms Together

Properties

- **Termination**: sum of sizes of formulas on ax and cut decreases
- **Confluence**: local confluence + Newman's lemma

$\Longrightarrow$ **unique canonical form**

Reducing Cuts and Expanding Axioms Together

Properties

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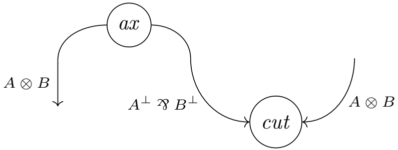
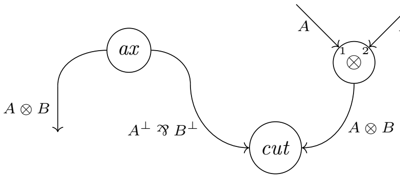
  $\Longrightarrow$ **unique canonical form**



Properties

- **Termination**: sum of sizes of formulas on *ax* and *cut* decreases
- **Confluence**: local confluence + Newman's lemma

$\Longrightarrow$ **unique canonical form**



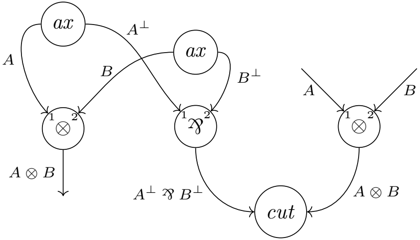
Reducing Cuts and Expanding Axioms Together

4 Rewrite Rules

Properties

- **Termination**: sum of sizes of formulas on *ax* and *cut* decreases
- **Confluence**: local confluence + Newman's lemma

$\Longrightarrow$ **unique canonical form**



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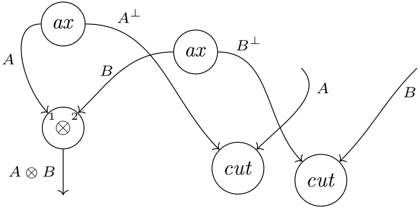

Reducing Cuts and Expanding Axioms Together

Properties

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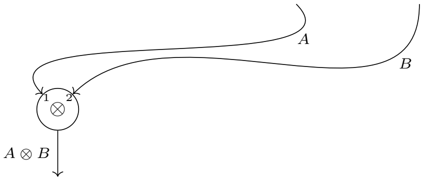

Cut Admissibility in the Sequent Calculus

Proof Sketch

- start from a proof π with cuts
- translate π into proof structure $\mathcal{S}$
- reduce all cuts in $\mathcal{S}$
- we obtain a cut-free proof

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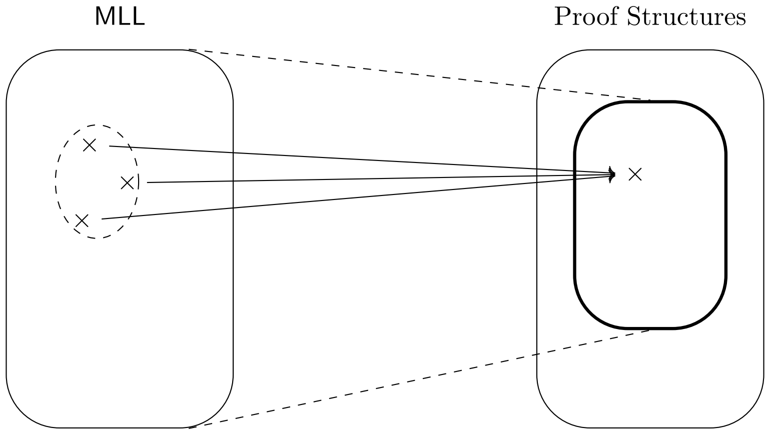
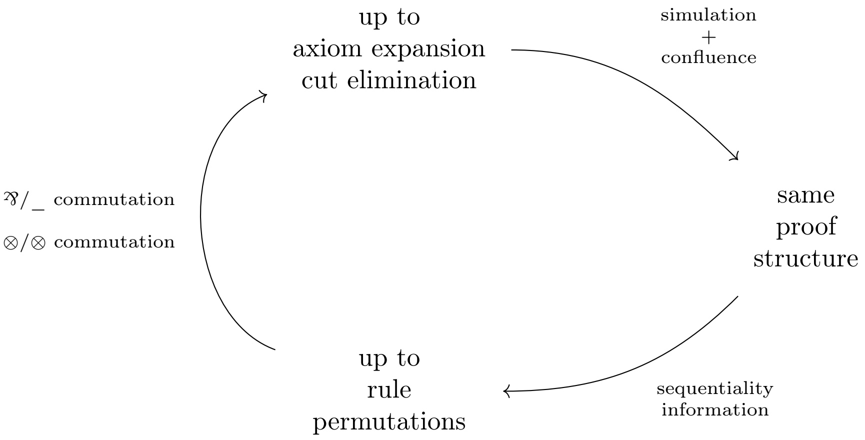
NO!!!

