O-minimal expansions of $\mathbb{R}$

Lecture II : asymptotics

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Outline

Hardy field of an o-minimal structure

Definition

A structure expanding $(\mathbb{R}, +, \times, \leq)$ is the given of all its definable sets, that satifies:

1. for all n, the definable subsets of $\mathbb{R}^n$ form a boolean algebra
2. definable sets are stable by projection and cartesian product
3. all semi-algebraic sets are definable.

Definition

An expansion of $(\mathbb{R}, +, \times, \leq)$ is o-minimal if the definable sets of $\mathbb{R}$ are the finite unions of points and intervals.

We fix an o-minimal structure expanding $(\mathbb{R}, +, \times, \leq)$ and focus on the asymptotic behavior of unary functions.

Proposition

If $f : \mathbb{R} \to \mathbb{R}$ is definable, f has ultimately a sign: there exists $\sigma \in \{<, =, >\}$ and $A \in \mathbb{R}$,

$$\forall x > A,\ f(x)\,\sigma\,0$$

Proof.

$\{f > 0\}$, $\{f = 0\}$, $\{f < 0\}$ is a partition of $\mathbb{R}$ in definable sets. □

Corollary

The order at infinity, given by

$$f \leq_\infty g \Leftrightarrow \exists A \in \mathbb{R}, \forall x \geq A,\ f(x) \leq g(x)$$

is a total ordering on the germs of definable functions at $+\infty$.

Corollary

The germs at infinity of definable function form a Hardy field, this is, a totally ordered field closed by derivation.

Proof.

It is obviously a ring. If $f \not\equiv 0$, f does not ultimately vanish then has an inverse, that is definable $(\{(x, y); f(x)y = 1\})$. Finally, f is piecewise C^1 so f' is defined on a neighborhood of $+\infty$. Being defined by an (ε, δ) formula, f' is definable. □

Definition

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H gets more structure than a general Hardy field.

If $f, g \in H$ and $f \to +\infty$, we got $g \circ f \in H$ (composition), and $f^{-1} \in H$ (compositional inverse).

Moreover, if $P(y) = p_0(x)y^d + p_1(x)y^{d-1} + \cdots + p_0(x) \in H[y]$ has odd degree, $\{(x, y);\ P(x, y) = 0\}$ is definable and its projection contains a neighborhood of ∞. By curve selection, P has a root in H. So H is real closed.

(picture)

Valuation

Units:

$$U := \{u \in H;\ \exists n \in \mathbb{N},\ \frac{1}{n} < |u| < n\}$$

form a subgroup of $H^{\times}$.

Value group:

$$G := H^{\times}/U,$$

is a totally ordered group, with $v(f) \leq v(g)$ if $|f| \geq |g|$.

1. $v(f) = v(g)$ iff $0 < c < \frac{|f|}{|g|} < C$.
2. $v(f) \leq v(g)$ iff $\frac{g}{f}$ bounded, iff $g = O(f)$.
3. $v(f) < v(g)$ iff $\frac{g}{f} \to 0$ iff $g = o(f)$.
4. $v(fg) = v(f) + v(g)$
5. $v(f + g) \geq min(v(f), v(g))$
6. L'Hospital rule applies : $\frac{f'}{g'}$ always have a limit in $\mathbb{R} \cup \{\pm\infty\}$ so $\frac{d}{dx}$ is increasing on $v \neq 0$.

Definition

An o-minimal structure is polynomially bounded if, for all definable $f : (a, +\infty) \to \mathbb{R}$,

$$\exists n \in \mathbb{N}, \lim_{x \to +\infty} \frac{f(x)}{x^n} = 0$$

Theorem (Miller)

Let S be an o-minimal structure. Then either S defines the exponential function, or S is polynomially bounded.

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Principle of proof.
Rosenlicht : if H contains a smaller archimedean class than $f \in H$, the log derivative of f is equivalent to a derivative:
$\exists h \in H, \ \frac{f'}{f} \sim h'$.
Set $g := h \circ f^{-1}$. Then $g' \sim \frac{1}{x}$.
$G(t,x) = g(tx) - g(x)$
$G(t) = \lim_{x \to \infty} G(t,x)$.
G is finite and non zero on $(1, +\infty)$.
G satisfies $G(st) = G(s) + G(t)$.
$G(t) = k \ln(t)$.
G is definable.

Asymptotic in polynomially bounded structures

We now fix S to be polynomially bounded.

Proposition

Let $f \in H$. Then there exists $k \in \mathbb{R}$, $\alpha \in \mathbb{R}$ such that

1. $f(x) \sim kx^\alpha$
2. $x \mapsto x^\alpha$ *is definable.*

Field of exponents:

$$\Lambda := \{\alpha \in \mathbb{R};\ x \mapsto x^\alpha \text{ definable } \}$$

A definable function has an initial asymptotic development in $\mathbb{R}[[x^*]]$.

Łojasiewicz inequality

Theorem (Łojasiewicz inequality)

Let S be polynomially bounded, $f, g : K \to \mathbb{R}$ be definable continuous functions on a compact set $K \subset \mathbb{R}^n$. Suppose $f^{-1}(0) \subset g^{-1}(0)$. Then there exist $\alpha \in \mathbb{R}$, $c > 0$,

$$\forall x \in K, \ |f(x)| \geq c|g(x)|^{\alpha}.$$

(Proof)

Quasianalyticity

Proposition

Let S be polynomially bounded, and let

$$T : C^\infty(0_{\mathbb{R}^n}) \to \mathbb{R}[[x_1, \ldots, x_n]]$$

be the Tayor map. Then, the restriction of T to germs of definable C^∞ functions is injective.

Proposition

Let S be polynomially bounded, and let $f : U \to \mathbb{R}$ be definable and C^∞ on a connected open subset U of $\mathbb{R}^n$. Then, either $f^{-1}(0)$ has empty interior, or $f^{-1}(0) = U$.

Preparation Theorem

Theorem

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ *be definable. Then there exists a cell decomposition* $\mathcal{C}$ *of* $\mathbb{R}^{n+1}$*, and for each cell* $C \in \mathcal{C}$*, there exists :*

- *a definable unit* $u : C \to \mathbb{R}$*, this means:*
  $\exists 0 < a < b,\ \forall (x, y) \in C,\ c < u(x, y) < C$
- *definable functions:* $\begin{cases} a : \pi_n^{n+1}(C) \to \mathbb{R}, \\ \theta : \pi_n^{n+1}(C) \to \mathbb{R} \end{cases}$
- *an exponent* $\alpha \in \Lambda$

such that

$$f(x, y) = (y - \theta(x))^{\alpha} a(x) u(x, y)$$