Equidistribution of weakly special subvarieties, o-minimality and the geometric André–Oort conjecture
Joint work with Rodolphe Richard

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CIRM, O-minimality and foliations

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- $\mathbf{H} \simeq \mathbf{R} \rtimes \mathbf{F}_1 \times \mathbf{F}_2 \times \ldots \mathbf{F}_r$ with $\mathbf{R}$ unipotent, $\mathbf{F}_\alpha$ $\mathbb{Q}$-simple and $\mathbf{F}_\alpha(\mathbb{R})$ is not compact.

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- In this situation $\Gamma_H := \Gamma \cap H$ is lattice in $\mathbf{H}$.

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- Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of elements of G and and μ_{H_n,g_n} be the translate by g_n of μ_{H_n}. $\mathrm{Supp}(\mu_{H_n,g_n}) = \Gamma_{H_n}\backslash H_n.g_n \subset S$

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- **Main Question**: What can be said about the convergence of μ_{H_n,g_n}?

Ratner theory and Mozes-Shah theorem

Let $\mathbf{H} \subset \mathbf{G}$ be of type $\mathcal{H}$, $g \in G$ and $\gamma \in \Gamma$. Then $\mu_{H,g} = \mu_{\gamma H \gamma^{-1}, \gamma g}$. We fix a fundamental set $\mathcal{F}$ for the action of Γ on G. We say that the writing of $\mu_{H,g}$ is normalised if $g \in \mathcal{F}$.

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Theorem (Mozes-Shah)

Let $(\mu_{H_n,g_n})_{n\in\mathbb{N}}$ be a sequence of homogeneous measures in a normalised writing.
(i) If g_n varies in a bounded subset of $\mathcal{F}$, then up to passing to a subsequence, $g_n \to g_\infty$ and there exists a subgroup $\mathbf{H}_\infty$ of type $\mathcal{H}$ such that $\mathbf{H}_n \subset \mathbf{H}_\infty$ for all n and such that

$$\mu_{H_n,g_n} \longrightarrow \mu_{H_\infty,g_\infty}.$$

(ii) Let $\overline{S} = S \cup \{\infty\}$ be the one point compactification of S. Let μ_∞ a weak limit of μ_{H_n,g_n}. Then $\mu_\infty = \delta_\infty$ or $\mu_\infty = \mu_{H_\infty,g_\infty}$ as in (i).

Main result

Let $S = \Gamma\backslash G$, $\mu_n := \mu_{H_n,g_n}$ weakly converging to $\mu_\infty = \mu_{H_\infty,g_\infty}$ with $\mathbf{H}_n \subset \mathbf{H}_\infty$ and $g_n \to g_\infty$. Let $\theta : G \longrightarrow S = \Gamma\backslash G$ be the projection map.

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Theorem (Richard-U)

Let $\widehat{V} \subset S$ be a mesurable set and $U \subset G$ be **bounded** *and* **definable**. *Assume that*

- $\liminf \mu_n(\widehat{V} \cap \theta(U)) > 0$ *(in particular $\widehat{V} \cap U$ contains an open subset of $\Gamma\backslash\Gamma H_n g_n$)*
- $\theta^{-1}(\widehat{V}) \cap U \subset G$ *is* **definable**.

Then $\widehat{V} \cap \theta(U)$ contains a non empty open set of

$$\Gamma\backslash\Gamma H_\infty g_n = \Gamma_{H_\infty}\backslash H_\infty g_n \subset S.$$

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locally symmetric spaces, Shimura varieties, Period Domain

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- All the previous results apply to S_M. When $M = \{1\}$, $S_M = S$.
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- If G/K_∞ is hermitian, S_{K_∞} is a quasi-projective algebraic variety (Baily-Borel). (**Shimura Case**).