Sequent Calculus and Proof Structures

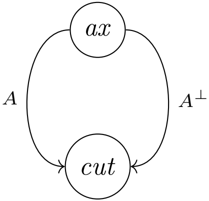
From Sequent Calculus to Proof Structures



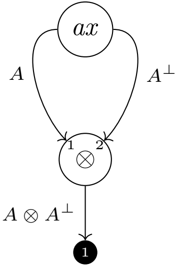
Proof Equalities on Normal Forms



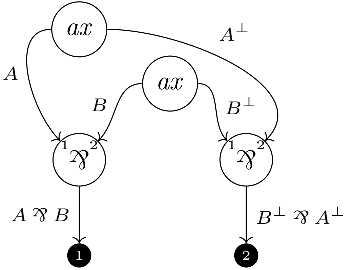
Surjectivity



$\vdash$



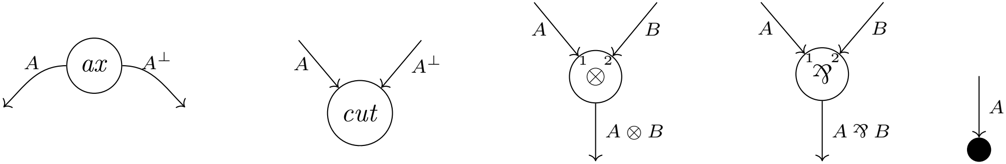
$\vdash A \otimes A^\perp$



$\vdash A \parr B, B^\perp \parr A^\perp$

Correctness

Danos-Regnier criterion



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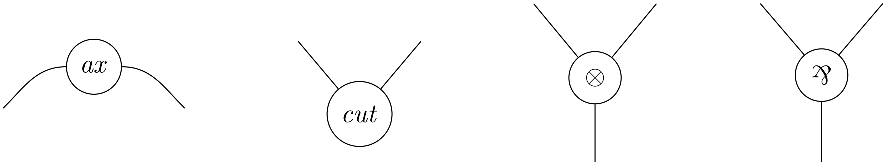

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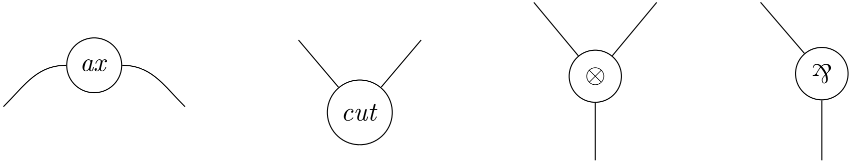
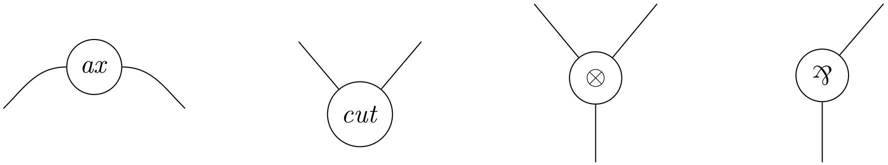


Proof Net = Correct Proof Structure

Switching graphs (*i.e.* induced non-directed graphs) are **acyclic and connected**.

