Non oscillating trajectories of o-minimal vector fields in dimension 3.

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Joint work with O. Le Gal (Chambéry) and F. Sanz (Valladolid)

O-minimality and foliations,
CIRM, 31 May - 4 June 2021

The setting of our work.

We consider:

- a system of two ODEs:

$$(S_F) \quad \begin{cases} {y_1}' &= f_1(x, y_1, y_2) \\ {y_2}' &= f_2(x, y_1, y_2) \end{cases}$$

  where $F = (f_1, f_2) : \Omega \to \mathbb{R}^2$ is C^1 on some open $\Omega \subset \mathbb{R}_+ \times \mathbb{R}^2$ with $(0, 0, 0) \in \overline{\Omega}$;

- two distinct C^1-maps $\gamma = (\gamma_1, \gamma_2) : (0, a) \to \mathbb{R}^2$ and $\delta = (\delta_1, \delta_2) : (0, a) \to \mathbb{R}^2$ that are *solutions*

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Corresponds after possible reparameterization by x if $dx(\xi) > 0$ to the case of:

- a vector field
$$\xi : U \to T_0\mathbb{R}^3$$
of class C^1 in a neighborhood U of $0 \in \mathbb{R}^3$;

- two distinct *trajectories* Γ and Δ (i.e. images $\Gamma = c((0, a))$ and $\Delta = d((0, a))$ of *integral curves* $c : (0, a) \to \mathbb{R}^3$ and $d : (0, a) \to \mathbb{R}^3$)

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$$(S_F) \quad \begin{cases} y_1{}' = f_1(x, y_1, y_2) \\ y_2{}' = f_2(x, y_1, y_2) \end{cases}$$

We consider $F = (f_1, f_2) : \Omega \to \mathbb{R}^2$ *definable in a polynomially bounded o-minimal structure $\mathcal{R}$ expanding $\mathbb{R}$* .

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- *polynomially bounded*: any 1-var definable function is ultimately bounded by a power of x;
- *expanding* $\mathbb{R}$: the real constant functions are definable.

Context and motivation.

General problem.

Describe the local dynamics of a vector field at a singular point.

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interlacement vs separation

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non-oscillating case $\rightsquigarrow$ relative behavior of *pencils* of trajectories

Known results.

Dimension 2.

$$oscillating = spiraling$$

$$vs$$

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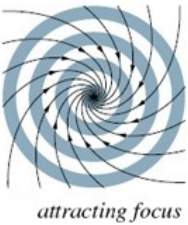
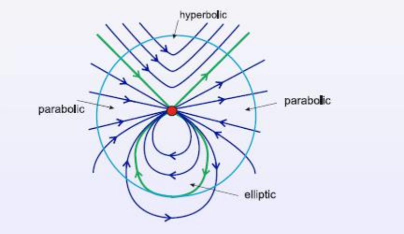


VS



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in fact *o-minimal* (Lion-Rolin 1998, Speissegger 1999)

Known results.

Dimension 3, real analytic case.

For 1 trajectory having *iterated tangents*:

$$\text{oscillating} = \text{twisting around an analytic axis } \Gamma_0$$

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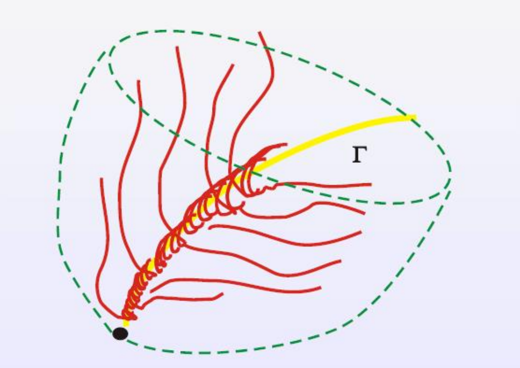
$$\text{in fact } \underline{\textit{generates a Hardy field}}$$

$$\mathcal{H} := \{h(t) := (f/g) \circ \Gamma(t) : \ (f/g) \in \mathrm{Fr}(\mathbb{R}\{x, y, z\})\}.$$

Known results.

In the oscillating case, $\Gamma \subsetneq \mathrm{Sing}(\xi)$ is a twisting axis.



