# Preparation Theorem in o-minimal structures. Episode II.

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# Preparation Theorem for subanalytic functions

**Theorem**

*Let $f(x,y)$ be a global subanalytic function on $\mathbb{R}^{n+1} \ni (x_1,\dots,x_n,y)$. Then $f$ is* **reducible.**

A function $f(x,y)$ is **reducible** if on cells of a cellular decomposition $\mathcal{C}$ of $\mathbb{R}^{n+1}$, it is reduced with **translation** $\theta$

$$f(x,y) = A(x)\ |y-\theta(x)|^{\nu}\ U(x,y) = A(x)\ |y_1|^{\nu}\ V(\psi(x,y)),$$

where $A(x)$ and $\theta(x)$ are subanalytic, $\nu \in \mathbb{Q}$, and $V(\psi(x,y))$ is a **reduced unit**, i.e. $\psi(x,y)$ called **reduction morphism** is of the form

$$\psi(x,y) = (\phi_1(x),\dots,\phi_s(x), |y_1|^{1/p}/a(x), b(x)/|y_1|^{1/p}),$$

and $V$ is non-vanishing and real analytic on a neighbourhood of the closure of the image of $\psi$, that is bounded.

# Remarks and additional properties.

- We may suppose that $A(x)$ and $\theta(x)$ are also reducible (nested reducible)


- The cells (cylinders) are of the form: $\{(x,y)|\, x \in B, \eta_1(x) < y < \eta_2(x)\}$ or $\{(x,y)|\, x \in B, y = \eta(x)\}$.
- one may assume that on each cell either $\theta = 0$ or $y$ and $\theta$ are comparable.
- we may reduce simultaneously, with the same translation and reduction morphism, and the same cell decomposition, a finite number of functions.
- The reduced form is far from being unique, we may often replace the translation by a different one.
- . . .

# Strategy of the analytic proof.

1. Theorem is local (on a correct compactification).

2. Suppose $\exists F(x, y, w)$, monic polynomial in $w$, s.t. $F(x, y, f(x)) = 0$, and s.t. the discriminant $\Delta_F(x, y)$ times the last coefficient is regular in $y$. Then the proof follows from a version of Jung's strategy of resolution of singularities.

$$F(x,y,w) = w^{d} + \sum_{1} a_i(x,y)\, w^{d-i} \quad , \quad a_0(x,y)$$

$$a_0(x,y)\,\Delta_F(x,y) = \mathrm{unit}(x,y)\left(y^{d'} + \sum_{1} b_i(x)\, y^{d'-i}\right)$$

3. The general case can by reduced to the above one by the local flattening theorem of Hironaka.

### Theorem (Local flattening theorem in real analytic settings)

*Let $\Phi : X \to W$ be a morphism of real analytic spaces, with $W$ reduced. Fix $p \in W$ and a compact $L \subset \Phi^{-1}(p)$.*

*Then there is a covering family $\sigma_\alpha : U_\alpha \to U$ of a neighborhood $U$ of $p$, where $\sigma_\alpha$ are compositions of local blowings-up with nonsingular analytic centers, such that*

**the stricts transforms** $\Phi_\alpha : X_\alpha \to U_\alpha$ *of (complexifications of) $\Phi$ by (complexifications of) $\sigma_\alpha$ are* **flat** *at every point of $X_\alpha$ corresponding to $L$.*

Flat morphism $\simeq$ 'weak algebraic triviality'

$/\mathbb{C}$ flat $\Rightarrow$ equidimensional fibers

Example

$D(x,y) = \sum_i a_i(x) y^i$, s.t. $a_i(0) = 0 \quad \forall i$

$\mathbb{C}^{n+1} \supset X = \{ D = 0 \}$

$\downarrow$

$x \in \mathbb{C}^n$

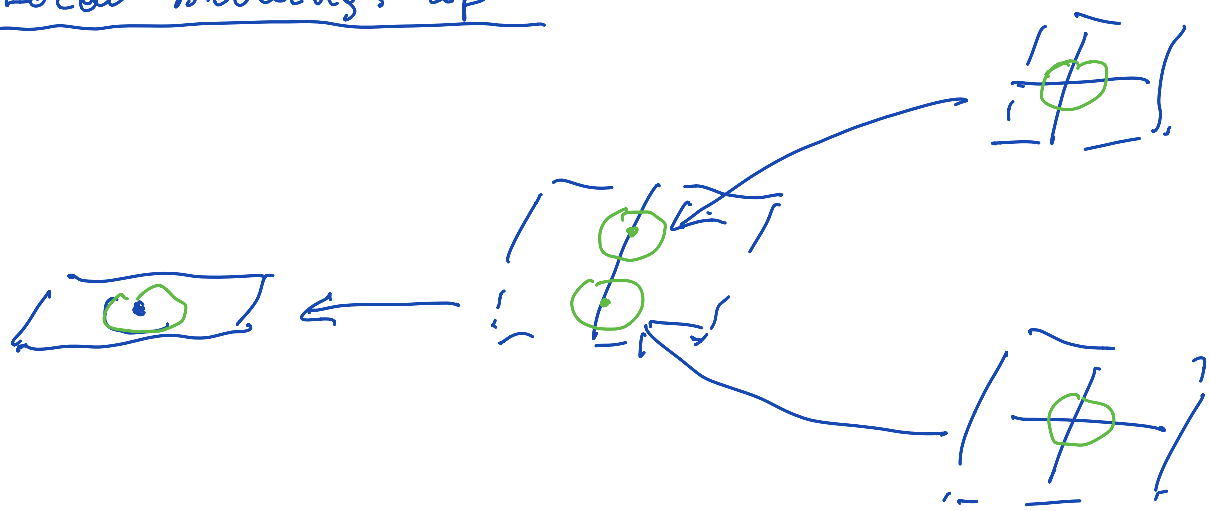
Blow-up the ideal $\mathcal{J} = (a_i)_{i \in N}$