Correctness

Danos-Regnier criterion



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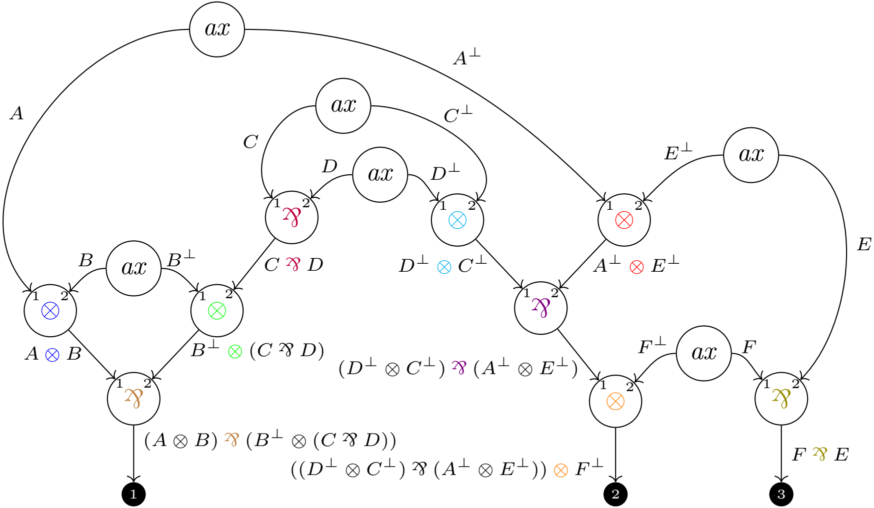
Example



ax
A
A^⊥
ax
C
C^⊥
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ax
D^⊥
E^⊥
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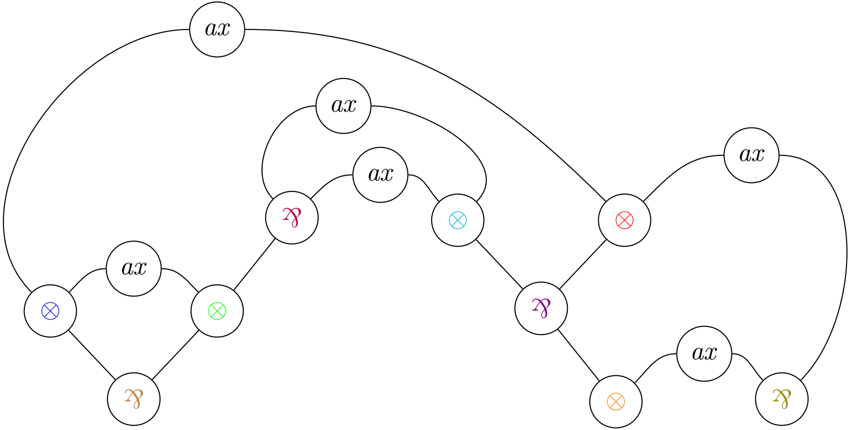
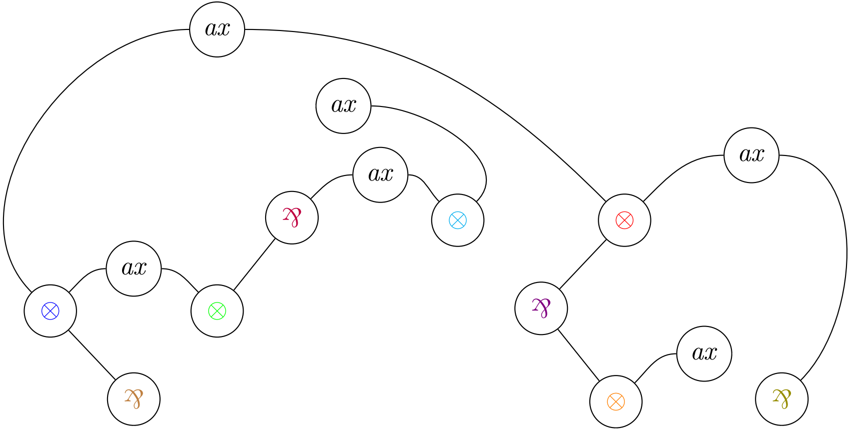
1 ⊗ 2
1 ⊗ 2
B
ax
B^⊥
C ⅋ D
D^⊥ ⊗ C^⊥
A^⊥ ⊗ E^⊥
1 ⊗ 2
1 ⊗ 2
1 ⅋ 2
A ⊗ B
B^⊥ ⊗ (C ⅋ D)
(D^⊥ ⊗ C^⊥) ⅋ (A^⊥ ⊗ E^⊥)
F^⊥
ax
F
1 ⅋ 2
1 ⊗ 2
1 ⅋ 2
(A ⊗ B) ⅋ (B^⊥ ⊗ (C ⅋ D))
((D^⊥ ⊗ C^⊥) ⅋ (A^⊥ ⊗ E^⊥)) ⊗ F^⊥
F ⅋ E
1
2
3

