

Preparation Theorem in o-minimal structures. Episode I.

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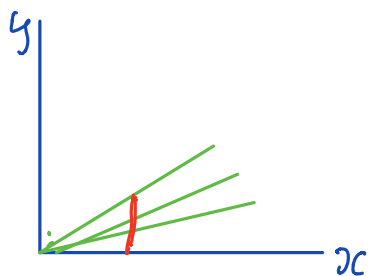
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Geometric motivation. Metric properties of singular spaces.

Given a singular (e.g. subanalytic) $X \subset \mathbb{R}^n$ or $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

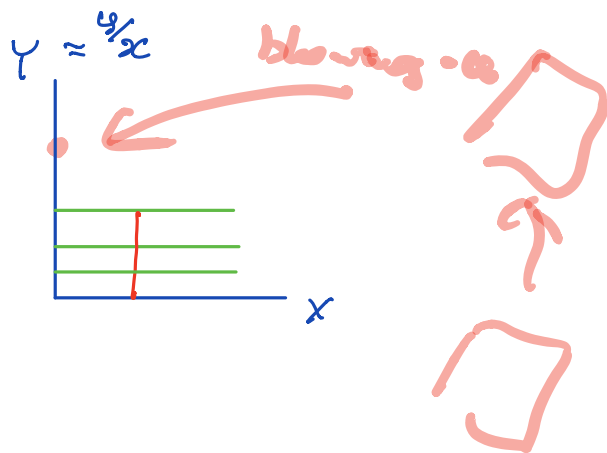
Can we study metric properties of X or f by means of the resolution of singularities?

Difficulty: A blowing-up destroys the distance and during the resolution process it is difficult to keep track of it.



$$\begin{aligned}x &= X \\ y &= XY\end{aligned}$$

←



Theorem

Let $P \in \mathbb{C}[Z]$. Then $\exists C > 0 \quad V(P) = P^{-1}(0) \quad , \quad (C = \deg P!)$

$$|P'(z)| \operatorname{dist}(z, V(P)) \leq C |P(z)|.$$

If $P = \prod (z - z_i)$

$$\left| \frac{P'}{P} \right| = |(\ln P)'| = \left| \sum (\ln(z - z_i))' \right| \leq \sum \frac{1}{|z - z_i|} \leq \frac{\deg P}{\operatorname{dist}(z, V(P))}$$

Question. Does it still hold for

- complex polynomials of many variables
 - real polynomials or real analytic functions
 - subanalytic functions
 - functions definable in o-minimal structures
 - with parameters, etc. ?
- must be polynomially bounded otherwise it is false

- Many variables. $P(z_1, \dots, z_n)$, $\deg P = d$

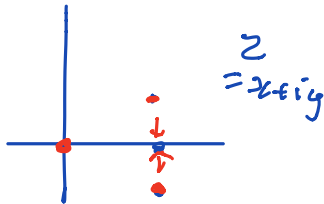
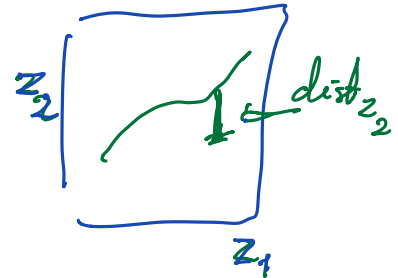
$$|P'_{z_i}| \operatorname{dist}(z, V(P)) \leq |P'_{z_i}| \operatorname{dist}_{z_i}(z, V(P)) \leq d |P_{z_i}|$$

$$\rightarrow |\operatorname{grad} P_{z_i}| \operatorname{dist}(z, V(P)) \leq d \cdot n |P_{z_i}|$$

- $P(x)$ real polynomial.

$$\exists X \subset \mathbb{R}^n, \dim X \leq n-1$$

$$|\operatorname{grad} P(x)| \operatorname{dist}(x, X) \leq d n |P_{z_i}|$$



for $n=1$ we may choose X the projection
or $V(P)$ (complex) to the real axis.

• in semi-algebraic case this is non trivial.

