# O-minimality and foliations (CIRM 2021)

## Talk I: from Pila-Wilkie to Wilkie's conjecture

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# Bombieri-Pila on analytic curves

Let $\mathbf{f} := (f_1, f_2) : I \to \mathbb{R}^2$ be an analytic map and write $\Gamma = \mathbf{f}(I)$.

**Theorem (Bombieri-Pila Duke '89 (+Pila Duke '91))**

*Suppose $\Gamma$ does not belong to an algebraic curve in $\mathbb{R}^2$. Then for every $\varepsilon > 0$ there exists $C = C(\Gamma, \varepsilon)$ such that*

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$$\#\Gamma(\mathbb{Q}, H) \leqslant C \cdot H^{\varepsilon}.$$

- $\Gamma(\mathbb{Q}, H) := \{x \in \Gamma : H(x) \leqslant H\}$.
- Can certainly fail if $\Gamma$ is algebraic!
- Asymptotic basically optimal for general analytic curves.

**Main step**

All points of $\Gamma(\mathbb{Q}, H)$ belong to $\sim H^{\varepsilon}$ algebraic curves of degree $d = d(\varepsilon)$.

# Interpolation determinants

**Question**

*How do we show a collection of points $P = \subset \mathbb{R}^2$ all satisfy an algebraic relation $P(x, y) = 0$ of degree $d$?*

- Let $\mathbb{P}_d$ be the space of polynomials of degree $\leqslant d$.
- $\mu = \dim \mathbb{P}_d = (d+1)(d+2)/2$.
- $e_{\mathbf{p}} : \mathbb{P}_d \to \mathbb{C}, \qquad e_{\mathbf{p}}(P) = P(\mathbf{p})$.

**Q:** When do $\{e_{\mathbf{p}} : p \in P\}$ have a common kernel?

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**Q:** When do $\{e_{\mathbf{p}} : p \in P\}$ have a common kernel?

## Answer

Iff for every $\mathbf{p}_1, \ldots, \mathbf{p}_\mu \in P$ the following determinant vanishes,

$$\Delta^d(\mathbf{p}_1, \ldots, \mathbf{p}_\mu) := \det(\mathbf{x}^\alpha(\mathbf{p}_j))_{\substack{j=1,\ldots,\mu \\ |\alpha|\leqslant d}}. \tag{1}$$

# Bombieri-Pila: the main estimate, holomorphic setting

**Proposition**

*Let* $z_1, \ldots, z_\mu \in D$ *with* $|z_j| < \delta$ *and let* $g_1, \ldots, g_\mu : D \to \mathbb{C}$ *be holomorphic and bounded by* $1$. *Then* $|\det(g_i(z_j))| \lesssim \delta^{\sim \mu^2}$.

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Expand each $g_i = \sum c_{ik} z^k = \sum_{k=0}^{\mu-1} c_{ik} z^k + r_i(z)$ as a Taylor series and expand

$$\det \begin{pmatrix} g_1(z_1) & \cdot & g_\mu(z_1) \\ & \ddots & \\ g_1(z_\mu) & \cdots & g_\mu(z_\mu) \end{pmatrix} \tag{2}$$

by multilinearity in the columns.

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- So each of $z^0, z^1, \ldots, z^{\mu-1}$ can appear at most once.
- Thus each summand has order at least $\sim \delta^{1+\cdots+\mu}$.

# Bombieri-Pila: the end

Let $J$ be an interval of length $\delta$, and $P = [\mathbf{f}(J)](\mathbb{Q}, H)$.

- With $(x, y) = (f_1, f_2)$, have $|\Delta^d| \lesssim \delta^{\sim\mu^2} = \delta^{\sim d^4}$.
- Either $\Delta^d = 0$ or $|\Delta^d| \gtrsim H^{\sim(-d^3)}$, because we know the common denominator of the determinant $\Delta^d$.

$\delta \lesssim H^{-1/d} \implies \Delta^d = 0 \implies P$ is contained in a curve of degree $d$.

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Subdivide $I$ into intervals of length $\delta = H^{-\varepsilon}$; apply the above to each.

**Conclusion**

$\Gamma(\mathbb{Q}, H)$ is contained in $H^\varepsilon$ algebraic curves $\{P_j = 0\}$, each of degree $d = d(\varepsilon) \sim 1/\varepsilon$.