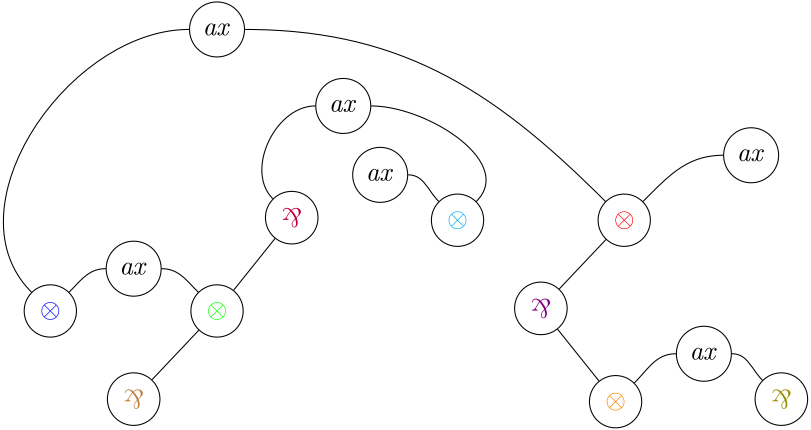
Example

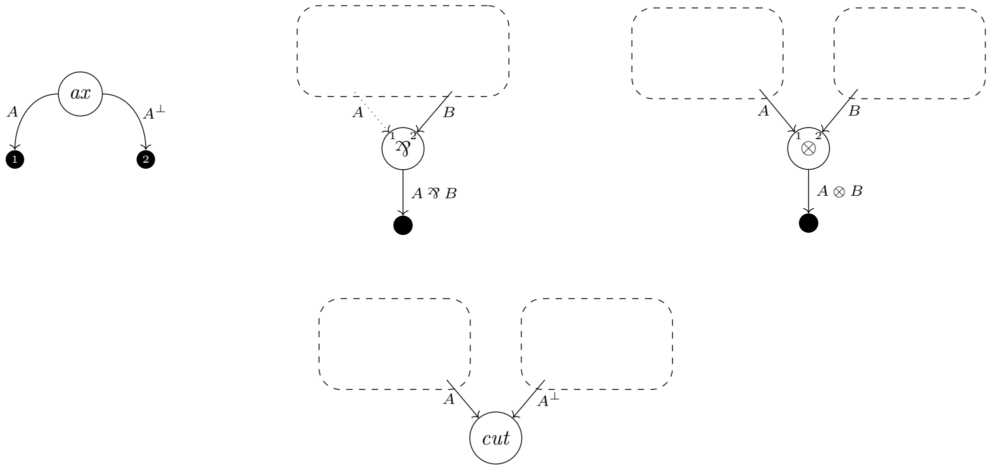


Example



Example





ax
A
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A ⊗ B
A
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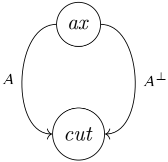
cut

