On the geometry of Hodge loci

1. Motivation:

k field

$f: X \longrightarrow S/k$ smooth proper family of algebraic varieties

Goal: describe the motivic locus

i.e: the set of $s \in S$ / the fiber X_s is "simpler" than the very general fiber

$\Updownarrow$

X_s and its powers contain more algebraic cycles than the very general fiber and its powers.

In this talk: $k = \mathbb{C}$

Example 1: $S \subset \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}(d))$ the Zariski open subset parametrizing smooth surfaces of degree d in $\mathbb{P}^3$

$f : X \longrightarrow S$ universal family of such surfaces.

Noether: for $d > 3$ a general X_s satisfies $\mathrm{Pic}\, X_s = \mathbb{Z}[\mathcal{O}_X(1)]$: every curve on X_s is a complete intersection in $\mathbb{P}^3$ of X_s with another surface

Motivic locus for $f \supset$ Noether-Lefschetz locus $= \{ s \in S \,/\, \rho(s) > 1 \}$

This Noether-Lefschetz locus is dense in S

Example 2 $\quad f : \mathcal{E} \longrightarrow Y_0(1) = SL(2,\mathbb{Z}) \backslash \mathbb{H} \underset{j}{\simeq} \mathbb{C}$ universal family of elliptic curves

$$E_\tau = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \longmapsto [\tau]$$

$$\operatorname{End} E_\tau = \begin{cases} \text{order in } \mathbb{Q}(\tau) & \text{if } \tau \text{ imaginary quadratic} \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

Special points = CM points = $\{ j(\tau),\ \tau$ imaginary quadratic in $\mathbb{H} \}$

Example 3 $\quad f : \mathcal{A}_g \longrightarrow A_g$ universal principally polarized Abelian variety of genus g

$$\text{Hodge locus of } f \supset A_{g,k} := \{ s \in A_g,\ A_s \text{ admits an Abelian subvariety of dim } k \}$$

$$(1 \leq k \leq g-1)$$

$$\cup$$

$$\{ \text{CM points of } A_g \}$$

Hodge incarnation :

We understand little about algebraic cycles.

We "linearize" the problem :

$(X_s\,,\ \text{algebraic cycle}) \rightsquigarrow (H^{2i}(X,\mathbb{Z}),\ \text{Hodge class})$

$f : X \longrightarrow S \quad \rightsquigarrow \quad \mathbb{V} = R^{2i} f_* \mathbb{Z} \longrightarrow S$ $\quad$ $\mathbb{Z}$-variation of Hodge structure on S

Motivic locus for f $\rightsquigarrow$ Hodge locus $HL(S, \mathbb{V}^{\otimes}) = \{ s \in S \,/\, \mathbb{V}_s$ has exceptional Hodge tensors $\}$

Short reminder in Hodge theory

A Hodge structure on a $\mathbb{Z}$-module $V_{\mathbb{Z}}$ is:

a Hodge decomposition $V_{\mathbb{C}} (:= V_{\mathbb{Z}} \otimes \mathbb{C}) = \bigoplus_{p,q \in \mathbb{Z}^2} V^{p,q}$ with $\overline{V^{p,q}} = V^{q,p}$

$\Longleftrightarrow$

Hodge filtration $F^p V_{\mathbb{C}} = \bigoplus_{r \geq p} V^{r,s}$ $\quad (V^{p,q} = F^p \cap \overline{F^q})$

Hodge classes for $V_{\mathbb{Z}}$: $V_{\mathbb{Q}} \cap V^{0,0} = V_{\mathbb{Q}} \cap F^0 = V_{\mathbb{Q}}^{x=1}$

Mumford-Tate group for $V_{\mathbb{Z}}$: $G(V_{\mathbb{Z}}) = \overline{x(\mathbb{C}^*)}^{Zar, \mathbb{Q}} \subset GL(V_{\mathbb{Q}})$

= fixator in $GL(V_{\mathbb{Q}})$ of all Hodge tensors in $V_{\mathbb{Z}}^{\otimes}$.

