Rel

The relationnal model of Linear Logic.

1) What is denotational semantics ?
2) MRel, a model of λ-calculus : CCC
3) Rel, a model of linear logic : SMCC + Exponential
4) Models stemming from Rel.

Bibligraphy:

* T. Ehrhard teaching notes on his web page. 2022
* Girard - Linear Logic 1987
* Girard - Linear Logic its syntax, its semantics. 1995
* Amadio Curien
* Not enough points is enough - Manzonetto, Bucciarelli, Ehrhard.

Why Rel ?

① many models stem from Rel
Fin, Köthe, Coh, CVS, PCoh, ...

② the interpretation of proofs are the same even if the models allow to underline different properties or constructions :
determinism, probabilities, finite non determinism

1) What is denotational semantics ?

A formal way of giving a meaning to types or formulae and to proofs or programs and to give an account of computation.

An invariant of calculus.

Theorem (correction). mandatory

LL: if $\pi : \Gamma \vdash A \rightsquigarrow \pi' : \Gamma \vdash A$ by cut elimination
then $[\![\pi]\!] = [\![\pi']\!]$

λ-calculus: if $\Gamma \vdash t : A$ et $t \rightsquigarrow t'$ by beta reduction
then $[\![t]\!] = [\![t']\!]$

Purpose:

1) study property of programs independantly of their syntax

2) Distinguish 2 programs / proofs that are not computationaly equivalent:

$$[\![s]\!] \neq [\![t]\!] \implies s \not\simeq_\beta t \quad \text{by contrapositive}$$

3) a guide for introducing new syntactic constructions.

ex: linear logic is more granular than λ-calculus: $\boxed{A \Rightarrow B \simeq\ !A \multimap B.}$

ex: fin spaces and Rill, stable functions, coherent spaces and LL ...

$\boxed{\text{Linear logic}}$ (Intuitionistic)

Formula : $A, B ::= 1 \mid A \otimes B \mid !A \mid A \multimap B$

Proofs : Two-sided version with only one formula on the right
$\Gamma ::= 1 \mid A, \Gamma$

$$\frac{}{A \vdash A}\,Ax \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}\,cut$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \frac{}{\vdash 1} \qquad \frac{\Gamma \vdash C}{\Gamma, 1 \vdash C}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}$$

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\,prom \qquad \frac{\Gamma, A \vdash C}{\Gamma, !A \vdash C}\,der \qquad \frac{\Gamma, !A \otimes !A \vdash C}{\Gamma, !A \vdash C}\,contr \qquad \frac{\Gamma \vdash C}{\Gamma, !A \vdash C}\,weak$$

Rk: ① you can derive one sided proof system by adding negation and $A \multimap B = A^{\perp} \parr B$.

② Additive part is beyond this introduction.

Semantics: of LL in Rel

- For any formula $[A]$ an object of Rel.
- For any proof : $[\pi : \Gamma \vdash A] : [\Gamma] \longrightarrow [A]$
  a morphism of Rel.

such that if $\pi \rightsquigarrow \pi'$ by cut-elimination
then : $[\pi] = [\pi']$ are the same morphisms.

Lambda - Calculus

Types : $A, B ::= T \mid A \& B \mid A \Rightarrow B$

Terms : $s, t ::= x \mid \lambda x.s \mid (s)t \mid$ if s then t else u $\mid$ fix s ...