Cut Elimination in Proof Nets

- No cyclic redex



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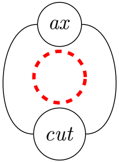

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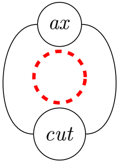
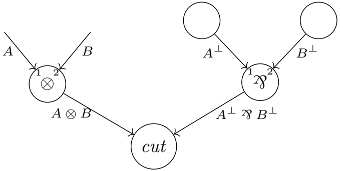


- Preservation of correctness by reduction



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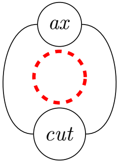

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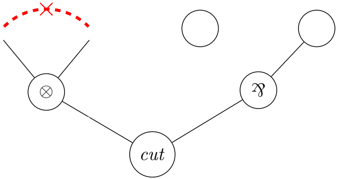

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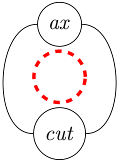

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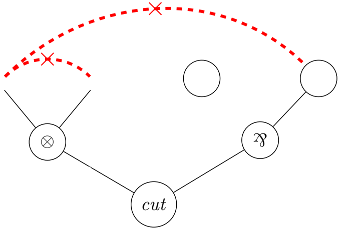

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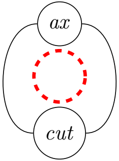

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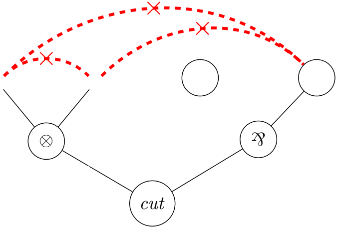

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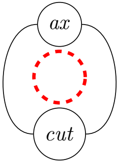

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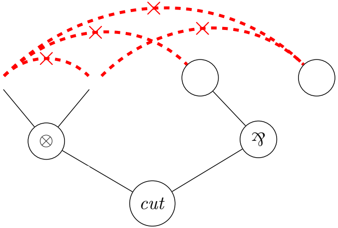

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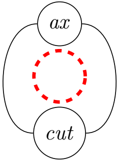

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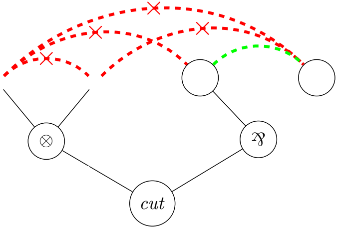

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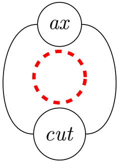

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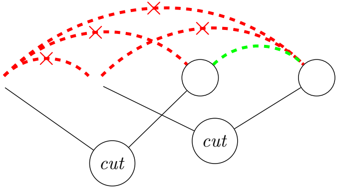

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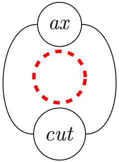

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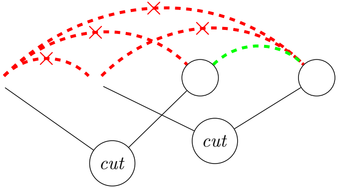

- Cut admissibility in sequent calculus: still a missing block: **sequentialization**

Sequentialization

Cut-Free Case

By induction on the size of the proof net.

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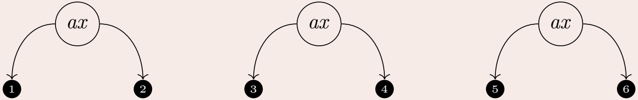
Analysis of Conclusions

- *ax* only



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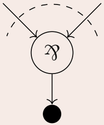
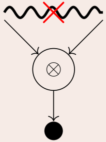


By induction on the size of the proof net.