Fact: if X smooth projective $/\mathbb{C}$ then $H^i(X,\mathbb{Z})$ admits a functorial polarizable $\mathbb{Z}HS$ (of weight i)

A $\mathbb{Z}$-variation of Hodge structure ($\mathbb{Z}VHS$) on a smooth quasi-projective $S/\mathbb{C}$ is

$$\mathbb{V} = (\mathbb{V}_{\mathbb{Z}}, (\mathcal{V}, \nabla, F^\bullet), Q)$$

$\mathbb{Z}$-local system of finite rank

filtered flat connection
$\nabla F^\bullet \subset \Omega^1_S \otimes F^{\bullet-1}$

$Q : \mathbb{V}_{\mathbb{Z}} \times \mathbb{V}_{\mathbb{Z}} \longrightarrow \mathbb{Z}_{S^{an}}$

such that: $\forall s \in S^{an}$, $(\mathbb{V}_{\mathbb{Z},s}, F^\bullet \mathcal{V}_s, Q)$ polarized $\mathbb{Z}HS$.

Example (geometric case): $f : X \longrightarrow S$ smooth projective

$$\mathbb{V} = (R^{2p} f_* \mathbb{Z}_{X^{an}}(p),\ R^{2p} f_* \Omega^\bullet_{X/S},\ \nabla^{GM},\ F^\bullet_{\text{stupid}},\ Q)$$

$HL(S, \mathbb{V}^{\otimes}) := \{ s \in S \,/\, \mathbb{V}_s$ has "exceptional Hodge tensors $\}$

$:= \{ s \in S,\ G_s := G(\mathbb{V}_s)$ is not generic $\}$

$:= \{ s \in S,\ \dim G_s$ not maximal $\}$

<u>Thm</u> (Cattani - Deligne - Kaplan) : for any $\mathbb{V}$ $HL(S, \mathbb{V}^{\otimes})$ is a countable union of irreducible algebraic subvarieties of S : the (strict) special subvarieties of S for $\mathbb{V}$.

$\{$ Special subvarieties $\} \supset \{$ Special points $\} \supset \{$ CM points : G_s torus $\}$

3. Period maps: special subvarieties as intersection loci

$\begin{array}{c}\mathbb{V}\\ \downarrow\\ S\end{array} \longleftrightarrow$ for each $s \in S^{an}$, $F_s^{\bullet} \subset V_{\mathbb{C}}$ Hodge filtration

$\longleftrightarrow$ for each $s \in S^{an}$, $x_s : \mathbb{C}^\times \longrightarrow GL(V_{\mathbb{R}})$

$\longleftrightarrow$ $\boxed{\Phi : S^{an} \longrightarrow \Gamma\backslash D \text{ holomorphic period map classifying } F_s^{\bullet}, \text{ or more precisely } x_s}$

Here: $D_S = \underline{G}_S(\mathbb{R})$-conjugacy class of $x_0 : \mathbb{C}^\times \longrightarrow \underline{G}_S(\mathbb{R})$, where $\underline{G}_S = MT_{gen}(S, \mathbb{V})$

$D_S \hookrightarrow D_S^\vee = G_S(\mathbb{C})/P$ flag variety $\qquad \Gamma_S = \underline{G}_S(\mathbb{Z})$

• Φ is horizontal: $d\Phi(TS) \subset T_h(\Gamma_S\backslash D_S)$ $\quad (\longleftrightarrow \nabla F^{\bullet} \subset F^{\bullet-1} \otimes \Omega^1_S)$

