Preparation Theorem in o-minimal structures. Episode III.

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Volumes and densities of subanalytic sets.

Theorem (Lion-Rolin '97, Comte-Lion-Rolin '00)

Consider globally subanalytic $X \subset \mathbb{R}^m \times \mathbb{R}^n$ as a family $X_t \subset \mathbb{R}^n$, parameterized by $t \in \mathbb{R}^m$, s.t. $\dim X_t \leq k$ for all t. Then the set of t where $Vol_k(X_t) < \infty$ is globally subanalytic and on this set

$$Vol_k(X_t) = P(A_1(t), \dots, A_r(t), \log A_1(t), \dots, \log A_r(t)),$$

where A_i are globally subanalytic and P is a polynomial.

$$\begin{tikzcd} \mathbb{R}^m \times \mathbb{R}^n \arrow[d] & \supset X \arrow[dl] \\ \mathbb{R}^m & \ni t = (t_1, \dots, t_m) \end{tikzcd}$$

$t \rightsquigarrow$ Hausdorff k-volume $Vol_k(X_t)$

Volumes and densities of subanalytic sets.

Corollary

Let $X \subset \mathbb{R}^n$, $\dim X \le k$ be subanalytic global. Then the local k-density of X is of the form

$$\Theta_k X(x) = P(A_1(x), \ldots, A_r(x), \log A_1(x), \ldots, \log A_r(x)),$$

where A_i are globally subanalytic and P is a polynomial.

$x \in X$. k-density $\Theta_k X(x) = \lim_{\varepsilon \to 0} \frac{\mathrm{Vol}_k(X \cap B(x,\varepsilon))}{\sigma_k \cdot \varepsilon^k}$

This limit always exists Kurdyka-Raby.

$x \longrightarrow \Theta_k X(x)$ is not necessarily subanalytic.

Theorem

Let $f(t,x)$, $t=(t_1,\dots,t_m), x=(x_1,\dots,x_n)$, *be a bounded subanalytic function defined on subanalytic* $X \subset \mathbb{R}^m \times \mathbb{R}^n$. *Suppose that the fibres* $X_t = X \cap (\{t\} \times \mathbb{R}^n)$ *are bounded and of dimension at most* k. *Then the integral with parameter*

$$\underline{\varphi(t) = \int_{X_t} f(t,x)\,\mathrm{dvol}_k}$$

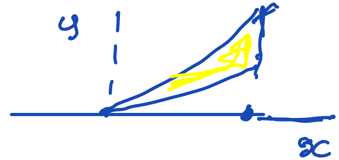
with respect to the k*-dimensional volume is of the form*

$$P(\widetilde{t}_1,\dots,\widetilde{t}_d,\log\widetilde{t}_1,\dots,\log\widetilde{t}_d),$$

where $\widetilde{t}_1,\dots,\widetilde{t}_d$ *are subanalytic functions in* t *and* P *is a real polynomial of degree at most* k *with respect to the logarithms.*

$\mathbb{R}^m \ni t \longrightarrow X_t \subset \mathbb{R}^n$

Idea of the proof



$\eta_1(x) < y < \eta_k(x)$

- Important case $k = n$
- For induction we consider finite sums of $f_D = \prod_{i=1}^{p} \ln f_i(t, x, y)$
  Integrating with respect to y does not change the form but introduces (maybe) one more logarithm.
- prepare f_i, and work on the 'blow-space' in local coordinates. obtained from $\varphi_1, \dots, \varphi_{n-1}(t,x)$, $y_1 = y - \Theta(t,x)$ by a combinatorial argument.
  - $\varphi_1, \dots, \varphi_{n-2}$, $\varphi_+ = y_1/a(t,x)$, $\varphi_- = b(t,x)/y_1$,
  - f_i' are fractional normal crossings in these coordinates.

$$f_i = A(x) \cdot y_1^{\nu} \cdot \mathrm{unit}(\;)$$

$$\ln f_i = \ln A(x) + \nu \ln y_1 + \ln \mathrm{unit}(\;)$$

We can suppose $f = A(t,x) \cdot \underbrace{h(\varphi(t,x,y))}_{\text{fractional power series}} \cdot \prod_{i=1} \ln y_i$

Splitting Lemma of Liou-Rubin.

$g \in R\{u, w_-, w_+\}$ it splits.

$$g(u, w_-, w_+) = g_+(u, w_-w_+, w_+) + w_- g_0(u, w_-w_+) + g_-(u, w_-w_+, w_-)$$

Apply it to h. to get.

$$f = A(t, x)\cdot\left(h_+ + y^{-1} h_0 + h_-\right)\cdot(\ln y)^p$$

- $\int y^{-1} h_0 \ln y^p \, dy$ introduce one extra logarithm.

- in the case of h_+ (or h_-) we integrate with respect to φ_+ (resp φ_-)

Proposition

The asymptotic expansion of $\mathrm{vol}_k(X \cap B(x_0, r))$ *is of the form*

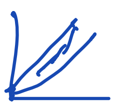
$$\mathrm{vol}_k(X \cap B(x_0, r)) = \sum_{j=0}^{l_0} a_{p_0,j} r^{p_0} (\ln r)^j + \dots, \qquad a_{p_0,l_0} \neq 0,$$

where either $p_0 = k$ *and* $l_0 = 0$ *or* $p_0 > k$ *and then* $l_0 \leq k - 2$.