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For a pair of *non-oscillating* trajectories
(Cano-Moussu-Sanz 2004):

interlaced = twisting around each other with formal axis $\hat{\Gamma}$

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in fact *each is o-minimal* (Le Gal-Sanz-Speissegger 2017)

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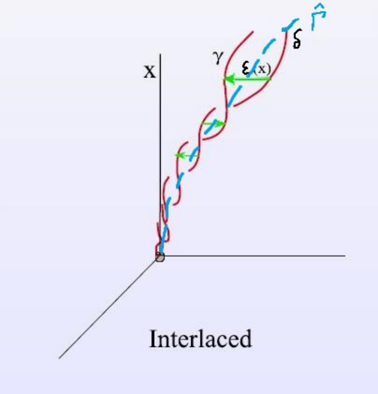

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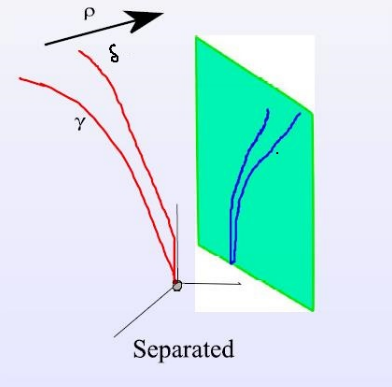

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O-minimal case for a *linear* system of two ODEs.

(Le Gal-Sanz-Speissegger 2012)

Pair of solutions :

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- In fact, true for *all* the solutions together;
- O-minimality $\Rightarrow$ they all *generate a Hardy field over the Hardy field of* $\mathcal{R}$ = Hardy-compatible.

$$\mathcal{H} := \{h(x) := f \circ (x, \gamma(x)) : \ f \text{ dfble in } \mathcal{R}, \gamma \text{ solution}\}.$$

Open problems

Real analytic vector field with a pair of non-oscillating separated trajectories.

- Is there a common Hardy field for both trajectories?
- Is there an o-minimal structure in which one or possibly both are definable?
- Same questions for all the trajectories of the integral pencil.

O-minimal polynomially bounded vector field with a pair of non-oscillating trajectories.

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Our main result.

We consider a system of ODEs:

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with $F = (f_1, f_2) : \Omega \to \mathbb{R}^2$ is C^1 and definable in $\mathcal{R}$ polynomially bounded o-minimal expansion of $\mathbb{R}$
and two distinct solutions $\gamma, \delta : (0, a) \to \mathbb{R}^2$ such that: .

- γ, δ have flat contact: $\|\gamma(x) - \delta(x)\| < x^n$ for some n ultimately;
- γ has the regular separation property: $\forall f : \mathbb{R}^3 \to \mathbb{R}$ definable, $f(x, \gamma(x)) \equiv 0$ or $|f(x, \gamma(x))| > x^n$ for some n ultimately.

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Regular separation property.

Regular separation property = Łojasiewicz inequality:

$\forall f : \mathbb{R}^3 \to \mathbb{R}$ definable, either $f(x, \gamma(x)) \equiv 0$ or $|f(x, \gamma(x))| > x^n$ for some n ultimately.

$\rightsquigarrow$ Holds in poly. bdd. o-minimal structures (see Ta Lê Loi 1995, van den Dries-Miller 1996).

$\rightsquigarrow$ Implies non-oscillation wrt $\mathcal{R}$.

Our main result.

Theorem (Le Gal-M.-Sanz, Arxiv 2020).

We consider a system of ODEs C^1 and definable in $\mathcal{R}$:

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and two distinct solutions $\gamma, \delta : (0, a) \to \mathbb{R}^2$ such that: .

- γ, δ have flat contact;
- γ has the regular separation property.

Then γ and δ are either interlaced,
or Hardy-compatible.

Idea of proof.

Consider:

- $\varepsilon := \gamma - \delta$;
- $\Psi := \mathrm{graph}(\gamma, \varepsilon) = \{(x, \gamma(x), \varepsilon(x)), x \in (0, a)\} \subset \mathbb{R}^5$, trajectory of a definable vector field in $\mathbb{R}^5$:

$$\xi = \frac{\partial}{\partial x} + F(x, \underline{y})\frac{\partial}{\partial \underline{y}} + (F(x, \underline{y}) - F(x, \underline{y} - \underline{z}))\frac{\partial}{\partial \underline{z}}.$$

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It suffices to prove that:

- either $\varepsilon(x)$ indefinitely rotates;
- or, for any definable $f(x, \underline{y}, \underline{z})$, $x \mapsto f(x, \gamma(x), \varepsilon(x))$ ultimately has a constant sign.