Typing System: $\Gamma = x_1 : A_1, \dots, x_n : A_n$

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma, x : A \vdash s : B}{\Gamma \vdash \lambda x.s : A \Rightarrow B} \qquad \frac{\Gamma \vdash s : A \Rightarrow B \quad \Gamma \vdash t : A}{\Gamma \vdash (s)t : B}$$

Computation: $(\lambda x.s)t \rightarrow s[x := t]$ every occurence of x in s is substituted by t.
β-reduction

Rk: ① Cartesian product can be encoded through CPS

$A \times B = (A \Rightarrow B \Rightarrow C) \Rightarrow C$

$\pi_1 : \lambda k.(k)(\lambda x \lambda y\, x) : A \times B \Rightarrow A$

$\pi_2 : \quad \dots \quad : A \times B \Rightarrow B$

$\langle s, t\rangle = \lambda z\, (z)\, s\, t \; : A \times B$

$(\pi_1)\langle s, t\rangle \xrightarrow[\beta]{} s$

$(\pi_2)\langle s, t\rangle \xrightarrow[\beta]{} t$

② Cartesian product can be added with constants $()$ and π_i

$$\frac{}{\Gamma \vdash () : T} \qquad \frac{\Gamma \vdash s : A \quad \Gamma \vdash t : B}{\Gamma \vdash \langle s, t\rangle : A \times B} \qquad \frac{\Gamma \vdash s : A \times B}{\Gamma \vdash \pi_i s : A}$$

Semantics: of λ-calculus in $\mathbb{R}$Rel

- For any type $[\![A]\!]$ an object of $\mathbb{R}$Rel.
- For any term : $[\![\Gamma \vdash t : A]\!] : [\![\Gamma]\!] \rightarrow [\![A]\!]$ a morphism of $\mathbb{R}$Rel.

such that if $t \rightsquigarrow t'$ by β-reduction then : $[\![t]\!] = [\![t']\!]$ are the same morphisms.

From $\mathcal{M}$Rel a model of λ-calculus
to Rel a model of Linear Logic

Finite Multisets $\mathcal{M}_{fin}(X)$ where X is a set

Def ① a finite multiset is a function $m : X \to \mathbb{N}$ with finite support: $\{ a \mid m(a) \neq 0 \}$ finite

empty multiset $[\,]$: $a \mapsto 0$

$m = [a, a, a, b]$ $a \mapsto 3$, $b \mapsto 1$

Rk the order does not matter!

② $\forall m, n \in \mathcal{M}_{fin}(X)$, $m + n : X \to \mathbb{N}$, $a \mapsto m(a) + n(a)$

$[a, a, b] + [a, b, c] = [a, a, a, b, b, c]$

Prop ③ $\mathcal{M}_{fin}(X \uplus Y) \simeq \mathcal{M}_{fin} X \times \mathcal{M}_{fin} Y$

$[(1, x_1), \dots, (1, x_k), (2, y_1), \dots, (2, y_\ell)] \mapsto ([x_1, \dots, x_k], [y_1, \dots, y_\ell])$

Relation $X \overset{R}{\nrightarrow} Y$

R : relation de X dans Y $R \subseteq X \times Y$ $(R \in \mathcal{P}(X \times Y))$

Composition of relations. $X \overset{R}{\nrightarrow} Y \overset{R'}{\nrightarrow} Z$

$x \rightsquigarrow \exists y \rightsquigarrow z$

$(x, z) \in R' \circ R$ iff $\exists y \in Y$ such that $(x, y) \in R$ and $(y, z) \in R'$

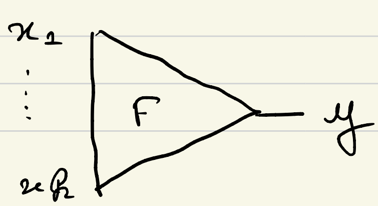
Multirelation $X \xrightarrow{F} Y$

Definition

a multirelation F is a relation from $\mathcal{M}_{fin} X$ to Y.

