Motivic Vitushkin invariants

Georges COMTE
(Savoie Mont Blanc - UMR CNRS 5127)

Immanuel HALUPCZOK
(Düsseldorf)

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Introduction: real integral geometry as motivation

$X \subset \mathbb{R}^n$ a compact definable set in a given o-minimal structure over $\mathbb{R}$.

Denote for $\varepsilon > 0$

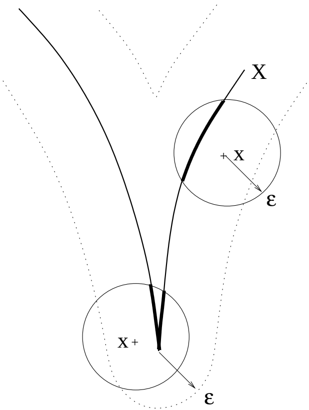
$$\boxed{\mathrm{Vol}_\varepsilon := \int_{x\in\mathbb{R}^n} \chi(X \cap B(x,\varepsilon))\ \mathrm{d}x}$$

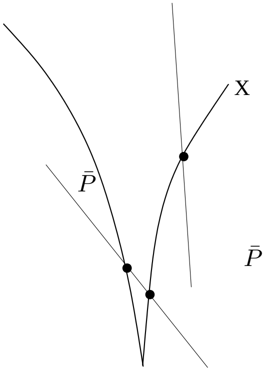
- In case X is convex or smooth, $\mathrm{Vol}_\varepsilon(X) = \mathrm{Vol}_{\mathbb{R}^n}(T_\varepsilon(X))$.
- Vol_ε is additive, since χ and $\int$ are so.

Theorem. $$\boxed{\mathrm{Vol}_\varepsilon(X) = \sum_{i=0}^{n} \Lambda_i(X)\varepsilon^{n-i}} \qquad \boxed{\Lambda_i(X) = \int_{\bar{P}\in\bar{G}_n^{n-i}} \chi(X \cap \bar{P})\ \mathrm{d}\bar{P}}$$

$\bar{G}_n^{n-i}$: affine $(n-i)-$vector planes $\bar{P} \subset \mathbb{R}^n$.

Introduction: real integral geometry as motivation





$$\mathrm{Vol}_\varepsilon := \int_{x \in \mathbb{R}^n} \chi(X \cap B(x,\varepsilon))\ \mathrm{d}x \qquad \Lambda_i(X) = \int_{\bar{P} \in \bar{G}_n^{n-i}} \chi(X \cap \bar{P})\ \mathrm{d}\bar{P}$$

Introduction: real integral geometry as motivation

Replace χ by $\boxed{V_0 := \#\text{ connected components}}$ in the definition of Λ_i.

- $V_0(X \cup Y) \geq V_0(X) + V_0(Y) - V_0(X \cap Y)$.

Vituskin invariants. $\boxed{V_i(X) = \int_{\bar{P} \in \bar{G}_n^{n-i}} V_0(X \cap \bar{P}) \, \mathrm{d}\bar{P}}$

- $i > \dim(X)$: $V_i(X) = 0$
- $i = \dim(X)$: Cauchy-Crofton formula $V_i(X) = \Lambda_i(X) = \mathrm{Vol}_{\dim(X)}(X)$

Introduction: real integral geometry as motivation

Metric entropy. $\boxed{M_\varepsilon(X)}$ the minimal number of ε-balls needed to cover X.

Theorem [Ivanov 1975]. $$\boxed{M_\varepsilon(X) \leq C(n) \sum_{i=0}^{n} V_i(X)\varepsilon^{-i}}$$

$$\updownarrow$$

$$\boxed{\mathrm{Vol}_\varepsilon(X) = \sum_{i=0}^{n} \Lambda_i(X)\varepsilon^{n-i}}$$

Applications (Yomdin). Quantitative Sard theorem for the entropy dimension.

Introduction: from the real field to nonarchimedean fields?

Motivation. Find nonarchimedean analogues of V_i and M_ϵ, and relate them to obtain a nonarchimedean analogue of Ivanov's theorem.

Difficulties.

- Find a notion of nonarchimedean connected components.
- Find a notion of preorder $\preceq$ on the ring of integrable motivic constructible functions & compatible wrt motivic integration.

  Basically one would compare the motive $[X^2 - 2] \in SK_0(\mathrm{RDef_k})$ and 1 in this way: $1 \preceq [X^2 - 2]$.

1- Nonarchimedean setup

Notation. K a valued field, k its residual field and Γ its value group ($\simeq \mathbb{Z}$), π a uniformizer of the valuation ring $\mathscr{O}_K$, $\mathscr{M}_K$ the maximal ideal, $|x| := \mathbb{L}^{-\mathrm{val}(x)}$.