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Let $f : \mathbb{R}^5 \to \mathbb{R}$ be definable.

Key tool: definable cell decomposition of (the corresponding nbhd of 0 in) $\mathbb{R}^5$ adapted :

- to z_1, z_2 and f;
- to the vector field ξ: for any cell, either ξ is tangent to it, or it is transverse to it.

Observations:

- by regular separation, $\exists!$ (induced) cell $C \subseteq \mathbb{R}^3$ s.t.

$$\mathrm{graph}(\gamma) \subseteq C;$$

- $C \times \{(0,0)\}$ is a cell and γ, δ are distinct $\Rightarrow$

$$\Psi \cap C \times \{(0,0)\} = \emptyset.$$

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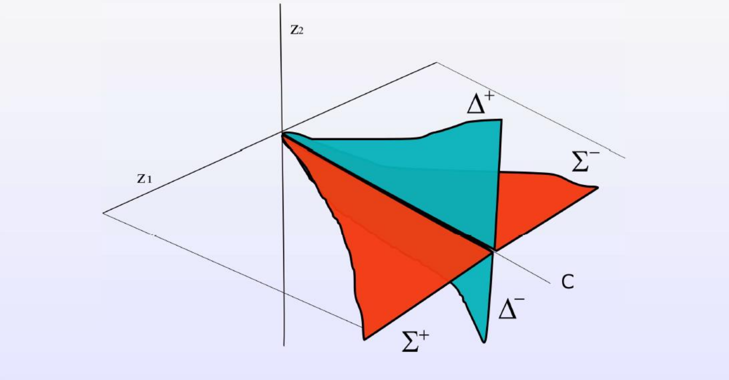
Consider the (induced) cells over C in $\mathbb{R}^4$:

- Σ^+ and Σ^- in $z_2 = 0$;
  (e.g. $\Sigma^+ = \{(x, \underline{y}, z_1, 0) : (x, \underline{y}) \in C,\ 0 < z_1 < \varphi^+(x, \underline{y})\}$)
- Δ^+ and Δ^- in $z_1 = 0$.



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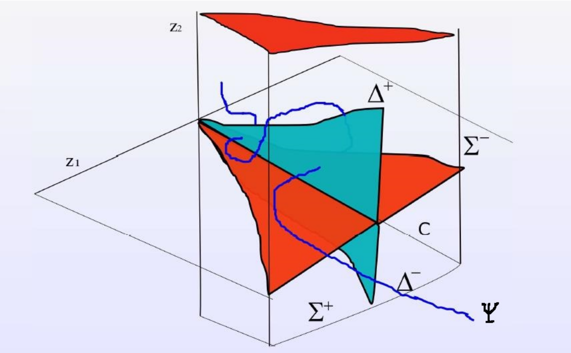
Lemma: $\Psi \subseteq (\Sigma^+ \cup (C \times \{0\}) \cup \Sigma^-) \times \mathbb{R}$
and

$$\Psi \cap ((C \times \{0\}) \times \mathbb{R}) \subseteq \Delta^+ \cup \Delta^-.$$

(Indeed, by regular sep + flatness: $\varphi^+(x, \gamma(x)) > x^N > \varepsilon_1(x)$...)



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Claim 1. If $\varepsilon_1(x)$ has ultimately a constant sign, then so does $f(x, \gamma, \varepsilon)$.

Proof.

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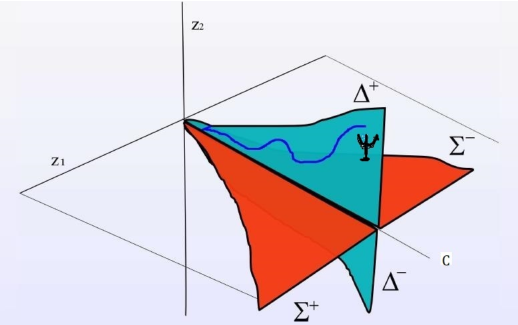
Proof. If $\varepsilon_1(x) \equiv 0$, then $\Psi \subseteq \Delta^+ \times 0$ or $\Psi \subseteq \Delta^- \times 0$.