Representation:

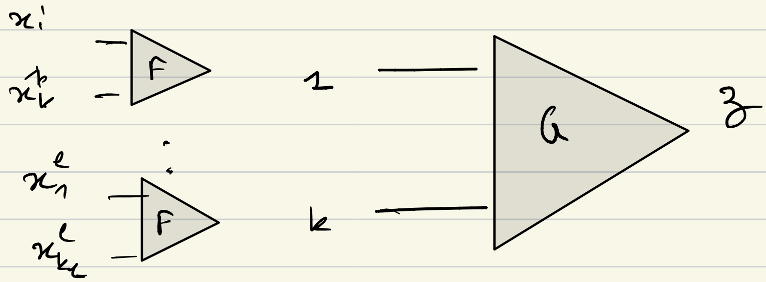
coming from operads



means $([x_1, \dots, x_k], y) \in F$

composition

$$X \xrightarrow{F} Y \xrightarrow{G} Z$$



$\forall m \in \mathcal{M}_{fin} X,\ z \in Z,\ (m, z) \in G \circ F$ iff

$\exists k\ \exists\ y_1 \ldots y_k \in \mathcal{M}_{fin} Y.$

$\exists\ m_1 \ldots m_k \in \mathcal{M}_{fin} X$

$m = m_1 + \ldots + m_k$

$\forall i, (m_i, b_i) \in F \qquad ([y_1, \ldots, y_k], z) \in G.$

$\boxed{\mathcal{M}Rel}$ a model of λ-calculus.

Category objects : Set X, Y

morphisms : $X \to Y$ multirelation

Cartesian : terminal object : $\top = \emptyset$

$A \times B$ product : $X \& Y = X \uplus Y$ Disjoint union of Sets.

$$\begin{tikzcd}
 & & X \\
Z \arrow[urr, bend left, "F"] \arrow[r, "\langle F, G\rangle"] \arrow[drr, bend right, "G"] & X \& Y \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"] & \\
 & & Y
\end{tikzcd}$$

$\pi_1 \in \mathcal{M}_{fin}(X \uplus Y) \times X$
$\cong \mathcal{M}_{fin} X \times \mathcal{M}_{fin} Y \times X$

$\pi_1 = \{([(1,x)], x) \mid x \in X\} = \{([x],[\,], x) \mid x \in X\}$

$\langle F, G\rangle \subseteq \mathcal{M}_{fin} Z \times (X \& Y)$

$\langle F, G\rangle = \{(m, (1,x)) \mid (m,x) \in F\} \cup \{(m, (2,y)) \mid m, y \in G\}$

Closed : Morphism object : $X \Rightarrow Y = M_{fin}(X) \times Y$

$A \Rightarrow B$

Rk : $MRel(X, Y) \simeq \mathcal{P}(M_{fin}(X) \times Y)$

$F : Z \,\&\, X \longrightarrow Y$ $\qquad$ $\Lambda F : Z \longrightarrow X \Rightarrow Y$

$F \subseteq M_{fin}(Z \uplus X) \times Y$ $\qquad$ $\Lambda F \subseteq M_{fin}(Z) \times (M_{fin}\, X \times Y)$

$\Lambda F = \{([z_1, \dots, z_k], [x_1, \dots, x_\ell], y) \mid ([(1, z_1), \dots, (1, z_k), (2, x_1), \dots, (2, x_\ell)], y) \in F\}$

$Ev : \quad X \Rightarrow Y \,\&\, X \longrightarrow Y.$

$Ev \subseteq M_{fin}(M_{fin}(X) \times Y \uplus X) \times Y$

$M_{fin}(M_{fin}(X)) \times Y) \times M_{fin}(X) \times Y$

$Ev = \{(M, m, y) \mid M = [(m, y)]\}$

Rk : MRel is large enough to interpret a programming language.

- $[\![Bool]\!] = \{t, f\}$

$$\frac{\Gamma \vdash u : Bool \qquad \Gamma \vdash s : A \qquad \Gamma \vdash t : A}{\Gamma \vdash \text{if } u \text{ then } s \text{ else } t : A}$$

$$\Gamma \longrightarrow \Gamma \times \Gamma \xrightarrow{u \times \Gamma} Bool \times \Gamma \simeq \Gamma \uplus \Gamma \xrightarrow[s \uplus t]{} A$$

$[\![\text{if } u \text{ then } s \text{ else } t]\!] = \{(m, a) \mid \exists\, m_1, m_2 \text{ st } m = m_1 + m_2$
$(m_1, t) \in u \quad (m_2, a) \in s$
or $(m_1, f) \in u \quad (m_2, a) \in t\}$

if tt then s else t $\to$ s
if ff then s else t $\to$ t.

