O-minimality and foliations (CIRM 2021)

Talk II: point counting with foliations

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Growth vs. zeros

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Many problems in dynamics/arithmetic can be reduced to counting roots of analytic functions (e.g. Poincaré maps, period integrals).

One common approach based on value distribution theory:

Lemma ("Jensen")

The number of zeros $N_K(f)$ of a holomorphic function $f : \bar{U} \to \mathbb{C}$ on a compact $K \subset U$ is bounded by

$$N_K(f) \leqslant \text{const}(U, K) \cdot \ln \frac{\max_{z \in \bar{U}} |f(z)|}{\max_{z \in K} |f(z)|}.$$

So, if you can get

- An upper bound for f in U; and
- A lower bound for f in K,

then you have an upper bound for $N_K(f)$.

A result for vector fields

Let ξ be a rational vector field in $\mathbb{C}^n$, defined over a number field $\mathbb{K}$.
Let γ be a trajectory and $K \subset \gamma$ be a compact subset.

For P polynomial, let $\Sigma_P :=$ the union of trajectories where P vanishes identically.

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Theorem (B. IMRN '2016)

For any polynomial P of degree $d = \deg P$ we have

$$\#[K \cap \{P = 0\}] \leqslant \text{const}(K, \xi) \cdot d^{2n^2} \log d \cdot \log \text{dist}^{-1}(K, \Sigma_P).$$

- The key point is that this grows polynomially with $\deg P$.
- In the paper most of the work went to showing that the $\log \text{dist}$ term can be eliminated under a certain differential Galois theoretic condition. But for this talk we do not need this.

Multiplicity estimates

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We need the following estimate for orders of polynomials.

Theorem

Suppose p *is a non-singular point of* ξ. *Then*

$$\operatorname{ord}_p^\xi P \leqslant C_\xi d^n.$$

- First proved by Neserenko. Different approach by Gabrielov, B.
- Important in transcendental number theory.
- Singular case also interesting but not for this talk.

Idea for the proof of zero-counting

Growth-vs-zeros is well suited for getting this kind of polynomial bound:

1. Easy to get $\log|P| \leqslant \text{poly}(d)$ on a big domain U.
2. Main difficulty: get lower bound for $\log|P|$ on the compact K.

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2. Main difficulty: get lower bound for $\log|P|$ on the compact K.
3. By Cauchy estimates, enough to find lower bounds for $\log|\xi^k P| \geqslant -\text{poly}(d)$ for some suitable k.
4. The common zeros of $P, \xi^1 P, \xi^2 P, \ldots$ are exactly Σ_P.
5. Moreover by *multiplicity estimates* (Nesterenko, Gabrielov) it is enough to consider $P, \xi P, \ldots, \xi^N P$ where $N \sim d^n$.

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6. By diophantine Łojasiewicz inequality of Brownawell, these guys are simultaneously small only near their common zero locus, i.e. Σ_P.
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A similar statement for higher-dimensional foliations would have many applications. But can we do growth-vs-zeros in several variables?

Higher dimension?

Now let $\boldsymbol{\xi} = \xi_1, \ldots, \xi_n$ be a collection of commuting vector fields in $\mathbb{M} := \mathbb{C}^N$ defined over a number field $\mathbb{K}$.

For a variety $V \subset \mathbb{M}$ of codimension n, denote by Σ_V the points where V has improper intersection with the leaf at p.

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For a variety $V \subset \mathbb{M}$ of codimension n, denote by Σ_V the points where V has improper intersection with the leaf at p.

- Denote by $\mathbb{B} \subset \mathbb{M}$ the unit ball.
- Denote by $\mathcal{B} \subset \mathcal{L}$ a ball in the $\boldsymbol{\xi}$-coordinates.
- Denote by δ_V the maximum of the degree $\deg V$ and log-height $h(V)$. Similarly for $\delta_{\boldsymbol{\xi}}$.