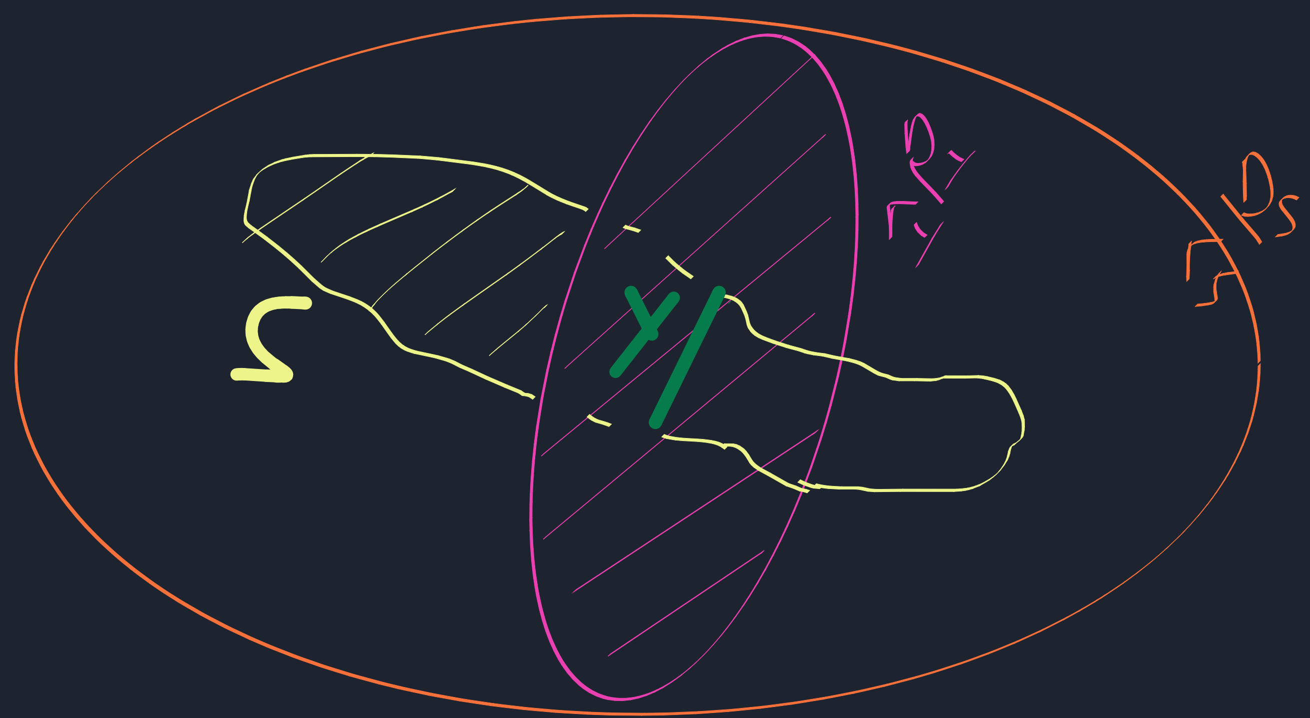
$T_h(\Gamma_S\backslash D_S) \subset T(\Gamma_S\backslash D_S)$ non integrable in general

Given $Y \subset S$, let $\underline{G}_Y := \mathrm{MT}_{gen}(Y, \mathbb{V}_{|Y}) \subset \underline{G}_S$

$$\begin{tikzcd} Y \arrow[r, "\Phi_S|_Y"] \arrow[d, hook] & \Gamma_Y\backslash D_Y \arrow[d] \\ S \arrow[r, "\Phi_S"'] & \Gamma_S\backslash D_S \end{tikzcd}$$

$\Longrightarrow Y \subset Y^{sp} := \Phi_S^{-1}(\Gamma_Y\backslash D_Y)^\circ$ special closure of Y for $\mathbb{V}$

In particular: $Y \subset S$ is <u>special</u> for $\mathbb{V}$ $\iff Y = \Phi_S^{-1}(\Gamma_Y\backslash D_Y)^\circ$



$Y \subset S$ is <u>atypical</u> for $\mathbb{V}$ if

$$\operatorname{codim}_{\Phi(S)}(\Phi(Y)) < H\text{-}\operatorname{codim}_{\Gamma_S \backslash D_S}\left(\Gamma_Y \backslash D_Y\right) := \operatorname{rg} T_h D_S - \operatorname{rg} T_h D_Y$$

<u>Questions</u>:

1/ Can we describe the distribution of the special subvarieties in S?