Definable sets. $\mathscr{L}$ the 3-sorted language (VF, RF, VG),

- ring language for VF and RF,
- Presburger language $(+, -, 0, 1, \leq, \cdot \equiv \cdot \bmod_n, n \geq 1)$ for VG,
- symbol val for the valuation $\mathrm{val} : \mathrm{VF} \to \mathrm{VG}$,
- symbol $\overline{\mathrm{ac}}$ for the angular component $\overline{\mathrm{ac}} : \mathrm{VF} \to \mathrm{RF}$, interpreted as $\overline{\mathrm{ac}}(x) = \pi^{-\mathrm{val}(x)}x \mod \mathscr{M}_K$.

Theory. We consider a complete $\mathscr{L}$-theory $\mathscr{T}$ of discrete valued fields of equi-characteristic 0, which is 1-h-minimal [Cluckers-Halupczok-Rideau 2020] ($\longleftrightarrow$ o-minimality for $\mathbb{R}$).

- Always $K \models \mathscr{T}$.

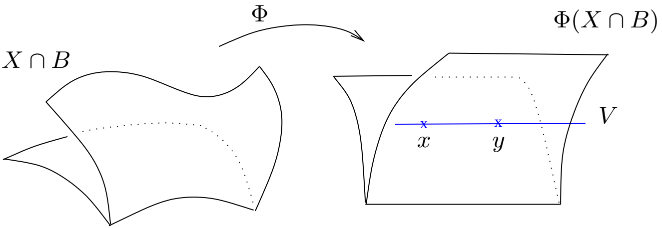
Concrete example. Henselian discrete valued field of equi-characteristic 0. For instance $K = k((t))$, $\mathrm{char} k = 0$, with symbols in $\mathscr{L}$ for all analytic functions $\mathscr{O}_K^n \to K$, $n \geq 1$.

2- t-stratifications and motivic connected components

Translatability on a ball. $X \subset K^n$ bounded definable, $B \subset K^n$ ball, $V \subset K^n$ vector subspace.

X is *V-translatable on B* if there exists a bijection $\Phi : B \to B$ s.t.

- $\forall x, y \in B$, if $x - y \in V$ then $x \in \Phi(X \cap B) \iff y \in \Phi(X \cap B)$,
- $\mathrm{val}(\Phi(y) - \Phi(x) - (y - x)) < \mathrm{val}(y - x)$.



One says that X is *d-translatable on B* if there exists V of dimension d s.t. X is V-translatable on B.

Existence of t-stratifications [Halupczok 2014]. $\forall d < n$, $\exists S_d \subset K^n$, $S_d = \emptyset$ or $\dim(S_d) = d$, such that: $\forall B \subset K^n \setminus S_d$, X is $(d+1)$-translatable on B.

3- t-stratifications and motivic connected components

- S_0 is finite ($S_0 \subset K^n$ and $\dim(S_0) \le 0$).
- In case $\dim(X) = 0$ one can take $S_0 = X$.
- Let $\ell := \min \#\{S;\ X \text{ is 1-translatable on } K^n \setminus S\}$,

$$\boxed{\Sigma_0 = \bigcup_{\#S=\ell} S}$$

Proposition. • Σ_0 is definable and $\Sigma_0 = \coprod_{i=1}^{\ell} B_i$, B_i : ball or singleton.

• X is 1-translatable on every $B \subset K^n \setminus S \iff S = \coprod_{i=1}^{\ell} \{p_i\}$, $p_i \in B_i$.

Definition. Let $\boxed{\Sigma_{0,mot} \subset k^m}$, $\#\Sigma_{0,mot} = \ell$, definably parameterizing Σ_0:

$$\Sigma_0 \twoheadrightarrow \Sigma_{0,mot} \text{ with fibers the } B_i\text{'s}$$

$$\boxed{V_0(X) := \mu_0(\Sigma_{0,mot})}$$

- If each point of $\Sigma_{0,mot}$ is definable $V_0(X) = \#\Sigma_{0,mot} = \ell \in SK_0(\mathrm{RDef}_k)$.

- But in general the situation is like $V_0(X) = [X^2 - 2] \in SK_0(\mathrm{RDef}_k)$.

4- Motivic integration

- $Z \subset K^n \times k^m \times \Gamma^r$ definable, $\boxed{SK_0(\mathrm{RDef}_Z)}$ the Grothendieck semiring of definable sets $\subset Z \times k^p \to Z$, $\quad [Z \times k] := \mathbb{L}$.
- $\boxed{A := \mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}, (\frac{1}{1-\mathbb{L}^i})_{i>1}]}$, $\boxed{A_+}$ semigroup of positive elements of A, after evaluating $\mathbb{L}$ at any $r > 1$.
- $\boxed{\mathscr{P}(Z)}$ subring of A^Z generated by constant functions, definable functions $Z \to \mathbb{Z}$, $\mathbb{L}^\beta$ with $\beta : Z \to \mathbb{Z}$ definable, $\boxed{\mathscr{P}_+(Z)}$ elements of $\mathscr{P}(Z)$ with values in A_+.

