O-minimal expansions of $\mathbb{R}$

Lecture III : Quasianalyticity and o-minimality

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Outline

Quasi-analyticity in polynomially bounded structures

Definition

S is polynomially bounded if, for all definable $f : (a, +\infty) \to \mathbb{R}$,

$$\exists n \in \mathbb{N}, \lim_{x \to +\infty} \frac{f(x)}{x^n} = 0$$

Proposition

Let S be polynomially bounded, and let

$$T : C^\infty(0_{\mathbb{R}^n}) \to \mathbb{R}[[x_1, \ldots, x_n]]$$

be the Tayor map. Then, the restriction of T to germs of definable C^∞ functions is injective.

Structures generated by QA classes

Definition

A class of C^∞ functions $\mathcal{A} = (\cup \mathcal{A}_U)_U$, is the given, for each n and each open subset U of $\mathbb{R}^n$, of an algebra $\mathcal{A}_U$ of C^∞ functions such that:

1. $\mathcal{A}$ is closed by composition : $f_1, \ldots, f_k \in \mathcal{A}_U,\ g \in A_V,\ (f_1, \ldots, f_k)(U) \subset V \Rightarrow g \circ (f_1, \ldots, f_k) \in \mathcal{A}_U$
2. $\mathcal{A}$ is closed by implicit functions: if $f \in A_U$, $\varphi : V \to \mathbb{R}$, $(x, \varphi)(V) \subset U$, $f(x, \varphi(x)) = 0$, $\frac{\partial f}{\partial y}(x, \varphi(x)) \neq 0$ then $\varphi \in \mathcal{A}_V$
3. $\mathcal{A}$ is closed by monomial division: $f \in \mathcal{A}_U,\ f(x, 0) \equiv 0 \Rightarrow f(x, y)/y \in \mathcal{A}_U$.
4. $\mathcal{A}_U$ contains polynomial functions
5. $\mathcal{A}$ is closed by gluing and restrictions : if $U = \bigcup_i U_i$,

$$f \in \mathcal{A}_i \Leftrightarrow \forall i \in I,\ f_{|U_i} \in \mathcal{A}_{U_i}$$

Structures generated by QA classes

Definition

A class of C^∞ functions $\mathcal{A} = (\cup \mathcal{A}_U)_U$, is the given of an algebra $\mathcal{A}_U, U \subset \mathbb{R}^n, n \geq 1$ of C^∞ functions such that:

1. $\mathcal{A}$ is closed by composition
2. $\mathcal{A}$ is closed by implicit functions
3. $\mathcal{A}$ is closed by monomial division
4. $\mathcal{A}_U$ contains polynomial functions
5. $\mathcal{A}$ is closed by gluing and restrictions

$\mathcal{A}_a := \{\text{germs at } a \text{ of } f \in \mathcal{A}_U,\ a \in U\}$

Definition

A class of C^∞ function is quasi-analytic if for all $n \geq 1$, the Taylor map $T : \mathcal{A}_{0\in\mathbb{R}^n} \to \mathbb{R}[[x_1, \ldots, x_n]]$ is injective.

Examples :

1. Analytic functions form a class of C^∞ functions, that is quasianalytic.
2. C^∞ functions form a class of C^∞ functions, that is not quasi-analytic.
3. Certain quasi-analytic Denjoy Carleman classes form a class of C^∞ functions that is quasianalytic.

Definition

For any family $\mathcal{F} = (f_i : U_i \in \mathbb{R}^{n_i} \to \mathbb{R})_{i \in I}$ of C^∞ functions, there exists a smallest class $\mathcal{A}_\mathcal{F}$ of C^∞ functions that contains all f_i, $i \in I$. We call $\mathcal{A}_\mathcal{F}$ the class generated by $\mathcal{F}$.

Proposition

Let S be a polynomially bounded o-minimal structure. Say a function $f : U \to \mathbb{R}$ is a locally definable C^∞ function if

$$\forall x \in U, \exists V \in Nbhd(x),\ f_{|V} \text{ is } C^\infty \text{ and definable.}$$

Then, the collection of all locally definable C^∞ functions form a C^∞ class which is quasi-analytic.

Theorem (Rolin, Speissegger, Wilkie)

Let $\mathcal{A}$ be a quasi-analytic class of C^∞ functions. Then the structure generated by all $f_{|B}$, $\overline{B} \subset U$, $f \in \mathcal{A}_U$ is a polynomially bounded o-minimal structure.

Remark : the two operations “Poly. bound. structure → QA class” and “QA class → Poly. bound structure” are not strictly reciprocal but after one cycle they do.

Examples:

1. Analytic functions
2. Borel multi-summable functions (van den Dries, Speissegger)
3. Denjoy-Carleman classes (Rolin, Speissegger, Wilkie)
4. Certain trajectories of singular analytic vector fields (Rolin, Sanz, Schäfke)
5. Generic C^∞-functions (leGal)

Getting quasi-analyticity

$\mathcal{F}$ family of functions, $\mathcal{A}_{\mathcal{F}}$ generated class of C^{∞} functions. Is $\mathcal{A}_{\mathcal{F}}$ quasi-analytic ?

$f \in \mathcal{A}_{\mathcal{F}},\ T_0 f = \hat{0} \in \mathbb{R}[[x]] \Rightarrow f \equiv 0$?

$f \in \mathcal{A}_{\mathcal{F}}$ means f is obtained from $\mathcal{F}$ after a sequence of composition, taking implicit functions, divisions, which all are formal operators (the operation can be made on the formal series).

Enumerate the fonctionnal equations formally satisfied by functions in $\mathcal{F}$ and see if they are truly satisfied.

Examples : generic functions, SAT condition.