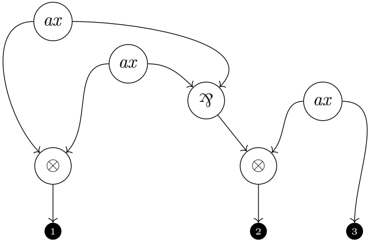
Analysis of Conclusions

- ax only
- at least one $⅋$
- no $⅋$ and at least one $\otimes$

$\Longrightarrow$ looking for:



Splitting Tensor



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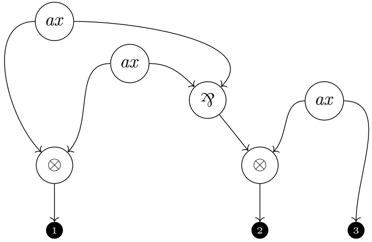
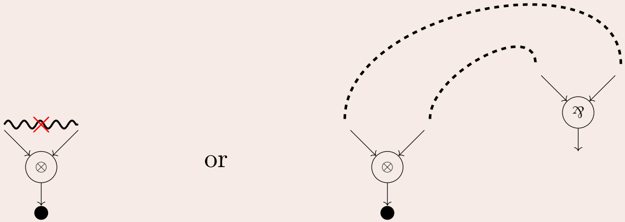
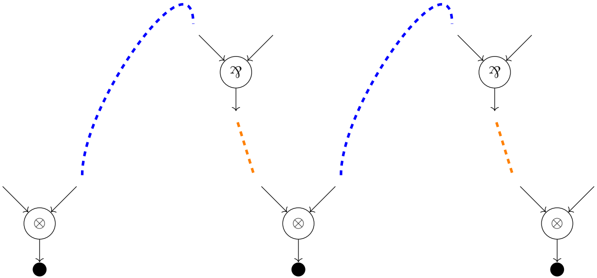


Conclusion Tensor



Finding a Splitting Tensor



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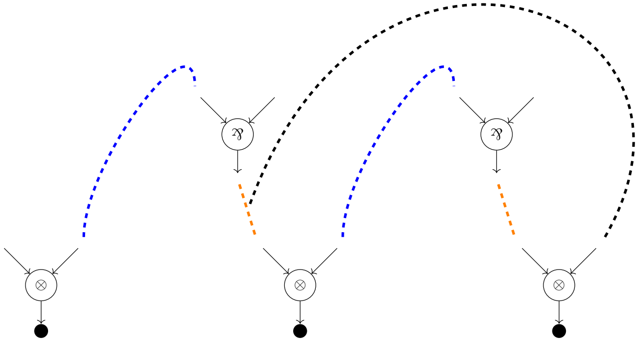

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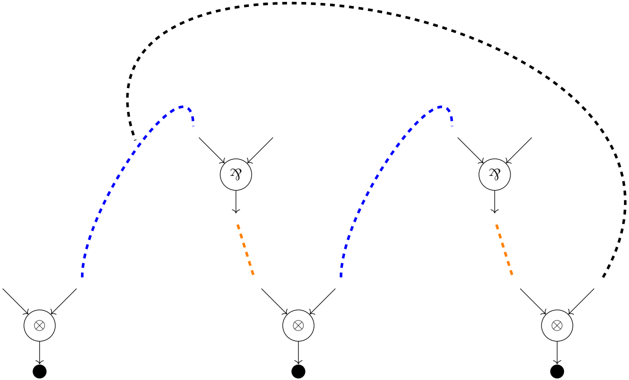

