Introduction to o-minimal geometry

Lecture I : o-minimality

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Outline

Structures on $\mathbb{R}$

A (real) algebraic set X is a subset of $\mathbb{R}^n$ defined by polynomial equations: $\{x \in \mathbb{R}^n,\ P_1(x) = 0, \dots, P_k(x) = 0\}$.
Example : quadratic equations with their solutions:

$$X = \{(a, b, c, x) \in \mathbb{R}^4;\ ax^2 + bx + c = 0\}.$$

With X comes the quadratic equations having a (real) solution:

$$Y = \{(a, b, c) \in \mathbb{R}^3;\ \exists x \in \mathbb{R},\ ax^2 + bx + c = 0\}$$

The quadratic formula gives a presentation of Y with polynomial equations and inequalities:

$$Y = \{(a, b, c);\ a \neq 0,\ b^2 - 4ac \geq 0\} \cup \{a = 0, b \neq 0\} \cup \{(0, 0, 0)\}$$

$$
\begin{aligned}
X &= \{(a,b,c,x) \in \mathbb{R}^4;\ ax^2+bx+c=0\} \\
Y &= \{(a,b,c) \in \mathbb{R}^3;\ \exists x \in \mathbb{R},\ ax^2+bx+c=0\} \\
&= \{(a,b,c);\ a \neq 0,\ b^2-4ac \geq 0\} \cup \{a=0, b \neq 0\} \cup \{(0,0,0)\}
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Y is the projection of X by $\pi : \mathbb{R}^4 \ni (a,b,c,x) \mapsto (a,b,c) \in \mathbb{R}^3$.

Y is not algebraic: $(1,b,1) \in Y \Leftrightarrow b \in (-\infty,-2] \cup [2,+\infty)$.

Y can be described by polynomial equations and inequalities.

Studying real algebraic sets drives us to consider sets that are defined not only by polynomial equations but inequalities: the semi-algebraic sets.

Semi-algebraic sets have usefull stability properties. Closed by :

- union, intersection, complement,
- cartesian product,
- projections (Tarski).

Corresponds to stability of formulas by:

- conjunction ("or"), disjunction ("and"), negation ("not"),
- cartesian product,
- adding existential quantifier ("$\exists x \in \mathbb{R}, \ldots$"),
- adding universal quantifier ("$\forall x \in \mathbb{R}, \ldots$").

Example : if $X = \{x \in \mathbb{R}^n;\ \Phi(x)\}$ is semi-algebraic, its tangent cone is semi-algebraic:

$$TC(X) = \left\{ (x,v) \in \mathbb{R}^{2n},\ \begin{array}{l} \forall \varepsilon > 0,\ \exists t > 0,\ \exists w \in \mathbb{R}^n, \\ t < \varepsilon,\ ||w - v|| < \varepsilon,\ \Phi(x + tw) \end{array} \right\}$$

$$\complement \left(\Pi^{x,v,\varepsilon}_{x,v} \left(\complement \left(\Pi^{x,v,\varepsilon,w,t}_{x,v,\varepsilon} \left\{ (x,v,\varepsilon,w,t) \in \mathbb{R}^{3n+2};\ \begin{array}{l} 0 < t,\ t < \varepsilon, \\ ||w - v|| < \varepsilon, \\ \Phi(x + tw) \end{array} \right\} \right) \right) \right)$$

We want a framework that mimics the good stability properties of semi-algebraic sets.

Definition

A structure $S = \bigcup_{n \in \mathbb{N}} S_n$ on $\mathbb{R}$ is the given for each $n \in \mathbb{N}$ of a collection $S_n \subset \mathcal{P}(\mathbb{R}^n)$ of subset of $\mathbb{R}^n$ such that:

1. $\forall n \in \mathbb{N}$, S_n is a boolean algebra (closed by $\cap, \cup, ^{\complement}$)
2. S is closed by projection and cartesian product: if $X \in S_n, Y \in S_m$, then $\pi_k^n(X) \in S_k$ and $X \times Y \in S_{m+n}$
3. the graph of $=$ belongs to S_2: $\{(x, y);\ x = y\} \in S_2$.

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The structure generated by $(A_i)_{i\in I}$, where $A_i \subset \mathbb{R}^{n_i}$ is the smallest structure that contains all A_i, $i \in I$.

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2. S is closed by projection and cartesian product: if $X \in S_n$, $Y \in S_m$, then $\pi^n_k(X) \in S_k$ and $X \times Y \in S_{m+n}$
3. S contains all semi-algebraic sets.