Theorem (Theorem A)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function definable in a polynomially bounded o-minimal structure. Then there is a constant C and a definable $X \subset \mathbb{R}^n$, $\dim X \leq n - 1$, such that f is C^1 on $\mathbb{R}^n \setminus X$ and for all $x \in \mathbb{R}^n \setminus X$

$$\|\operatorname{grad} f(x)\| \operatorname{dist}(x, X) \leq C|f(x)|.$$

Theorem (Theorem B)

$x = (x_1, \dots, x_n)$, y single variable

Let $f(x, y) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be a function definable in a polynomially bounded o-minimal structure. Then there is a constant C and a definable $X \subset \mathbb{R}^{n+1}$, $\dim X \leq n$, such that f is C^1 on $\mathbb{R}^{n+1} \setminus X$ and for all $(x, y) \in \mathbb{R}^{n+1} \setminus X$

$$|f'_y(x, y)| \operatorname{dist}_y((x, y), X) \leq C|f(x, y)|.$$

Given analytic f . By Hironaka's resolution

f can be rectilinearized $f: \mathbb{R}^n \rightarrow \mathbb{R}$

$\exists \sigma: M \rightarrow \mathbb{R}^n$ composition of blowings-up
with smooth centers. s.t. locally

$$\tilde{x} \in M, \quad f \circ \sigma(\tilde{x}) = \underbrace{\tilde{c}}_{\text{monomial}} \cdot \text{unit}(\tilde{x}) \quad \text{--- normal crossing}$$

Suppose f solution $F(x, w) = 0$, $F(x, f(x)) = 0$.

F analytic, $F(x, w) = w^d + \sum a_i(x) w^{d-i}$

x single variable Puiseux $\exists q$ $f(\pm x^q)$ analytic

$x = (x_1, \dots, x_n)$ Jung-Abhyankar
Assumption

$a_d(x), \Delta F(x)$ normal crossings

Conclusion $\exists q$ $f(\pm x_1^q, \dots, \pm x_n^q)$ analytic
+ normal crossing.

Rectilinearization of subanalytic functions.

Theorem [Brieskorn-Milman '91, A.P. '94]

Let $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous subanalytic function. Then there is a covering family $\sigma_\alpha : U_\alpha \rightarrow U$, where σ_α are compositions of local blowings-up with nonsingular analytic centers and local substitutions of powers

$$\psi(x_1, \dots, x_n) = (\pm x_1^{r_1}, \dots, \pm x_n^{r_n}),$$

such that each $f \circ \sigma_\alpha$ is a normal crossing.

Remarks

Related to Jung's idea of resolution of singularities:

- $F(x_1, \dots, x_n, x_{n+1})$. $F = x_{n+1}^d + \sum_{i=1}^r a_i(x_1, \dots, x_n) x_{n+1}^{d-i}$
- rectilinearize Δ_F to make it normal crossings
(by inductive assumption)
- combinatorial algorithm to make F normal crossings

Keeping track of one fixed variable.

Suppose now that $f(x, y)$ satisfies an equation

$$F(x, y, f(x, y)) = 0,$$

polynomial with respect to the last variable, and let $D(x, y)$ be the discriminant of F times the last coefficient.

- ① Using additional blowings-up in x we may assume D is regular in y and then, by Weierstrass Preparation Theorem, that it is a monic polynomial in y (times a function of x).
- ② Then rectilinearize the discriminant of D times the last coefficient. Substitute the powers. We get (in new x variables)

$$D(x, y) = A(x) \prod_i (y - \theta_i(x)) \quad , \quad \theta_i \text{ analytic.}$$

- ③ $D(x, y)$ is not yet normal crossings but a purely combinatorial resolution algorithm makes it normal crossings.

Coordinates on this "blow-up" space

The local coordinates on the blow up space are of the form

- functions of x ,
- functions of x times a power of one of $y - \theta_i(x)$.

Therefore in the old coordinates on pieces.

$$f(x, y) = A(x) |y - \theta(x)|^v \text{ unit}(x, y)$$

Preparation Theorem for subanalytic functions

Theorem (A.P. 94, Lion-Robin 97)

Let $f(x, y)$ be a global subanalytic function on $\mathbb{R}^{n+1} \ni (x_1, \dots, x_n, y)$.
Then f is **reducible**.

A function is **reducible** if it is piecewise of the form

$$f(x, y) = A(x) |y - \theta(x)|^\nu \text{ unit}(x, y).$$

Comments:

- No preliminary change of variables is required.
- $\nu \in \mathbb{Q}$ but negative exponents ν may appear and cannot be avoided.

Precise definition of a reducible function.