$[\![\lambda x. \text{if } x \text{ then if } x \text{ then tt else ff else if } x \text{ then tt else ff}]\!] =$

$\{([tt, ff], ff), ([ff, tt], tt),$
$([tt, tt], tt), ([ff, ff], ff)\}$

- <u>Fixpoint</u> exists in $\mathcal{M}Rel$ $\quad$ $\mathcal{M}Rel(X, Y)$ ordered by inclusion

directed family of relations have lowest upper bounds.

$$\overline{\vdash Y : (A \Rightarrow A) \Rightarrow A}$$

$[Y] \subseteq \mathcal{M}fin(\mathcal{M}fin A \times A) \times A.$

smallest set such that

$$([([\,], a)], a) \in [Y]$$

$$([([a_1, \dots, a_n], a)] + m_1 + \dots + m_k, a) \in [Y]$$

$\forall k,\ a_1, \dots, a_n, a \in X,\ m_1, \dots, m_k \in \mathcal{M}fin(\mathcal{M}fin X \times X)$

$$(m_i, a_i) \in [Y]$$

- <u>Pure λ-calculus</u> $\quad \mathcal{D} = \mathcal{D} \Rightarrow \mathcal{D}$

$$X \,\&\, (\mathbb{N} \times X) \simeq \mathbb{N} \times X$$

$$(1, x) \mapsto (0, x)$$

$$(2, (n, x)) \mapsto (n+1, x)$$

Fixpoint of $F(X) = \mathcal{M}fin(\mathbb{N} \times X)$ exists in Rel

$$\begin{cases} D_0 = \emptyset \\ D_{n+1} = F(D_n) \end{cases}$$

$$D_\infty = \bigcup D_n$$

$$D_\infty = \mathcal{M}fin(\mathbb{N} \times D_\infty)$$

$$\begin{aligned} D_\infty \Rightarrow D_\infty &= \mathcal{M}fin(D_\infty) \times D_\infty. \\ &= \mathcal{M}fin(D_\infty) \times \mathcal{M}fin(\mathbb{N} \times D_\infty) \\ &= \mathcal{M}fin(D_\infty \,\&\, (\mathbb{N} \times D_\infty)) \\ &= \mathcal{M}fin(\mathbb{N} \times D_\infty) \\ &= D_\infty. \end{aligned}$$

Notice that $\boxed{X \Rightarrow Y = \mathcal{M}_{fin}\, X \times Y.}$

we recognize the same decomposition as in coherent spaces.

- $\mathcal{M}_{fin}$ is a comonad interpreting !
- $X \multimap Y = X \times Y$ so linear multirelations are just relations.

$\boxed{\text{Rel}}$ is a model of linear logic.

<u>Category</u>: <u>objects</u> sets X, Y

<u>morphisms</u> $\text{Rel}(X,Y) = \mathcal{P}(X \times Y)$

Relations $R \subseteq X \times Y$

$Id_X = \{(x,x) \mid x \in X\}$

- <u>symmetric monoidal</u>: $X \otimes Y = X \times Y$ $\quad 1 = \{*\}$

associativity $\quad \alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$

$\alpha_{X,Y,Z} = \{((x,(y,z)),((x,y),z)) \mid x \in X, y \in Y, z \in Z\}$

unit $\quad e : 1 \otimes X \to X \qquad e' : X \otimes 1 \to X$

$e = \{((*,x),x) \mid x \in X\} \qquad e' = \{((x,*),x) \mid x \in X\}$

bifunctor $\quad X \overset{R}{\nrightarrow} X' \quad Y \overset{R'}{\nrightarrow} Y' \qquad X \otimes Y \nrightarrow X' \otimes Y'$