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**Conclusion**

$\Gamma(\mathbb{Q}, H)$ is contained in $H^\varepsilon$ algebraic curves $\{P_j = 0\}$, each of degree $d = d(\varepsilon) \sim 1/\varepsilon$.

Final claim: the number of intersections $\Gamma \cap \{P_j = 0\}$ is bounded by some constant depending only on $d$. (o-minimality).

# Going beyond curves: new problems

Let $A \subset \mathbb{R}^n$ be a trascendental set of higher dimension.

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No! $A$ might still contain algebraic curves with many rational points.

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Let $A \subset \mathbb{R}^n$ be a trascendental set of higher dimension.

**Should we still expect $\#A(\mathbb{Q}, H) = O(H^\varepsilon)$?**

No! $A$ might still contain algebraic curves with many rational points.

- Let $A^{\mathrm{alg}}$ be the union of all semialgebraic curves in $A$.
- Let $A^{\mathrm{trans}} := A \setminus A^{\mathrm{alg}}$. Hope: extend everything to $A^{\mathrm{trans}}$.

**Theorem (Pila-Wilkie)**

*Let $A$ be definable in an o-minimal structure. Then for every $\varepsilon > 0$ there exists $C = C(A, \varepsilon)$ such that*

$$\#A^{\mathrm{trans}}(\mathbb{Q}, H) \leqslant C \cdot H^\varepsilon. \tag{3}$$

# Idea of proof for P-W Theorem

By induction on $m := \dim A$:

1. Cover $A$ by finitely many images of maps $\mathbf{f} : I^m \to A$.
2. For each map, construct $H^\varepsilon$ hypersurfaces $S_\alpha$ of degree $d = d(\varepsilon)$ containing $[\mathbf{f}(I^m)](\mathbb{Q}, H)$.
3. Continue inductively for each intersection $A \cap S_\alpha$.

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One should understand how to cover each of the intersections $A \cap S_\alpha$ by images $\mathbf{f}(I^m)$, uniformly over $S_\alpha$.

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P-W's breakthrough:

1. One can take $\mathbf{f}$ to be $C^r$ smooth instead of analytic (for some large $r$);
2. One can construct $C^r$ parameterizations uniformly over $\alpha$: in the semialgebraic case this is the "Yomdin-Gromov algebraic lemma", generalized to o-minimal structures by P-W.

# Restricted Wilkie's Conjecture

## Conjecture (Wilkie Conjecture)

*Suppose $A$ is definable in $\mathbb{R}_{\mathrm{exp}}$. Then there exists $C = C(A)$ and $\kappa = \kappa(A)$ such that*

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Consider the structure $\mathbb{R}^{\mathrm{RE}}$: similar to $\mathbb{R}_{\exp}$, but instead of allowing $e^x$ we allow *restricted* elementary functions: $\exp|_{[0,1]}$ and $\sin|_{[0,\pi]}$.

**Theorem (B., Novikov.)**

*Let $A$ be definable in $\mathbb{R}^{\mathrm{RE}}$. Then there exists $C = C(A)$ and $\kappa = \kappa(A)$ such that*

$$\#A^{\mathrm{trans}}(\mathbb{Q}, H) \leqslant C \cdot (\log H)^{\kappa}. \tag{4}$$

The constant $\kappa$ is computed and $C$ is computable in principle.

# What's so special about restricted elementary functions?

These functions and their complexification are *Pfaffian*: can expressed as polynomial combinations of solutions of a triangular system of algebraic differential equations:

$$(e^x)' = e^x$$
$$(\tan x)' = 1 + \tan^2 x$$
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**Khovanskii's fewnomial theory**

A kind of "Bezout theorem" for Pfaffian functions: number of solutions for systems of equations depend polynomially on degrees.

We'll hear a lot more about these in other talks!

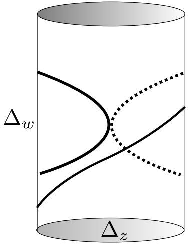
# Weierstrass Polydiscs

The main difficulty in sharpening the P-W proof is the construction of $C^r$-smooth parametrizations.

To overcome this we introduce the following complex-analytic notion:

**Definition (Weierstrass Polydisc)**

*If $X \subset \mathbb{C}^n$ is a purely m-dimensional analytic set, we say that a polydisc $\Delta = \Delta_z \times \Delta_w$ is a Weierstrass polydisc for X if* $\dim \Delta_z = m$ *and* $X \cap (\Delta_z \times \partial \Delta_w) = \emptyset$.