$\sigma : \widetilde{M} \to \mathbb{C}^n$ , $\sigma^*(\mathcal{J})$ is locally principal

on $\widetilde{M}$ in local coordinates $\tilde{x}$ , $x = \sigma(\tilde{x})$

$$D(\sigma(\tilde{x}), y) = \underbrace{a_{i_0}(\sigma(\tilde{x}))}_{\text{exceptional divisor.}} \cdot \left( \sum_{i=1}^{\infty} \underbrace{a_i / a_{i_0}}_{a_{i_0}/a_{i_0} \neq 0} y^i \right)$$

## Local blowings-up



# Motivation of Lion & Rolin

1. Elimination of quantifiers: Projection of a subanalytic set is subanalytic.

2. Preparation Theorem for logarithmic-analytic functions and for logarithmic-exponential functions, that imply, in particular, the elimination of quantifiers in these classes.
   (proven before by Model Theory by van den Dries, MacIntire and Marker)

3. Preparation Theorem for $x^{\lambda}$-functions that implies the elimination of quantifiers for this class.
   (proven before by Model Theory by Miller)

# Motivation of Lion & Rolin

- Elimination of quantifiers: Projection of a subanalytic set is subanalytic.

$$\begin{array}{cc} \mathbb{R}^{n+1} & X = \bigcup \{ \{ f_{ij} = 0 \} \cap \{ g_{ik} > 0 \} \} \\ \downarrow & \downarrow \\ \mathbb{R}^n & \end{array}$$

$f_{ij}, g_{ik}$ subanalytic

suban. cellular decomposition

# Proof of Lion & Rolin

Follows the approach of Denef & van den Dries.

1. Blowings- up are replaced by the $D$-operator: $D(x, y) = x/y$.
2. Flattening is replaced by Finiteness Lemma.

$F(x, w)$ analytic function germ. $\exists_d \; \exists \; F_0(x), \ldots, F(x)$

$$F(x,w) = \sum_{i=0}^{d} F_i(x)\, y^i \; \mathrm{unit}_i(x,y)$$

$\underset{\text{work}}{\rightsquigarrow}$ $\quad |F_i| \leq |F_{i_0}|$

# Preparation Theorem for functions definable in a polynomially bounded o-minimal structures

## Theorem (van den Dries, Speissegger)

*Let $\mathcal{S}$ be a polynomially bounded o-minimal structure and let $f(x, y)$ be an $\mathcal{S}$-definable function.*
*Then there is a definable cellular decomposition $\mathcal{C}$ of $\mathbb{R}^{n+1}$, compatible with a cellular decomposition of $\mathbb{R}^n$, such that on each cell of $\mathcal{C}$*

$$f(x, y) = A(x)\ |y - \theta(x)|^{\nu}\ unit(x, y),$$

*where $A(x)$ and $\theta(x)$ are definable, $\nu \in \Lambda$, and $|u(x, y) - 1| < \frac{1}{2}$.*

Proof by Model Theory.

(Proof of Addendum on the unit by Nguyen & Valette is a fairly elementary, though not entirely trivial, application of the above theorem.)

# Are the proofs constructive ?

1. Lion & Rolin's proof is constructive.
2. Model Theory proof is not.
3. Analytic proof a priori not, but it is.

# Hironaka's proof of local flattening theorem.

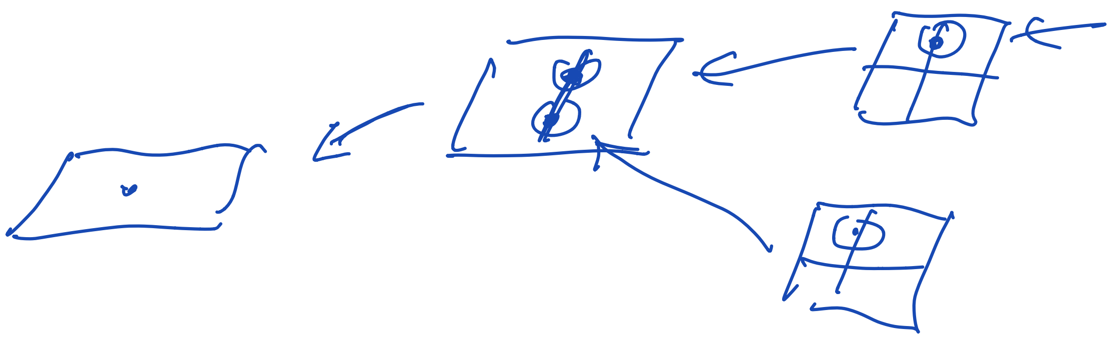
Hironaka: La voûte étoilée = the starry vault

(version of Zariski's Riemann surface)

$$\begin{array}{ccc} (X, x) & & \\ \downarrow & & \\ (W, w) & \longleftarrow & (\tilde{W}, \tilde{w}) \end{array}$$

blow-up flattener

defined locally.



Une étoile is "such a sequence of choices of points and centers" (with some identification)

$\mathcal{E}(W)$ la voûte étoilée.

$\rightsquigarrow$ Hausdorff topology

basis of open sets étoiles that factors through a fixed composition of local blowings-up.

$\mathcal{E}(W) \longrightarrow W$ is proper.

For our purpose one needs only equidimensionality (after complexification).

one can choose centers differently so that the dimension of the special fiber drops after each blowing-up.

# References.

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