O-minimality and foliations (CIRM 2021)

Talk III: applications of foliation point-counting

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Plan of the talk

We'll focus on two flavors of applications:

Galois orbit lower bounds

- For torsion points in abelian varieties.
- For special points in Shimura varieties.

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- For relative Manin-Mumford.
- An application to Pell's equation $A^2 - DB^2 = 1$ over $\mathbb{C}[t]$.

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Foliations everywhere!

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Let A be a genus g abelian variety over number field $\mathbb{K}$.

Theorem (David)

Let $P \in A$ be torsion of order n. Then

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- Same holds for $2P, 3P, \ldots, (n-1)P \in \mathbb{K}(P)$.
- We get n different $([\mathbb{K}(P) : \mathbb{K}], h_{\text{Fal}}(A) + \log n)$-points on Γ.

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- We get n different $([\mathbb{K}(P) : \mathbb{K}], h_{\text{Fal}}(A) + \log n)$-points on Γ.
- Γ contains no algebraic curves.
- Counting: $n = \text{poly}(h_{\text{Fal}}(A), \log n, [\mathbb{K}(P) : \mathbb{Q}])$.

Crucial point: counting is polynomial in $g = [\mathbb{K}(P) : \mathbb{Q}]$.

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- $\mathcal{F}$ generated by $\partial_{z_j} + \xi_j$ and ∂_{u_i}.
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- $V = \{z = \Omega \cdot u\}$.
- $\Phi : \mathbb{M} \to \mathbb{C}^{2g}, \quad \Phi(z, x, u, \Omega) = (x, u)$.
- $A := \Phi(V \cap \mathcal{B})$ is Γ with u in “period coordinates”. (But note u is complex).

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- Foliation counting: (g, h)-points contained in few blocks W_α.
- Func. trans.: each block W_α corresponds to a single $z = \Omega \cdot u$.
- Periods are $\mathbb{R}$-independent $\implies$ block contains at most one real u.

Conclusion: bounds for number of $(x, u) \in \Gamma(g, h)$.

Galois lower bounds for special points

Theorem (Tsimerman)

If $p \in \mathcal{A}_g$ *is special,* $[\mathbb{Q}(p) : \mathbb{Q}] \geqslant \mathrm{disc}(p)^c$ *for some* $c = c_g > 0$.

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- Using Pila's strategy this implies André-Oort for $\mathcal{A}_g$.
- Proof uses Masser-Wüstholz isogeny estimates.
- Also uses discriminant-negligible height property (DNH): $h(p) < \mathrm{disc}(p)^{\varepsilon}$.
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Theorem (B.-Schmidt-Yafaev)

For every Shimura S, DNH $\implies$ Galois lower bound.

- In particular new proof of Tsimerman's theorem.
- Replace Masser-Wüstholz by counting, works for every S.

Proof using counting

$\pi : \Lambda \to S$ the universal cover of a Shimura variety S. There is:

- a principal G-bundle $P \to S$, a connection ∇ and a leaf $\mathcal{L} \subset P$,
- a map $\Phi : P \to \check{X} \times S$ such that

$$\Phi(\mathcal{L}) = \mathrm{graph}\ \pi$$

Can use foliations to count algebraic point on the graph of π.

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- If p is special of disc. d and degree g, then there are at least d^c specials p_i with the same d, g.
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Under the rug

- Difficulty connected to non-compactness of Λ.
- Resolved by asymptotic analysis near the boundary (crucial that ∇ is regular-singular).

André-Oort for modular curves

Denote

- $\mathbb{H}$ the upper half-space, $\mathcal{F} \subset \mathbb{H}$ the usual $\mathrm{SL}_2(\mathbb{Z})$ fund. domain.
- $Y(1) := \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \cong \mathbb{C}$ the modular curve.
- $j : \mathbb{H} \to Y(1)$ the Klein modular invariant map.

Some points are more special than others

- $p \in Y(1)$ represent an elliptic curve E_p. If E_p has endomorphisms other than $\mathbb{Z}$ then p is called *CM* or *special*.
- $p = j(\tau)$ is special iff τ is a quadratic number.

Consider a curve $V = \{P(p, q) = 0\} \subset Y(1)^2$ over $\mathbb{Q}$.

Some curves are more special than others

V is called *special* if P is one of the *modular polynomials* Φ_N.

André-Oort for modular curves (cont.)

André-Oort conjecture for $Y(1)^2$

If V is not special then it contains finitely many special pairs (p, q).

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This case is due to André. Rough idea of Pila's proof:

- Write $\pi := (j, j) : \mathcal{F}^2 \to Y(1)^2$. Consider $X = \pi^{-1}(V)$.
- X contains no semialgebraic curves (not easy, also uses P-W).
- P-W $\implies$ $\#X(4, H) = O_{X,\varepsilon}(H^\varepsilon)$ for any $\varepsilon > 0$.
- On the other hand, let $(p, q) = j(\tau_1, \tau_2) \in V$ be special and $H = H(\tau_1, \tau_2)$.
- Class field theory $\implies$ (p, q) has $c_1 H^{c_2}$ Galois conjugates.
- Each conjugate corresponds to a point in $X(4, H)$. Contradiction for $H \gg 1$.