$R \otimes R' = \{((x,y),(x',y')) \mid (x,x') \in R, (y,y') \in R'\}$

- <u>closed</u>: $X \multimap Y = X \times Y$

<u>Rk</u>: $\text{Rel}(X,Y) = \mathcal{P}(X \times Y)$

$F : Z \otimes X \nrightarrow Y \qquad \Lambda F : Z \nrightarrow X \multimap Y$

$F \subseteq (Z \times X) \times Y \qquad \Lambda F \subseteq Z \times (X \times Y)$

$\Lambda F = \{(z,(x,y)) \mid ((z,x),y) \in F\}$

$Ev : X \multimap Y \otimes X \nrightarrow Y.$

$Ev \subseteq (X \times Y) \times X \times Y$

$Ev = \{((x,y),x,y) \mid x \in X, y \in Y\}$

$$\frac{\pi_1 : \Gamma, A \vdash B}{\pi : \Gamma \vdash A \multimap B}$$

$\pi = \Lambda(\Gamma \otimes A \xrightarrow{\pi_1} B)$

$[\![\pi]\!] = \{(\gamma, (a,b)) \mid ((\gamma, a), b) \in \pi_1\}$

$$\frac{\pi_1 : \Gamma \vdash A \qquad \pi_2 : \Delta, B \vdash C}{\pi : \Gamma, \Delta, A \multimap B \vdash C}$$

$\pi = \Gamma \otimes \Delta \otimes A \multimap B \xrightarrow{\pi_2} A \otimes \Delta \otimes A \multimap B \xrightarrow{EV} \Delta \otimes B.$

$[\![\pi]\!] = \{(\gamma, \delta, a, b, c) \mid (\gamma, a) \in \pi_1,$
$(\delta, b, c) \in \pi_2\}$

$$\frac{\pi_1 : \Gamma \vdash A \qquad \pi_2 \Delta, A \vdash C}{\pi : \Gamma, \Delta \vdash C}$$

$\pi = \Gamma \otimes \Delta \xrightarrow{\pi_1 \otimes \Delta} A \otimes \Delta \xrightarrow{\pi_2} C$

$[\![\pi]\!] = \{(\gamma, \delta, c) \mid \exists a (\gamma, a) \in [\![\pi_1]\!] \ (\delta, a, c) \in [\![\pi_2]\!]\}$

$$\frac{\pi_1 : \Gamma, A, B \vdash C}{\pi : \Gamma, A \otimes B \vdash C}$$

$\pi = \Gamma \otimes (A \otimes B) \xrightarrow{\alpha} \Gamma \otimes A \otimes B \xrightarrow{\pi_1} C$

$[\![\pi]\!] = \{(\gamma, (a, b), c) \mid (\gamma, a, b, c) \in [\![\pi_1]\!]\}$

$\pi : \vdash 1$

$[\![\pi]\!] = \{*\}.$

$$\frac{\pi_1 : \Gamma \vdash C}{\pi : \Gamma, 1 \vdash C}$$

$\pi = \Gamma \otimes 1 \xrightarrow{e} \Gamma \xrightarrow{\pi_1} C$

$[\![\pi]\!] = \{(\gamma, *, c) \mid (\gamma, c) \in [\![\pi_1]\!]\}$

$\boxed{\mathcal{M}_{fin}}$ is an exponential modality

* <u>comonad on Rel</u>

<u>Functor</u> $\mathcal{M}_{fin} : X \mapsto \mathcal{M}_{fin} X$

$\mathcal{M}_{fin} R \subseteq \mathcal{M}_{fin} X \times \mathcal{M}_{fin} Y$

$\mathcal{M}_{fin} R = \{([a_1, \dots, a_k], [b_1, \dots, b_k]) \mid \forall i\, (a_i, b_i) \in R\}$

comultiplication (digging) $\delta_X : !X \nrightarrow !!X$

$\delta_A \subseteq \mathcal{M}_{fin} X \times \mathcal{M}_{fin}(\mathcal{M}_{fin} X)$