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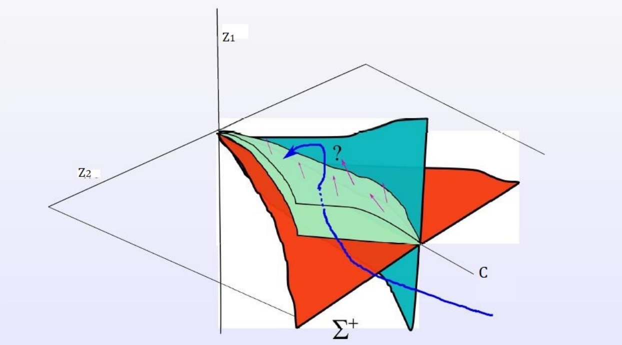
Claim 1. If $\varepsilon_1(x)$ has ultimately a constant sign, then so does $f(x, \gamma, \varepsilon)$.

Proof. If e.g. $\varepsilon_1(x) > 0$, then Ψ cannot cross twice any cell of type "graph over Σ^+.



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Proof of the theorem.

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$\Rightarrow$ Ψ intersects $\Delta^+ \cup \Delta^-$ **infinitely many times** and **transversely**.

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Suppose that $\varepsilon_1(x)$ has not ultimately a constant sign.

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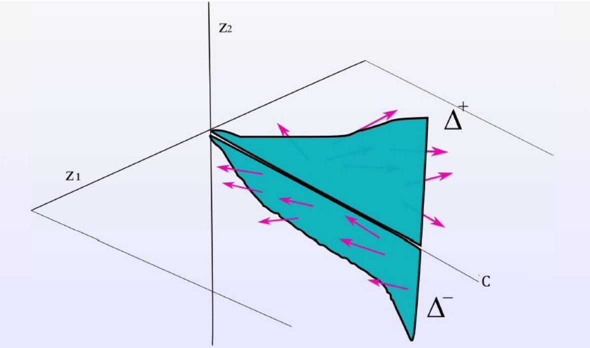
$\Rightarrow$ the vector field ξ has **opposite orientation** in Δ^+ and in Δ^-.



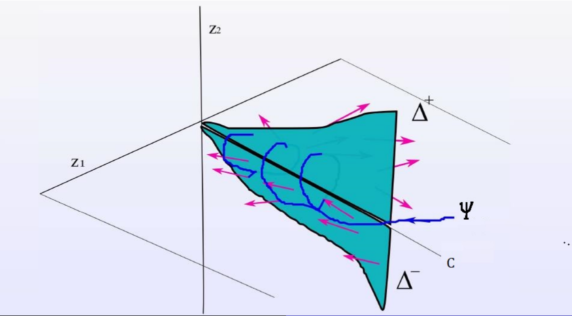
Idea of proof.

Proof of the theorem.

Suppose that $\varepsilon_1(x)$ has not ultimately a constant sign.

$\Rightarrow \Psi$ intersects $\Delta^+ \cup \Delta^-$ **infinitely many times** and **transversely**.

$\Rightarrow$ the vector field ξ has **opposite orientation** in Δ^+ and in Δ^-.
$\Rightarrow \varepsilon(x)$ rotates around 0, qed.



Applications.

- Vector fields in $\mathbb{R}^3$ definable over $\mathcal{R}$.
- Analytic vector fields in $\mathbb{R}^3$: separated pencils of with formal non-degenerate axis.

  $\rightarrow$ Existence of trajectories with regular separation;

  $\rightarrow$ Case of a subanalytically transcendental formal axis.

  $\rightarrow$ Several examples.

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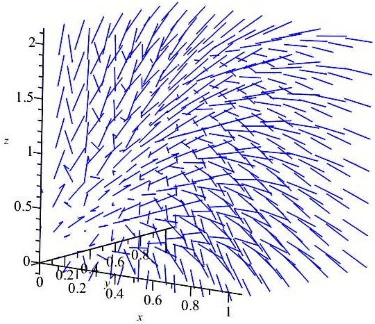
An example.

$$\xi = 2x^2 \frac{\partial}{\partial x} + 2(y-x)\frac{\partial}{\partial y} + (z-2x)\frac{\partial}{\partial z}$$

has a formal curve subanalytically transcendental:

$$\hat{\Gamma} = \{(x, E(x), E(2x))\} \text{ where } E(x) := \sum_n n! x^{n+1}.$$



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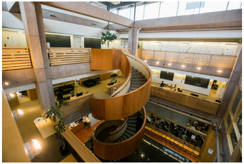

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