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- When $M = K_\infty$, S_M is a locally symmetric space.
- If G/K_∞ is hermitian, S_{K_∞} is a quasi-projective algebraic variety (Baily-Borel). (**Shimura Case**).
- If V is a quasi projective variety supporting a $\mathbb{Z}$-variation of Hodge structure, there is an associated holomorphic period map

$$\psi : V \longrightarrow \Gamma\backslash G/M$$

for some associated, (G, Γ, M). (**Period domain case**)

Shimura case

When $S = \Gamma\backslash G/K_\infty$ is a Shimura variety. A weakly special subvariety Z of S is a complex totally geodesic subvariety of S. Moreover

$$\Gamma_H\backslash HgK_\infty/K_\infty = \Gamma_H\backslash X_H \to Z \subset S$$

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for some semi-simple subgroup $\mathbf{H} \subset \mathbf{G}$ of type $\mathcal{H}$, such that $X_H = H.gK_\infty/K_\infty$ is hermitian symmetric. The main theorem implies the "geometric part of the André-Oort conjecture" that was proven before using o-minimality and functional transcendence (Ax-Lindemann):

Theorem

Let V be an algebraic subvariety of S containing a Zariski dense set of weakly special subvarieties. Then V is weakly special or there is a special subvariety

$$\psi : \Gamma_{H_1}\backslash X_{H_1} \times \Gamma_{H_2}\backslash X_{H_2} \longrightarrow S$$

and an algebraic subvariety W of $\Gamma_{H_2}\backslash X_{H_2}$ such that

$$V = \psi(\Gamma_{H_1}\backslash X_{H_1} \times W) \subset S.$$

sketch of proof of the "geometric part of André-Oort"

Let $Z_n = \Gamma\backslash\Gamma H_n g_n K_\infty/K_\infty$ be a sequence of maximal weakly special subvariety of V. Assume for simplicity that the sequence of measures μ_{Z_n} has no "loss of mass". We may also assume that V is not contained in any proper special subvariety.

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- By Mozes-Shah, up to passing to a sub-sequence we may assume that μ_{H_n,g_n} weakly converges to μ_{H_∞,g_∞} for some subgroup $\mathbf{H}_\infty$ of type $\mathcal{H}$ containing all the $\mathbf{H}_n$.

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- Then using that $\pi : \mathcal{F} \longrightarrow S$ is definable in $\mathbb{R}^{an,\exp}$ and the main theorem we deduce that that

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- In general Z'_n is not weakly special: $\mathbf{H}_\infty$ may have a unipotent part, $H_\infty.g_n.K_\infty$ could be not a symmetric space or a non hermitian symmetric space.

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- If the mass of μ_{Z_n} escape at ∞, we use some recent results of Daw-Gorodnik-U to adapt the proof.

The period domain case

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- By Cattani, Deligne, Kaplan if Z is weakly special, analytic components of $\psi^{-1}(Z)$ are algebraic subvarieties of V.
- We define the **horizontal weakly special positive dimensional locus** $WS^h(V)$ of V to be the union of all the positive dimensional analytic components of $\psi^{-1}(Z)$ for Z varying among horizontal weakly special subvarieties of S_M.

The period domain case

Theorem (Chen-Richard-U)

Let V be a quasi-projective variety supporting a $\mathbb{Z}$-variation of Hodge structures. Assume that the horizontal weakly special positive dimensional locus $WS^h(V)$ is Zariski dense in V. Then the period map $\psi : V \longrightarrow \Gamma\backslash G/M$ factorises:

$$S \xrightarrow{(\psi_1,\psi_2)} \Gamma_1\backslash G_1/M_1 \times \Gamma_2\backslash G_2/M_2 \xrightarrow{\varphi} \Gamma\backslash G/M,$$

where $\Gamma_1\backslash G_1/M_1$ is a Shimura variety, φ is a finite Hodge morphism, ψ_1 is a dominant algebraic morphism and $\psi(V)$ is dense in

$$\varphi(\Gamma_1\backslash G_1/M_1 + \times\psi_2(V)).$$

Remark

Assume that μ_{H_n,g_n} converges to μ_{H_∞,g_∞}. Let $Z_n := \mathrm{Supp}(\mu_{H_n,g_n})$ and $Z_\infty := \mathrm{Supp}(\mu_{H_\infty,g_\infty})$. Then $\dim(Z_\infty) \geq \dim(Z_n)$ with equality iff for n big enough, $\mathbf{H}_n = \mathbf{H}_\infty$.

Remark

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- In homogeneous dynamics the dimension grows at the limit.
- In o-minimal families the dimension decreases at the limit.