$\delta_X = \{(m, [m_1, \dots, m_k]) \mid m_1 + \dots + m_k = m\}$

counit (dereliction) $\varepsilon_X : !X \nrightarrow X$

$\varepsilon_X \subseteq \mathcal{M}_{fin} X \times X$

$\varepsilon_X = \{([x], x) \mid x \in X\}$

* <u>comonoid structure</u> (structure).

contraction $c_X : !X \nrightarrow !X \otimes !X$

$c_X \subseteq \mathcal{M}_{fin} X \times \mathcal{M}_{fin} X \times \mathcal{M}_{fin} X$

$c_X = \{(m, m_1, m_2) \mid m = m_1 + m_2\}$

weakening $w_X : !X \nrightarrow 1$

$w_X \subseteq \mathcal{M}_{fin} X \times \{*\}$

$w_X = \{([\,], *)\}$

* <u>Seely Isomorphisms</u> (monoidal strength)

unit: $m_0 : 1 \nrightarrow !\top$

$m_0 \subseteq \{*\} \times \mathcal{M}_{fin}\{\emptyset\}$

$m_0 = \{(*, [\,])\}$

strength: $m_2 : !X \otimes !Y \nrightarrow !(X \& Y)$

$m_2 \subseteq \mathcal{M}_{fin} X \times \mathcal{M}_{fin} Y \times \mathcal{M}_{fin}(X \uplus Y)$

$m_2 = \{([x_1 \dots x_k], [y_1, \dots, y_l]), [(1, x_1), \dots, (1, x_k), (2, y_1), \dots, (2, y_l)]\}$

Rk: m_0 and m_2 are bijections.

$$\frac{\Pi_0 \quad \Gamma, A \vdash \Delta}{\Pi \quad \Gamma, !A \vdash \Delta}$$

$$\Pi : \quad \Gamma \otimes !A \xrightarrow{\Gamma \otimes \varepsilon} \Gamma \otimes A \xrightarrow{\Pi_1} \Delta$$

$$\Pi = \{ (\gamma, [a], \delta) \mid (\gamma, a, \delta) \in \Pi_0 \}$$

$$\frac{\Pi_0 : \; !\Gamma \vdash B}{\Pi : \; !\Gamma \vdash !B}$$

$$\Pi : \; !A_1 \otimes \cdots \otimes !A_n \xrightarrow[m_n]{} !(A_1 \& \cdots \& A_n) \xrightarrow{\delta} !!(A_1 \& \cdots \& A_n) \xrightarrow[!m_m^{-1}]{} !(!A_1 \otimes \cdots \otimes !A_m) \xrightarrow[![\Pi_0]]{} !B$$

$$\Pi = \{ ((m_1, \dots, m_m), [b_1, \dots, b_k]) \mid \exists\, m_1^1, \dots, m_1^k \text{ st } m_1 = m_1^1 + \dots + m_1^k$$
$$\dots$$
$$m_m^1, \dots, m_m^k \text{ st } m_m = m_m^1 + \dots + m_m^k$$
$$\forall i \quad (m_1^i, \dots, m_m^i), b_i \in \Pi_0 \}$$

$$\frac{\Pi_0 \quad \Gamma \vdash \Delta}{\Pi \quad \Gamma, !A \vdash \Delta}$$

$$\Pi : \quad \Gamma \otimes !A \xrightarrow{\Gamma \otimes \Pi_0} \Delta \otimes 1 \xrightarrow{e} \Delta$$

$$\Pi = \{ (\gamma, [\,], \delta) \mid (\gamma, \delta) \in \Pi_0 \}$$

$$\frac{\Pi_0 : \; \Gamma, !A, !A \vdash \Delta}{\Pi : \; \Gamma, !A \vdash \Delta}$$

$$\Pi : \quad \Gamma \otimes !A \xrightarrow{\Gamma \otimes c} \Gamma \otimes !A \otimes !A \xrightarrow{\Pi_0} \Delta$$

$$\Pi = \{ (\gamma, m, \delta) \mid \exists\, m_1, m_2 \text{ s.t } m = m_1 + m_2 \quad (\gamma, m_1, m_2, \delta) \in \Pi_0 \}$$