For instance: $\overline{HL(S, \mathbb{V}^{\otimes})}^{Zar}$?

2/ Same question for the atypical locus $HL(S, \mathbb{V}^{\otimes})_{atyp} = \bigcup_{\substack{Y \subset S \\ Y \text{ atypical}}} Y$

Main conjecture (K., 2016): Let $\mathbb{V} \to S$ be an (admissible, graded polarizable) $\mathbb{Z}$VMHS. Then S contains only finitely many maximal atypical subvarieties for $\mathbb{V}$.

Example 1: $S \hookrightarrow Sh = \Gamma_S \backslash D_S$, $\mathbb{V}$ standard

$Y \subset S$ atypical $\iff \operatorname{codim}_S Y < \operatorname{codim}_{S^{sp}} Y^{sp}$

Main conjecture $\iff$ Zilber-Pink conjecture

Example 2: Main conjecture $\implies$

Conjecture (K., André-Oort for $\mathbb{Z}$VHM): Let $\mathbb{V} \to S$ a $\mathbb{Z}$VMHS.

Suppose the union of CM points is Zariski-dense in S.

Then $\Phi_S : S \xrightarrow[\text{dominant}]{} Sh = \Gamma_S \backslash D_S$

4/ Results

The first result deals with general special subvarieties.

Thm 1 (K; Otwinowska 2020) Let $\mathbb{V} \longrightarrow S$ ZVHS (pure). Suppose G_S^{ad} is simple.

Let $HL(S, \mathbb{V}^{\otimes})_{pos} :=$ union of (strict) special subvarieties whose image under the period map is positive dimensional.

Then $HL(S, \mathbb{V}^{\otimes})_{pos}$ is a finite union of strict special subvarieties of S for $\mathbb{V}$, or it is Zariski-dense in S.

Cor: $S \hookrightarrow A_g$ Hodge generic.

Then either $(S \cap HL(A_g))_{pos}$ is a finite union of special subvarieties of S, or it is Zariski-dense in S.

(Colombo-Pirola; Izadi; Chai): if $\mathrm{codim}_{A_g} S \leq g$ then $HL(S, \mathbb{V}^{\otimes})_{pos}$ is dense in S for the usual topology).

The second result deals with atypical intersections of positive period dimension:

"Thm 2 (Baldi - K. - Ullmo)": $\mathbb{V} \longrightarrow S$; G_S^{ad} simple

Then $HL(S, \mathbb{V}^{\otimes})_{pos, atyp}$ is a finite union of special subvarieties.

(Daw-Ren for Shimura)

"Cor": $\mathbb{V} \longrightarrow S$

Suppose the union of positive dimensional special subvarieties $\begin{cases} \cdot \text{ of Shimura type} \\ \cdot \text{ with dominant period map} \end{cases}$ is Zariski-dense in S.

Then S is of Shimura type with dominant period map.

- Notice that in both Thm 1 and Thm 2 we are considering only "geometric" objects, namely special subvarieties of positive period dimension.

  We are thus excluding <u>special points</u>, as our methods are purely geometric.

- When S and $\mathbb{V}$ are defined over $\overline{\mathbb{Q}}$, one expects the special subvarieties to be defined over $\overline{\mathbb{Q}}$ (in the geometric case this follows from the conjecture that Hodge classes are absolute Hodge)

<u>Thm</u> (K.-Otwinowska-Urbanik) $\mathbb{V} \to S$ with $\mathbf{G}_S^{ad}$ simple

Then any $Y \subset S$ special of positive period dimension and maximal for these properties is defined over $\overline{\mathbb{Q}}$.

The case of special points seems much harder!

5/ Methods: o-minimality and functional transcendence

Thm A (conj. by K.; Bakker-Tsimerman; Gao-Klingler, Chiu)

Let $\mathbb{V} \longrightarrow S$ $\mathbb{Z}$VHS

$$\begin{tikzcd}
S^{an} \times \check{D}_S & S^{an} \times D_S \arrow[l, hook] & \Delta := S^{an} \times_{\Gamma_S \backslash D_S} D_S \arrow[l, hook] \arrow[r, "f"] \arrow[d] & D_S \arrow[d] \\
 & & S \arrow[r, "\Phi_S"] & \Gamma_S \backslash D_S
\end{tikzcd}$$

Let $Z \subset \Delta$ $\mathbb{C}$-analytic irreducible.