Definition

Let $B \subset \mathbb{R}^r$ be a definable subset (in some fixed o-minimal theory). A family of subsets $(A_b)_{b\in B}$ with A_b a definable subset of $\mathbb{R}^s$ is a **definable family** if

$$\tilde{A} := \cup_{b\in B}(\{b\} \times A_b) \subset \mathbb{R}^r \times \mathbb{R}^s$$

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It's well known that there are some uniformity results for definable families. For exemple if A_b is finite for all $b \in B$ then $|A_b|$ is uniformly bounded.

- Let $K \subset \mathbb{R}^n$ be a compact and $\mathcal{K}(K)$ the set of compact subsets of K
- Then $\mathcal{K}(K)$, endowed with the Hausdorff distance $\delta_H(\ , \ )$ is sequentially compact.

$$\delta_H(F_1, F_2) = \max\{\sup_{y \in F_2} d(y, F_1), \sup_{x \in F_1} d(x, F_2)\}$$

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Proposition

Let $(A_b)_{b \in B}$ a definable family with $A_b \subset K \subset \mathbb{R}^s$ for all $b \in B \subset \mathbb{R}^r$. Let $(b_i)_{i \in \mathbb{N}}$ be a sequence in B. We assume that we have a Hausdorff limit L for the sequence $(\overline{A_{b_i}})_{i \in \mathbb{N}}$ of $\mathcal{K}(K)$. Then L is a **definable** *compact subset of K and*

$$\dim(L) \leq \liminf_{i \in \mathbb{N}} \dim(A_{b_i}).$$

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- *For all bounded subsets U and V of G, there exists a finite subset$F$ of Γ such that*

$$\forall g \in V, (\Gamma H g) \cap U = (F H g) \cap U.$$

- *If moreover U and V are semi-algebraic, the family*

$$((\Gamma H g) \cap U)_{g \in V}$$

is semi-algebraic.

main technical tool in o-minimality and Hausdorff convergence

Theorem (Richard-U)

Let $n \in \mathbb{N}$, *let* $K \subseteq \mathbb{R}^n$ *be a compact and* $(\nu_i)_{i\in\mathbb{N}}$ *a sequence of finite measures with support in* K *such that* $\lim_{i\in\mathbb{N}} \nu_i = \nu_\infty$ *is a finite measure,*

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- $\forall i \in \mathbb{N}, \mathrm{Supp}(\nu_i) \subset A(b_i)$
- $\liminf_{i\in\mathbb{N}} \nu_i(A') > 0.$

Then for n *big enough,* A' *contains a non empty open subset of* $A(b_i)$*.*

Sketch of proof of the main thm

Let $S = \Gamma\backslash G$, $\mu_n := \mu_{H_n,g_n}$ weakly converging to $\mu_\infty = \mu_{H_\infty,g_\infty}$ with $\mathbf{H}_n \subset \mathbf{H}_\infty$ and $g_n \to g_\infty$. Let $\theta : G \longrightarrow S = \Gamma\backslash G$ be the projection map. $\widehat{V} \subset S$ measurable, $U \subset G$ bounded and definable. We assume that

- $\liminf \mu_n(\widehat{V} \cap \theta(U)) > 0$ (in particular $\widehat{V} \cap U$ contains an open subset of $\Gamma\backslash\Gamma H_n g_n$)
- $\theta^{-1}(\widehat{V}) \cap U$ is definable.

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- $\liminf \mu_n(\widehat{V} \cap \theta(U)) > 0$ (in particular $\widehat{V} \cap U$ contains an open subset of $\Gamma\backslash\Gamma H_n g_n$)
- $\theta^{-1}(\widehat{V}) \cap U$ is definable.

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- $B = U$, $(A_b)_{b\in B} = (\Gamma H_\infty b \cap U)_{b\in U}$ which is a definable family (using reduction theory).

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Conclusion $\widehat{V} \cap U$ contains an open subset of $\Gamma\backslash\Gamma H_\infty g_n$. If $\widehat{V}$ is moreover analytic then $\Gamma\backslash\Gamma H_\infty g_n \subset \widehat{V}$.

THANKS FOR YOUR ATTENTION!