Rel and MRel LNL model.

$$\mathrm{Rel}_! = \begin{tikzcd} \mathrm{MRel}, \&, \top \arrow[rr, bend left] \arrow[rr, phantom, "\top"] & & \mathrm{Rel}, \otimes, 1 \arrow[ll, bend left] \end{tikzcd}$$

$$X \longrightarrow \mathcal{M}_{fin}\, X$$

$$R \subseteq \mathcal{M}_{fin} X \times Y \qquad R^{?} \subseteq \mathcal{M}_{fin} X \times \mathcal{M}_{fin} Y$$

$$x \longleftarrow X$$

$$\{([x], y) \mid x, y \in R\} \subseteq \mathcal{M}_{fin} X \times Y \qquad R \subseteq X \times Y$$

Classical LL

- $\bot = 1 = \{*\}$ $\qquad X^{\bot} = X \multimap \bot$ $\qquad X^{\bot\bot} = (X \multimap \bot) \multimap \bot$

$$= X \times \{*\} \times \{*\}$$
$$\simeq X.$$

$$\frac{\pi_0 : \Gamma, A \vdash \Delta}{\pi : \Gamma \vdash A^{\bot}, \Delta} \qquad \pi = \{(\gamma, (a, \delta)) \mid ((\gamma, a), \delta) \in \pi_0\}$$

Rk: deduce the interpretation of monolater sequents.

$$A \oplus B = (A^{\bot} \& B^{\bot})^{\bot} \qquad [A \oplus B] = [A] \uplus [B]$$

$$\pi \subseteq \Gamma \times (A \uplus B)$$

$$\frac{\pi_0 \vdash \Gamma, A}{\pi \vdash \Gamma, A \oplus B} \qquad \pi = \{(\gamma, (1, a)) \mid (\gamma, a) \in \pi_0\}$$

$$\pi \subseteq (A \uplus B) \times \Delta$$

$$\frac{\pi_1 : \vdash A, \Delta \qquad \pi_2 \vdash B, \Delta}{\pi \vdash A \& B, \Delta} \qquad \pi = \{((1, a), \delta) \mid (a, \delta) \in \pi_1\} \cup \{((2, b), \delta) \mid (b, \delta) \in \pi_2\}$$

Rel is the basic ingredient

Biorthogonality

Bibliography.

Π proof of MALL then $[\Pi]$ the same in Coh, Fin, Rel, PCoh
of MELL PCoh, Fin, Rel.
Non coh^t spaces.

Principle: object: $|A|$, $\mathcal{C}_A$ where
- $|A|$ is a set
- orthogonality relation on $\mathcal{C}$: $x \perp y$
- $\mathcal{C}^\perp = \{x' \mid \forall x \in \mathcal{C},\ x' \perp x\}$

morphisms: $R : |A| \to |B|$ that "preserves" $\mathcal{C}$
R is a relation.

Example: $\boxed{\text{Coherent spaces:}}$

obj: $|A|$, $\mathcal{C}_A \subseteq \mathcal{P}(|A|)$ $\quad x \perp y$ iff $\#(x \cap y) \leq 1$

morph: $R \subseteq |A| \times |B|$ s.t $\forall x \in \mathcal{C}_A$, $R \cdot x = \{b \mid \exists a \in x\ (a,b) \in R\} \in \mathcal{C}_B$.
$\forall y' \in \mathcal{C}_B^\perp$ $\quad R^\perp \cdot y' = \{a \mid \exists b \in y'\ (a,b) \in R\} \in \mathcal{C}_A^\perp$
$\mathcal{C}_{A \multimap B} = R \subseteq A \times B$.