Theorem (B. '18)

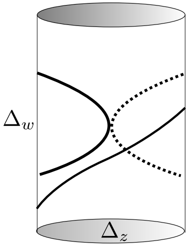
Suppose $\operatorname{codim} V = n$ *and* $\mathcal{B} \subset \mathbb{B}$. *Then*

$$\#[\mathcal{B} \cap V] \leqslant \operatorname{poly}(\delta_V, \delta_{\boldsymbol{\xi}}, \log \operatorname{dist}^{-1}(\mathcal{B}, \Sigma_V)).$$

Weierstrass Polydiscs

Weierstrass Polydisc

If $X \subset \mathbb{C}^n$ is a purely m-dimensional analytic set, we say that a polydisc $\Delta = \Delta_z \times \Delta_w$ is a *Weierstrass polydisc* for X if $\dim \Delta_z = m$ and $X \cap (\Delta_z \times \partial\Delta_w) = \emptyset$.



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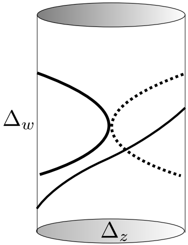

Consider n equations in n variables (*) $P_1 = \cdots = P_n = 0$.

Main idea

Suppose we manage to cover $X = \{P_1 = \cdots = P_{n-1} = 0\}$ by Weierstrass polydiscs. Then the solutions of (*) corresponds to zeros of the *resultant*

$$\mathcal{R}(z) = \prod_{w:(z,w)\in X\cap\Delta} P_n(z,w),$$

and we can try to apply growth-vs-zeros to $\mathcal{R}$.

Covering by Weierstrass polydiscs

For this to work, we must be able to cover the curve X by Weierstrass polydiscs. I illustrate for curves $X \subset \mathbb{C}^2$.

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Let $S^1 = \{|\lambda| = 1\}$ act on $\mathbb{C}^2$ by multiplication.

- We are looking for two "big" orthogonal discs Δ_z, Δ_w with $\Delta_z \times \partial\Delta_w$ disjoint from X.
- If B is any ball, $S^1 \cdot B$ contains a set $\Delta_z \times \partial\Delta_w$ of the required form!
- Enough to find a ball B with $S^1 \cdot B$ disjoint from X.
- Equivalently, we want B disjoint from $S^1 \cdot X$.

Question

How to find a ball disjoint from $S^1 \cdot X$?

So far strategy is similar to first talk.

Covering by Weierstrass polydiscs: complex ideas

Step 1: A bound on the volume of X

1. A complex variant of Crofton's formula says that $\mathrm{vol}(X)$ can be estimated in terms of the intersection of X with complex lines.
2. Can think of these lines as leafs of a one-dimensional foliation.
3. By induction we get a bound on $\mathrm{vol}(X)$.

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Step 2: Find a ball disjoint from $S^1 \cdot X$

1. Fact (Lelong): if X passes through the center of a complex ball B of radius ε then $\mathrm{vol}(X \cap B) \geqslant \mathrm{const} \cdot \varepsilon^2$.
2. We can choose ε^{-4} disjoint ε-balls in the unit ball of $\mathbb{C}^2$ s.t. only $\sim \mathrm{vol}(X)\varepsilon^{-2}$ meet X.
3. Only $\sim \mathrm{vol}(X)\varepsilon^{-3}$ balls meet $S^1 \cdot X$.
4. Once $\varepsilon \lesssim 1/\mathrm{vol}(X)$ we find an ε-ball disjoint from $S^1 \cdot X$.

Multiplicity estimates: higher dimension

Let $P_1, \ldots, P_n$ be polynomials.

Definition

$\mathrm{mult}_p^{\mathcal{F}}(P_1, \ldots, P_n)$: *multiplicity of p as a common zero of* $P_1 = \cdots = P_n = 0$ *on* $\mathcal{L}_p$.

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We need the following estimate for multiplicities of polynomials.

Theorem

Suppose $p \in \mathcal{L}$. *Then*

$$\text{mult}_p^{\mathcal{F}}(P_1, \ldots, P_n)|_{\mathcal{L}} \leqslant C_{\mathcal{F}} d^{2n}.$$

- Proved by Gabrielov-Khovanskii.
- Recently improved to d^n (B.), also important in transcendental number theory.