Using foliation counting

The Schwarzian differential equation

The modular invariant $j : \mathbb{H} \to \mathbb{C}$ satisfies a differential equation

$$S(j) + R(j)(j')^2 = 0, \qquad R(j) := \frac{j^2 - 1968j + 2654208}{2j^2(j-1728)^2}.$$

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- The graph of $j^{\times 2} : \mathbb{H}^2 \to \mathbb{C}^2$ can be realized as a leaf $\mathcal{L}$ of a foliation.
- One can replace counting in $\mathcal{F}^2$ by counting in some big ball $\mathbb{B}$ (either using equidistribution or height bounds).
- Instead of P-W, we can apply our result to $\Phi(\mathcal{L} \cap \{P(p,q) = 0\})$ where $\Phi = (\tau_1, \tau_2)$.

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What do we gain? For instance, polynomial dependence on δ_P.

Theorem (Andé-Oort for $V \subset \mathbb{C}^n$)

There is a polynomial time algorithm for computing all special points in V.

We do not know what the algorithm is...

Unlikely intersections in abelian schemes

Consider a family of abelian surfaces $\lambda : A \to \mathbb{C}$ and $C \subset A$ an irreducible curve, both defined over some number field $\mathbb{K}$.

Theorem (Masser-Zannier)

If C is not contained in a proper subgroup of A, it contains finitely many torsion points.

- Many extension by Masser, Zannier, Barroero, Capuano, Schmidt...
- Interesting applications: Pell, Elementary integration, Billiards...

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Theorem (B.)

In fact, every torsion $c \in C$ has order at most $\mathrm{poly}(\delta_C, [\mathbb{K} : \mathbb{Q}])$. In particular, set of torsion points effectively computable in polynomial time.

- This is the natural expected asymptotic.
- Very likely works for the various extensions...

Proof sketch

Let $A_\lambda := \mathbb{C}^2/\Lambda_\lambda$. Then

1. $c \in C$ is torsion in $A_{\lambda(c)} \iff$ its logarithm in $\mathbb{C}^2$ is a rational combination of the generators of $\Lambda_{\lambda(c)}$.
2. The order of torsion N is roughly the height of the coefficients.
3. P-W: the number of $c \in C$ that are N-torsion is $O_C(N^\varepsilon)$.
4. If $c \in C$ is N-torsion then $[\mathbb{Q}(c) : \mathbb{Q}] \geqslant \text{const}(C) N^d$ for $d > 0$.
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Foliations everywhere!

1. The generators of Λ_λ are given by complete hyperelliptic integrals, satisfy homogeneous Picard-Fuchs type equations.
2. The $A_{\lambda(c)}$-logarithm of c is an incomplete hyperelliptic integral, satisfies inhomogeneous Picard-Fuchs type equation.

Foliation point-counting replaces $O_C(N^\varepsilon)$ by $\text{poly}(\delta_C, [\mathbb{K} : \mathbb{Q}], \log N)$.

Pell's equation over $\mathbb{C}[t]$

Consider Pell's equation for fixed $D \in \mathbb{C}[X]$

$$A^2 - DB^2 = 1$$

and try to solve with non-zero $A, B \in \mathbb{C}[X]$. In the integers this is always solvable unless D is a perfect square. How about with polynomials?

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Theorem (Masser-Zannier '15)

Let $D = X^6 + X + t$. Pell's equation is only solvable for finitely many t.

Actually there is a criterion that works for any $D \in \mathbb{C}[X, t]$, we consider this example for simplicity.

Theorem (B.)

The set of t is effectively computable in polynomial time (with D as input).

Reduction to torsion points

Fox fixed $t \in \mathbb{C}$ consider the hyperelliptic curve $C_t := \{Y^2 = D(t)\}$. If a polynomial solution $A, B \in \mathbb{C}[x]$ to Pell exists,

$$(A - YB)(A + YB) = 1, \tag{1}$$

then $A - YB$ is a regular function on C_t without poles or zeros, except at the two points at infinity.

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then $A - YB$ is a regular function on C_t without poles or zeros, except at the two points at infinity.
Thus in the Jacobian A_t of C_t,

$$0 = [A - YB] = m(\infty_1 - \infty_2). \tag{2}$$

In particular, $s(t) := \infty_1 - \infty_2 \in A_t$ is torsion.

Conclusion

By effective M-Z we can find all t where this happens.

Thank you for your attention!

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...and hoping to see you at the Fields in Toronto!

Thematic Program on

Tame Geometry, Transseries and Applications to Analysis and Geometry

(Fields Inst., January–June, 2022)