Motivic constructible functions. $\boxed{C_+(Z) := SK_0(\mathrm{RDef}_Z) \otimes_{\substack{[Y]=1_Y \\ \mathbb{L}-1=\mathbb{L}-1}} \mathscr{P}_+(Z)}$

Theorem [Cluckers-Loeser 2008] Functor (+ axiomatic) of motivic integration:

$$\boxed{f_! : I_Z C_+(X) \to I_Z C_+(Y)} \quad \text{for} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

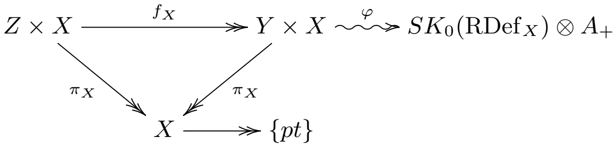
$$\boxed{f_!(\varphi)(y) \text{ ``}=\text{''} \int_{x \in f^{-1}(y)} \varphi}$$

5- A convenient partial preorder on $C(X)$

$X \subset K^n \times k^m \times \Gamma^r$ definable.

Preorder on $C(X)$. Say that $F \in C(X)$ is $\succeq 0$ if there exist

- $Y, Z \in k^{m'} \times \Gamma^{r'}$,
- a definable surjection $f : Z \twoheadrightarrow Y$ with finite fibres,
- $\varphi \in I_{\{pt\}} C_+(Y \times X)$



such that

$$\boxed{F = \pi_{X!} f_X^* \varphi - \pi_{X!} \varphi}$$

More intuitively $\boxed{F(x) \text{ “=” } \int_{z \in Z} \varphi \circ f_X(z, x) - \int_{y \in Y} \varphi(y, x).}$

Denote $\boxed{C_{\succeq 0}(X)} := \{F \in C(X); F \succeq 0\}$.

5- A convenient partial preorder on $C(X)$

Properties.

- $\succeq 0$ is compatible with positivity wrt A_+: $I_{\{pt\}}C_+(X) \subset C_{\succeq 0}(X)$.
- $\succeq 0$ is compatible with addition. In particular

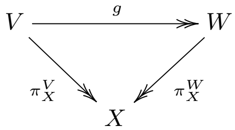
$$\boxed{F \preceq G \iff G - F \succeq 0}$$

  is a partial preorder.

- $\succeq$ is compatible with integration: $F, G \in I_V C(U \times V)$

$$F \preceq G \Longrightarrow \pi_{V!}F \preceq \pi_{V!}G.$$

- $V, W \subset X \times K^p$, $g : V \to W$ finite definable surjection,



  Then $F := \pi_{X!}^V(\mathbf{1}_V) \succeq \pi_{X!}^W(\mathbf{1}_W)$.

  In particular $[X^2 - 2] \succeq 1$ (as measures in $C(\{pt\})$).

6- Motivic Vitushkin invariants

Motivic Vitushkin invariants. $$\boxed{V_i(X) = \int_{\bar{P} \in \bar{G}_n^{n-i}} V_0(X \cap \bar{P}) \, \mathrm{d}\bar{P}}$$

Motivic Cauchy-Crofton formula. If $\dim(X) = d$, $\boxed{V_d(X) = C(n, d)\mu_d(X)}$.

It uses uniform Taylor approximation [Cluckers-Halupczok-Rideau 2020]

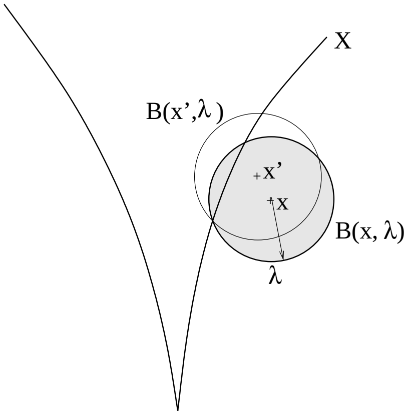
- For the real local C-C formula [C. 2000],
- For the motivic local C-C formula [Forey 2020].

Motivic metric entropy. For $\lambda \in \mathbb{L}^{\mathbb{Z}}$,

$$\boxed{M_\lambda(X) := \lambda^{-n}\mu_n\left(\{x \in K^n,\ B(x, \lambda) \cap X \neq \emptyset\}\right)}$$

Note that $\lambda^{-n} = \mu_n(B(0, \lambda))$.

7- Motivic metric entropy



Theorem.

$$\boxed{M_\lambda(X) \preceq \widetilde{C}(n) \sum_{i=0}^{n} V_i(X)\lambda^{-i}}$$

$$\updownarrow$$

$$\boxed{M_\varepsilon(X) \leq C(n) \sum_{i=0}^{n} V_i(X)\varepsilon^{n-i}}$$