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Definition

Given $S = (S_n)_{n \geq 1}$ a structure on $\mathbb{R}$, $A \subset \mathbb{R}^n$ is definable in S if $A \in S_n$. $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is definable in S if its graph belongs to S_{n+m}.

Sets definable in $(\mathbb{R}, +, \times, \leq)$ are exactly the semi-algebraic sets.
Maps definable in $(\mathbb{R}, +, \times, \leq)$ are exactly the semi-algebraic maps.

Examples.

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Even if there are intermediate behaviors, there is roughly an alternative: defining few "nice" sets or a lot of "complicated" sets.

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10. $\mathbb{R}_f$ with $f \in C^{\infty}([0, 1])$ generic for the Withney topology.

Regularities

We fix an o-minimal structure S.

Proposition

Let $f : \mathbb{R} \to \mathbb{R}$ be definable. Then f is piecewise continuous: there exists $x_1, \ldots, x_k \in \mathbb{R}$ such that f is continuous on $\mathbb{R} \setminus \{x_1, \ldots, x_k\}$.

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Proposition

Let $h : \mathbb{R}^n \times (a, b) \to \mathbb{R}$. Then for all x, $\ell(x) := \lim_{t \to b} h(x, t)$ exists in $\mathbb{R} \cup \{-\infty, +\infty\}$ and $x \mapsto \ell(x)$ is definable.

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Proposition

Let $f : \mathbb{R} \to \mathbb{R}$ be definable. Then f is piecewise monotone: there exists $x_1, \ldots, x_k \in \mathbb{R}$ such that f is monotone on $\mathbb{R} \setminus \{x_1, \ldots, x_k\}$.

Proposition

Let $f : \mathbb{R} \to \mathbb{R}$ *be definable. Then* f *is piecewise differentiable: there exists* $x_1, \dots, x_k \in \mathbb{R}$ *such that* f *differentiable on* $\mathbb{R} \setminus \{x_1, \dots, x_k\}$.

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Proposition

If f is definable and has an asymptotic development of order n at a then f is C^n at a.
$$f(a+x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + o((x-a)^n).$$

Cell decomposition and finiteness properties

Definition

- A cell decomposition of $\mathbb{R}$ is a finite partition of $\mathbb{R}$ into points and intervals.
- A cell decomposition $\mathcal{C} = (C_i)_i$ of $\mathbb{R}^{n+1}$ is given by a cell decomposition $\mathcal{D}$ of $\mathbb{R}^n$, and for each $D \in \mathcal{D}$, continuous definable functions $f_j : D \to \mathbb{R}$, $j = 1, \ldots, m_D$, with

$$f_0 := -\infty < f_1 < \cdots < f_{m_D} < +\infty =: f_{m_D+1}.$$

The C_i's are the graphs of the f_j, $j = 1, \ldots, m_D$ and the strips:

$$(f_j, f_{j+1}) = \{(x, y) \in D \times \mathbb{R};\ x \in D,\ f_j(x) < y < f_{j+1}(x)\}$$

for $j = 0, \ldots, m_D$, for all cells $D \in \mathcal{D}$.

Theorem (Cell decomposition)

Let $A_1, \dots, A_k \subset \mathbb{R}^n$ be definable. Then there exists a cell decomposition $\mathcal{C} = (C_i)_{i \in I}$ of $\mathbb{R}^n$ adapted to $A_1, \dots, A_k$, this is, each A_i is a union on cells.

Any definable set is a finite union of sets definably homeomorphic to some $(0;1)^d$. The dimension of a definable set is the maximal d that appears in a cell decomposition.

Theorem (Finiteness)

If A is definable, A has finitely many connected components.

Theorem (Uniform finiteness)

If $A \subset \mathbb{R}^n \times \mathbb{R}^m$ is definable, then there exists $N \in \mathbb{N}$, such that all fibers $A_x = \{y \in \mathbb{R}^m;\ (x,y) \in A\}$ have less than N connected components.

Theorem (Definable choice)

Let $A \subset \mathbb{R}^{n+m}$ be definable and Π be the projection $\mathbb{R}^{n+m} \ni (x, y) \to x \in \mathbb{R}^n$. Then there exists a definable map $\varphi : \Pi(A) \to \mathbb{R}m + n$ such that $\Pi \circ \varphi = Id$ on $\Pi(A)$.

Theorem (curve selection)

Let A be definable and $x \in \overline{A}$. Then there exists a continuous definable $\gamma : (0, 1] \to A$ with $\lim_{t\to 0} \gamma(t) = x$.

A connected definable set is arcwise connected (by definable arcs).