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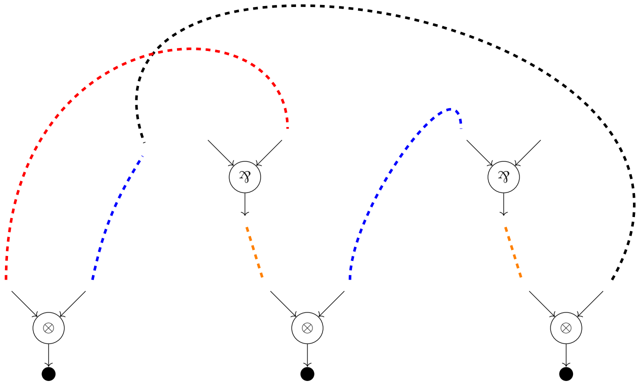
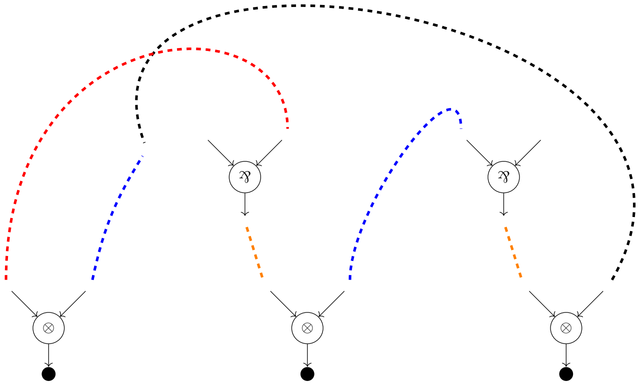


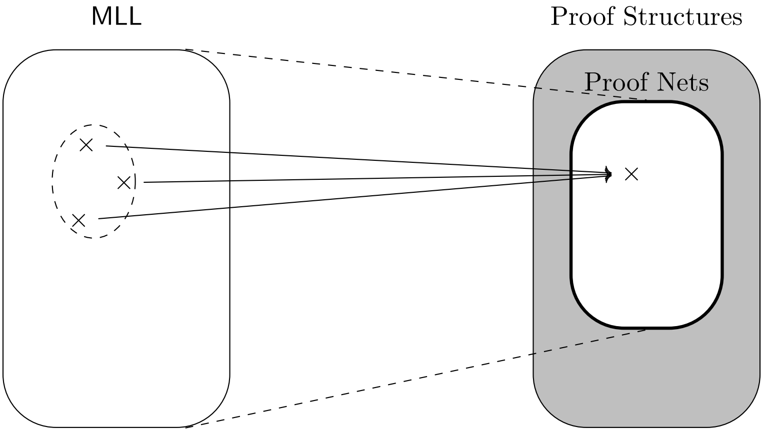
Finding a Splitting Tensor



Theorem (Sequentialization)

Any proof net is the image of a (not necessarily unique) sequent calculus proof.

The Big Picture



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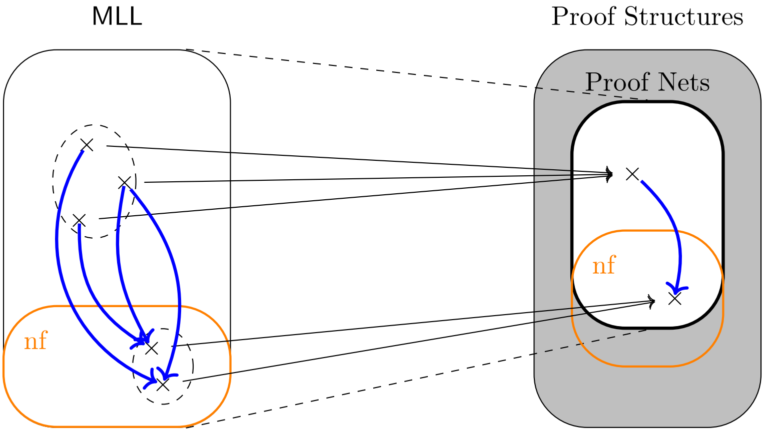

Conclusion

What is a proof?

- Sequent calculus
- **Proof net**: graph with correctness condition (linear time checking)

When are two proofs equal?

- Rules commutation
- Axiom expansion and cut elimination
- **Same proof net (normal form)**

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Extensions

units / exponentials / additives: lost paradise

- control of connectedness
- boxes *vs* equality of proofs