Then $\dim Z^{Zar} - \dim Z \geq \dim p(Z)^{ws}$

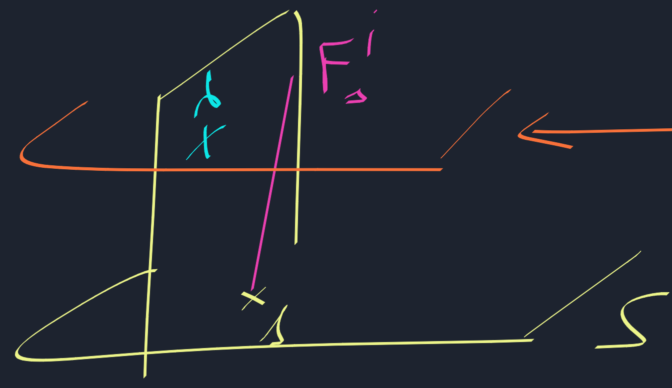
Thm B (Bakker-(Brunebarbe)-K.-Tsimerman)

Period maps $\Phi_S : S^{an} \longrightarrow \Gamma_S \backslash D_S$ are naturally definable in $\mathbb{R}_{an, exp}$.

Proof of Thm 1

$\mathbb{V} = (\mathbb{V}_{\mathbb{Z}}, \mathcal{V}, F^\cdot, \nabla, Q)$

$$\begin{array}{c} \mathcal{V} \\ p \big\downarrow \\ S \end{array}$$



$\mathbb{V}(\lambda)$ flat leaf of $\lambda \subset \mathcal{V}^{an}$

$\cup$

$\mathbb{V}^i(\lambda) := \mathbb{V}(\lambda) \cap F^i \subset \mathcal{V}^{an}$

$\downarrow$

$S^i(\lambda) := p(\mathbb{V}^i(\lambda)) \subset S$

$\boxed{\text{For } i=0,\ \lambda \in \mathbb{V}_{\mathbb{Q}}}$

$\mathbb{V}^0(\lambda) =$ locus in $\mathcal{V}^{an}$ where the flat transport of λ is Hodge

$\downarrow$

$S^0(\lambda) =$ locus of $s \in S^{an}$ at which some determination of λ is Hodge

$\boxed{\text{For general } i \in \mathbb{Z},\ \lambda \in \mathbb{V}_{\mathbb{C}}}$

Both $\mathbb{V}^i(\lambda) \subset \mathbb{V}(\lambda)$ have a natural $\mathbb{C}$-analytic structure:
analytic subsets of the étalé space of $\mathbb{V}_{\mathbb{C}}$

But if $\lambda \notin \mathbb{P}\mathbb{V}_{\mathbb{Q}}$ they are not in general $\mathbb{C}$-analytic subvarieties of $\mathcal{V}^{an}$.
A priori $S^i(\lambda)$ is geometrically ugly.

Two ingredients:

Thm a: Given $\mathbb{V} \longrightarrow S$ define

$$\begin{array}{l} \mathcal{V} \supset \mathbb{V}^i_{\geq d} := \bigcup_{\lambda \in \mathbb{V}_{\mathbb{C}}}^{n.t} \mathbb{V}^i(\lambda)_{\geq d} \\ \downarrow \\ S \supset S^i(\mathbb{V})_{\geq d} := p(\mathbb{V}^i_{\geq d}) \end{array}$$

Then
$$\begin{tikzcd} \mathbb{V}^i_{\geq d} \arrow[r, hook, "alg"'] \arrow[d] & F^i\mathcal{V} \arrow[d, "p"] \\ S^i(\mathbb{V})_{\geq d} \arrow[r, hook, "alg"'] & S \end{tikzcd}$$

$\Uparrow$

Thm a': $p : (\mathcal{V}, \nabla) \longrightarrow S$ algebraic flat connection

$F \subset \mathcal{V}$ algebraic subvariety. Fix $d > 0$

For $x \in F$, define $N_{F,x}$ = union of irreducible $\mathbb{C}$-analytic components of $(\mathbb{V}/x) \cap F)^{red}$ through x $\subset F^{an}$

Then $A_{F, \geq d} := \{ x \in F, \dim_x N_{F,x} \geq d \} \underset{alg}{\subset} F$.