A function $f(x, y)$ is **reducible** if there is a subanalytic cellular decomposition \mathcal{C} of \mathbb{R}^{n+1} , compatible with a cellular decomposition of \mathbb{R}^n , such that on each cell of \mathcal{C}

$$f(x, y) = A(x) |y_1|^\nu U(x, y),$$

where $y_1 = y - \theta(x)$, $A(x)$ and $\theta(x)$ are subanalytic, $\nu \in \mathbb{Q}$, and $U(x, y)$ is a **reduced unit**, i.e. , that is

$$U(x, y) = V(\psi(x, y)),$$

with

$$\psi(x, y) = (\phi_1(x), \dots, \phi_s(x), a(x)|y_1|^{1/p}, b(x)/|y_1|^{1/p}).$$

V is non-vanishing and real analytic on a neighbourhood of the closure of the image of ψ , that is bounded.

Preparation Theorem \implies Theorem B.

Theorem (Theorem B)

For a function $f(x, y)$ definable in a polynomially bounded o-minimal structure $\exists C$ and $\exists X$, $\dim X \leq n$, such that

$$|f'_y(x, y)| \operatorname{dist}_y((x, y), X) \leq C |f(x, y)|.$$

Proof $f(x, y) = A(x) |y - \theta(x)|^v \cup (x, y).$

$$\left| \frac{\partial}{\partial y} (|y - \theta(x)|^v) \right| = \left| \frac{v |y - \theta(x)|^{v-1}}{y - \theta(x)} \right| \leq \frac{|v| |y - \theta(x)|^{v-1}}{\operatorname{dist}((x, y), \Gamma_\theta)}, \quad \Gamma_\theta = \text{graph of } \theta$$

$$u(x, y) = V(\psi_1(x), \dots, \psi_s(x), \underbrace{a(x)}_{\alpha} \underbrace{(y - \theta(x))^k}_{\beta}, \underbrace{b(x)}_{\beta} / \underbrace{(y - \theta(x))^k}_{\beta})$$

since α, β are bounded and $u \geq \varepsilon > 0$

$$\left| \frac{\partial u}{\partial y} \right| = \left| \underbrace{\frac{\partial V}{\partial \alpha}}_{\text{bounded}} \frac{\partial \alpha}{\partial y} + \underbrace{\frac{\partial V}{\partial \beta}}_{\text{bounded}} \frac{\partial \beta}{\partial y} \right| \leq C \frac{|\alpha, \beta|}{|y - \theta(x)|} \leq \tilde{C} \frac{u(x, y)}{\operatorname{dist}((x, y), \Gamma_\theta)}$$

Preparation Theorem for functions definable in a polynomially bounded o-minimal structures

Theorem (van den Dries, Speissegger) (2001)

Let S be a polynomially bounded o-minimal structure and let $f(x, y)$ be an S -definable function.

Then there is a definable cellular decomposition \mathcal{C} of \mathbb{R}^{n+1} , compatible with a cellular decomposition of \mathbb{R}^n , such that on each cell of \mathcal{C}

$$f(x, y) = A(x) |y - \theta(x)|^\nu \operatorname{unit}(x, y),$$

where $A(x)$ and $\theta(x)$ are definable, $\nu \in \Lambda$, and $|u(x, y) - 1| < \frac{1}{2}$.

Λ field of exponents of S

Addendum by Nguyen and Valette

Theorem {2016}

A function $f(x, y)$ definable in a polynomially bounded o-minimal structure \mathcal{S} is \mathcal{S} -reducible.

that is it is piecewise of the form

$$f(x, y) = A(x) |y_1|^\nu U(x, y),$$

where $y_1 = y - \theta(x)$, with a **reduced unit**, i.e. , that is

$$U(x, y) = V(\psi(x, y)),$$

with $\psi(x, y) = (\phi_1(x), \dots, \phi_s(x), b_1(x)|y_1|^{1/p_1}, \dots, b_k(x)/|y_1|^{1/p_k})$.

The map V is non-vanishing and C^2 with bounded derivative on a neighbourhood of the closure of the image of ψ , that is bounded.

Theorem (Theorem C)

For a function $f(x, y)$ definable in a polynomially bounded o-minimal structure $\exists C$ and a definable stratification of \mathbb{R}^{n+1} such that \forall Lipschitz vector fields v on \mathbb{R}^{n+1} with Lipschitz constant L and tangent to strata

$$\left| \frac{\partial f}{\partial v}(x, y) \right| \leq CL|f(x, y)|.$$

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