Determinism. / For exponential, use the finite multiclique trick... Set of finite multicliques. Finite multisets whose support are cliques.

$\boxed{\text{Finiteness spaces}}$

obj: $|A|$, $\mathcal{F}_A \subseteq \mathcal{P}(|A|)$ $\quad x \perp y$ iff $x \cap y$ finite

morph: $R \subseteq |A| \times |B|$ st $R \cdot \mathcal{F}_A \subseteq \mathcal{F}_B$
$R^\perp \cdot \mathcal{F}_B^\perp \subseteq \mathcal{F}_A^\perp$

- Finite non determinism
- This model enjoys a linearized version = induces a vector space model
- This model denotes diff λ-calculus and DiLL

$\boxed{\text{Probabilistic Coherent Spaces}}$

objects: $|A|$, $\mathcal{P}_A \subseteq \mathbb{R}_+^{|A|}$ $\quad x, y \in \mathbb{R}_+^{|A|}$ $\quad x \perp y$ iff $\langle x,y\rangle = \sum_a x_a y_a \leq 1$

morph: $R \subseteq \mathbb{R}_+^{|A| \times |B|}$ st $\forall x \in \mathcal{P}_A$, $R \cdot x \in \mathcal{P}_B$.

- Discrete probability $[\text{coin}] = \left(\frac{1}{2}, \frac{1}{2}\right) \in \mathcal{P}_{1 \oplus 1}$
- Fixpoints.
- Conditional.

} Discrete probabilistic language.

Exercice

Compute the semantics of

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\vdash 1$}
\UnaryInfC{$\vdash \bot, \bot, 1$}
\UnaryInfC{$\vdash \bot, \bot, 1$}
\RightLabel{$\oplus R$}
\UnaryInfC{$\vdash \bot, \bot, 1 \oplus 1$}
\AxiomC{}
\UnaryInfC{$\vdash 1$}
\UnaryInfC{$\vdash \bot, \bot, 1$}
\UnaryInfC{$\vdash \bot, \bot, 1$}
\RightLabel{$\oplus R$}
\UnaryInfC{$\vdash \bot, \bot, 1 \oplus 1$}
\BinaryInfC{$\vdash \bot, \bot \& \bot, 1 \oplus 1$}
\AxiomC{}
\LeftLabel{$\bot \times 2$}
\UnaryInfC{$\vdash 1$}
\UnaryInfC{$\vdash \bot, \bot, 1$}
\UnaryInfC{$\vdash \bot, \bot, 1$}
\RightLabel{$\oplus R$}
\UnaryInfC{$\vdash \bot, \bot, 1 \oplus 1$}
\AxiomC{}
\RightLabel{$\bot \times 2$}
\UnaryInfC{$\vdash 1$}
\UnaryInfC{$\vdash \bot, \bot, 1$}
\UnaryInfC{$\vdash \bot, \bot, 1$}
\RightLabel{$\oplus R$}
\UnaryInfC{$\vdash \bot, \bot, 1 \oplus 1$}
\BinaryInfC{$\vdash \bot, \bot \& \bot, 1 \oplus 1$}
\RightLabel{$\&$}
\BinaryInfC{$\vdash \bot \& \bot, \bot \& \bot, 1 \oplus 1$}
\RightLabel{der $\times 2$}
\UnaryInfC{$\vdash ?(\bot \& \bot), ?(\bot \& \bot), 1 \oplus 1$}
\RightLabel{contr}
\UnaryInfC{$\vdash ?(\bot \& \bot), 1 \oplus 1$}
\end{prooftree}
$$

compare with the semantics of:

$$
\begin{aligned}
\vdash \lambda x^{\mathbb{B}}.\ & \text{if } x \text{ then if } x \text{ then } tt \\
& \qquad\qquad\qquad\qquad \text{else } ff \\
& \qquad \text{else if } x \text{ then } tt \\
& \qquad\qquad\qquad \text{else } ff
\end{aligned}
\quad : \mathbb{Bo}
$$