Cor A $\overline{\mathbb{V}^i_{\geq d} \cap \mathbb{V}_{\mathbb{Q}}}^{Zar}$ is "saturated" by the $\mathbb{V}^i(x)_{\geq d}$:

$$\exists\, U \underset{Z\text{-open}}{\subseteq} \overline{\mathbb{V}^i_{\geq d} \cap \mathbb{V}_{\mathbb{Q}}}^{Zar} \text{ such that } U \subset \bigcup_{x \in U} \mathbb{V}^i(x)_{\geq d} \subset \overline{\mathbb{V}^i_{\geq d} \cap \mathbb{V}_{\mathbb{Q}}}^{Zar}$$

Thm b: $\forall i \in \mathbb{Z}, \forall \lambda \in \mathbb{V}_{\mathbb{C}}$ $\overline{S^i(\lambda)^{\circ}}^{Zar}$ is weakly special

Cor B (of Cor A and Thm b): $\overline{S^i(\mathbb{V}_{\mathbb{Q}})_{\geq d}}^{Zar}$ is saturated by weakly special subvarieties

$$\exists\, U \subseteq \overline{S^i(\mathbb{V}_{\mathbb{Q}})_{\geq d}}^{Zar} \,/\, U \subset \bigcup_{x \in U} Y_x \subset \overline{S^i(\mathbb{V}_{\mathbb{Q}})_{\geq d}}^{Zar}$$

weakly special through x of dimension $\geq d$

Proof of Thm 1: for simplicity we assume Φ_S immersive

$$HL(S, \mathbb{V})_{>0} = S^{\circ}(\mathbb{V}^{\otimes}_{\mathbb{Q}})_{\geq 1} \underset{\text{Deligne}}{=} \bigcup_{a=1}^{n} S^{\circ}(\mathbb{V}_{a,\mathbb{Q}})_{\geq 1} \qquad \mathbb{V}_a \subset \mathbb{V}^{\otimes}$$

$$\underset{\text{WLOG}}{=} S^{\circ}(\mathbb{V}_{\mathbb{Q}})_{\geq 1}$$

Cor B $\Rightarrow \exists U \subsetneq \overline{HL(S,\mathbb{V}^{\otimes})_{>0}}^{Zar} \,/\, \forall x \in U,\ \exists / Y_x$ weakly special $\begin{cases} x \in Y_x \\ \dim Y_x \geq 1 \end{cases}$

with $Y_x \subset \overline{HL(S,\mathbb{V}^{\otimes})_{>0}}^{Zar}$

Either $\exists x \,/\, Y_x = S \Rightarrow HL(S,\mathbb{V}^{\otimes})_{>0} \underset{Zar-dense}{\subset} S$

Or: $\forall x \in X \quad Y_x \subsetneq S$

G_S^{ad} simple $\Rightarrow Y_x \subset W_x \underset{special}{\subsetneq} S$

But then $\overline{HL(S,\mathbb{V}^{\otimes})_{>0}}^{Zar} \supset HL(S,\mathbb{V}^{\otimes})_{>0} \supset \bigsqcup \underset{Z\text{-dense}}{\subsetneq} \overline{HL(Z,\mathbb{V}^{\otimes})_{>0}}$

$\Rightarrow \overline{HL(S,\mathbb{V}^{\otimes})_{>0}}^{Zar} = HL(S,\mathbb{V}^{\otimes})_{>0}$ is a finite union of special subvarieties.