Wrap up

Assume inductively that we can

Count interections $\mathcal{B} \cap V$ with $\dim \mathcal{B} = \operatorname{codim} V = n$.

Now consider $\dim \mathcal{B} = n + 1$ and $W = V \cap \{P = 0\}$ for $\operatorname{codim} V = n$.

1. Cover the curve $\mathcal{B} \cap V$ by Weierstrass polydiscs.
2. Construct $\mathcal{R}(z)$ with zeros corresponding to $\mathcal{B} \cap W$.
3. Upper bound for $\log |\mathcal{R}(z)|$: easy.
4. Lower bound: Multiplicity estimates + "multiplicity operators" (w. Novikov) $\implies$ bound depending on $\log \operatorname{dist}^{-1}(\mathcal{B}, \Sigma_W)$.
5. Growth-vs-zero: bound for $\#(\mathcal{B} \cap W)$.

Unlike first talk, we actually use Weierstrass polydiscs also to count zeros.

Counting algebraic points

For a set $A \subset \mathbb{C}^N$ let

$$A(g, h) := \{\mathbf{x} \in A : [\mathbb{Q}(\mathbf{x}) : \mathbb{Q}] \leqslant g, h(\mathbf{x}) \leqslant h\}$$

where $h(\cdot)$ is the logarithmic Weil height.

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Proposition (Bombieri-Pila, Wilkie and B.-Novikov)

Let Δ be a Weierstrass polydisc for X. Then there exists a polynomial $P \in \mathbb{Z}[\mathbf{x}]$ which vanishes on $[\Delta \cap X](g,h)$ and

$$\delta_P = \operatorname{poly}(g,h,\nu), \qquad \textit{where } \nu := \deg(X \to \Delta_z).$$

Main idea

- Using Thue-Siegel we cook up P that's very small on X.
- P turns out to be so small that it must vanish on (g,h) points.

Counting algebraic points (simple version)

Theorem

Suppose $V \cap \mathcal{L}$ contains no semialgebraic curves for any leaf $\mathcal{L}$. Fix a $\boldsymbol{\xi}$-ball $\mathcal{B} \subset \mathbb{B}$. Then

$$\#[\mathcal{B} \cap V](g, h) = \operatorname{poly}(\delta_{\boldsymbol{\xi}}, \delta_V, g, h).$$

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1. Pila-Wilkie works for fixed g and has $e^{o(h)}$ instead of $\mathrm{poly}(h)$.
2. Improving $e^{o(h)}$ to $\mathrm{poly}(h)$ is "Wilkie's conjecture".
3. Getting polynomial dependence on g has important consequences in nubmer theory.

Main point

Diophantine applications of Pila-Wilkie always involve transcendental sets coming from leafs of foliations!

Counting algebraic points (final version)

- $V \subset \mathbb{M}$ and $\Phi : \mathbb{M} \to \mathbb{C}^\ell$ a rational map, both over $\mathbb{K}$.
- Fix $\boldsymbol{\xi}$-ball $\mathcal{B} \subset \mathbb{B}$.
- Set $A = \Phi(\mathcal{B} \cap V)$.

Suppose $\Phi|_{V \cap \mathcal{L}}$ has zero-dimensional fibers, for every leaf $\mathcal{L}$.

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Definition (ε-nearby block)

$\mathbb{Q}$-*variety* $W \subset \mathbb{C}^\ell$ *such that* $W_p \subset \Phi(\mathcal{L}_p \cap V)$ *for some* $p : \operatorname{dist}(p, \mathcal{B}) < \varepsilon$.

Theorem

For every $\varepsilon > 0$ *there exists a collection of* ε*-nearby blocks* W_α *with*

$$A(g, h) \subset \cup_\alpha W_\alpha$$

and

$$\#\{W_\alpha\}, \max_\alpha \delta_{W_\alpha} \leqslant \operatorname{poly}(\delta_{\boldsymbol{\xi}}, \delta_V, \delta_\Phi, g, h, \log \varepsilon^{-1}).$$