Example

$X \subset \mathbb{R}^3$; $X = \{ (x,y,z) ; \ 0 \leq x \leq \frac{1}{2}, \ x^2 \leq y \leq x, \ 0 \leq z \leq \underline{x^d/y} \}$, $d \geq 4$

$$\text{volume} \simeq -(d+1)^{-1} \, \underline{r^{d+1} \ln r} + \dots.$$



Remark using nested preparation.

one may show that a compact subanalytic set can be decomposed into cells bi-Lipschitz equivalent to cells given by monomials inequalitie

L-regular cells (open).

A definable cell of dimension d of $\mathbb{R}^{n+1}$ is called ***L*-regular** (with respect to a given system of coordinates) if

- If $d = 0$ then C is a point.
- If $d = n + 1$ then $C = \{(x, y) | \, x \in B, \eta_1(x) < y < \eta_2(x)\}$, where η_i are C^2 definable with bounded first order derivatives and B is L-regular.
- If $d < n + 1$ then C is the graph of $\phi : D \to \mathbb{R}^{n-d+1}$, where ϕ is C^2 definable with bounded first order derivatives, and $D \subset \mathbb{R}^d$ is L-regular.

L-regular decomposition

Theorem (Kurdyka '90, A.P. '88, Kurdyka-'96., Pawłucki '08)

Given finite family Y_i of definable subsets of $\mathbb{R}^{n+1}$ there is a finite decomposition of $\mathbb{R}^{n+1}$ in L-regular cells, each with respect to a suitable linear system of coordinates, s.t. every Y_i is a union of cells.

This theorem holds in every, not necessarily polynomially bounded, o-minimal structure.

Lifting of Lipschitz vector fields.

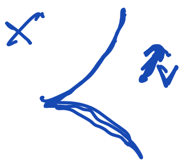
Theorem (Theorem C)

For a function $f(x, y)$ definable in a polynomially bounded o-minimal structure $\exists C$ and a definable stratification of $\mathbb{R}^{n+1}$ such that $\forall$ Lipschitz vector field v on $\mathbb{R}^{n+1}$ with Lipschitz constant L and tangent to strata

$$\left|\frac{\partial f}{\partial \mathrm{v}}(x, y)\right| \leq CL|f(x, y)|.$$

generalization of Theorem A

$$|\mathrm{grad}\, f| \cdot \mathrm{dist}(x, X) \leq C' |f(x,y)|$$



$$L = \frac{1}{\mathrm{dist}(x, X)}$$

$\longrightarrow$ Criterion for lifting of Lipschitz vector fields

Proof use the prepatation theorem.

with extra property: the translation θ

has bounded the first order partial derivatives

$f = A(x) \cdot |y - \theta(x)|^{v}$ unit( ) . show it for each factor.

$$\left| D_{V(x,y)} [y - \theta(x)] \right| = \left| w(x,y) - D_{\tilde{v}(x,y)} \theta \right|$$

$V = (\tilde{v}, w)$

$$\leq \underbrace{\left| w(x,y) - w(x, \theta(x)) \right|}_{\wedge\!\backslash\; L\,|y - \theta(x)|} + \underbrace{\left| \left(D_{\tilde{v}(x,\theta(x))} - D_{\tilde{v}(x,y)} \right) \theta \right|}_{\wedge\!\backslash\; L\,|y - \theta(x)|\,|D\theta(x)|}$$

Similar for the units.

Lipschitz stratification

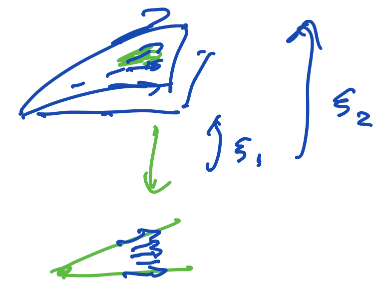
Let $X \subset \mathbb{R}^n$ be subanalytic closed. We say that a filtration

$$X = X^d \supset X^{d-1} \supset \cdots \supset X^l \neq \emptyset,$$

induces a **Lipschitz stratification** of X if

$\mathring{X}^j = X^j \setminus X^{j-1}$ is non-singular [C^2]

and Lipschitz vector fields tangent to strata can be extended from any $X^{j-1} \subset W \subset X^j$ to X, with the control on the Lipschitz constant of the extension.



apply Theorem C to $f = (\xi_1 - \xi_2)$

$|D\xi_1 v - D\xi_2 v| \le C' L |\xi_1 - \xi_2|$

Existence Theorem

Theorem

Any compact definable (in a polynomially bounded o-minimal structure) subset of $\mathbb{R}^n$ admits a definable Lipschitz stratification.

Mostowski's Definition of Lipschitz stratification

Let $P_q : \mathbb{R}^n \to T_q\mathring{X}^j$ denote the orthogonal projection onto the tangent space and $P_q^\perp = I - P_q$ the orthogonal projection onto the normal space $T_q^\perp\mathring{X}^j$. We say that the stratification $\{X^j\}$ satisfies Mostowski's Conditions if there is a constant $C > 0$ such that for all chains $\{q_m\}_{m=1,\dots,r}$ and all $2 \le k \le r$:

$$|P_{q_1}^\perp P_{q_2} \cdots P_{q_k}| \le C|q - q_2| / \operatorname{dist}(q, X^{j_k - 1}).$$

If, further, $q' \in \mathring{X}^j$ and $|q - q'| \le (\frac{1}{2c}) \operatorname{dist}(q, X^{j-1})$ then

$$|(P_q - P_{q'})P_{q_2} \cdots P_{q_k}| \le C|q - q'| / \operatorname{dist}(q, X^{j_k - 1}),$$

in particular,

$$|P_q - P_{q'}| \le C|q - q'| / \operatorname{dist}(q, X^{j_1 - 1}),$$

where $\operatorname{dist}(\cdot, \emptyset) \equiv 1$.