When restricting to a Weierstrass polydisc, the projection from $X$ to $\Delta_z$ is a finite ramified map. We denote its degree by $e(X, \Delta)$.

# Interpolation on Weierstrass polydiscs

Let $f : \Delta^{1/3} \to \mathbb{C}$ be holomorphic on a polydisc three times larger than $\Delta$.

**Proposition ("Weierstrass division")**

*There exists a function $P$ holomorphic in $\Delta_z$ and polynomial of degree at most $e(X, \Delta) - 1$ in each of the $w$ variables which agrees with $f$ on $X \cap \Delta$. Moreover $\|P\|_\Delta \leqslant 3^{n-m} \|f\|_{\Delta^{1/3}}$.*

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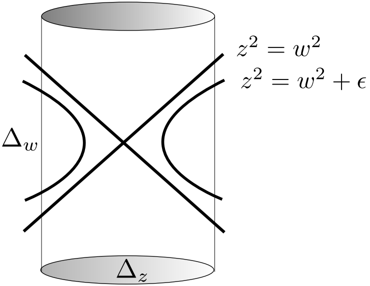
**Example**

*Suppose $X \subset \mathbb{C}^2$ is given by $z^2 = w^2 + \varepsilon$ with $|\varepsilon| < 1$. Then $D_1 \times D_2$ is a Weierstrass polydisc. Given $f(z, w)$, we can construct $P$ by replacing each $w^2$ in the Taylor expansion by $z^2 - \varepsilon$:*
$f(z, w) = wf_1(z, w^2) + f_0(z, w^2) \implies$
$P(z, w) = wf_1(z, z^2 - \varepsilon) + f_0(z, z^2 - \varepsilon)$.



# Exploring rational points in Weierstrass polydiscs

**Proposition (following Bombieri-Pila)**

$(X \cap \Delta^{2})(\mathbb{Q}, H)$ *is contained in an algebraic hypersurface of degree*

$$d = (e(X, \Delta) + \log H)^{O(1)}. \tag{5}$$

- Before we had $H^{\varepsilon}$ subcubes and $d = d(\varepsilon)$.
- Now we need $O(1)$ polydiscs, but $d$ depends on $H$.

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The proof is similar to Pila-Wilkie, but

## Main point

We can replace Taylor approximations by "Weiertrass division" for the key estimate – no longer need smooth charts.

# Bombieri-Pila: the main estimate for $X := z^2 - w^2 = \varepsilon$

## Proposition

*Let* $z_1, \dots, z_\mu \in X \cap \Delta^{1/\delta}$ *and let* $g_1, \dots, g_\mu : \Delta \to \mathbb{C}$ *be holomorphic and bounded by* 1*. Then* $|\det(g_i(z_j))| \lesssim \delta^{\sim \mu^2}$*.*

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# Proof of restricted Wilkie conjecture

1. Start with some set $A$ defined using restricted elementary formulas.
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Similar arguments give a proof of the P-W theorem for $\mathbb{R}_{\mathrm{an}}$.

Step 3 is the heart of the argument for polylog bounds.

# Covering by Weierstrass polydiscs

We consider the following elementary question which contains all the key ideas for the general statement.

## Question

*Let* $\Gamma \subset \mathbb{C}^2$ *be an algebraic curve of degree* $d$*. Find a Weierstrass polydisc centered at the origin, contained in the unit ball and containing a ball of radius* $1/\operatorname{poly}(d)$ *around the origin.*

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First main idea:

- Need: orthogonal discs $\Delta_z, \Delta_w$ with $\Delta_z \times \partial\Delta_w$ disjoint from $\Gamma$.
- If $B$ is any ball far from the origin, $S^1 \cdot B$ contains a set $\Delta_z \times \partial\Delta_w$ of the required form!
- Enough to find a ball $B$ with $S^1 \cdot B$ disjoint from $\Gamma$.
- Equivalently, we want $B$ disjoint from $S^1 \cdot \Gamma$.

# Covering by Weierstrass polydiscs (cont.)

**Question**

*How can we find a large ball disjoint from $S^1 \cdot \Gamma$?*

# Covering by Weierstrass polydiscs (cont.)

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- Related to metric entropy.
- We use standard tool: Vitushkin's formula.

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*How can we find a large ball disjoint from $S^1 \cdot \Gamma$?*

- Related to metric entropy.
- We use standard tool: Vitushkin's formula.

Consider a simpler question to make drawing easier:

**Question**

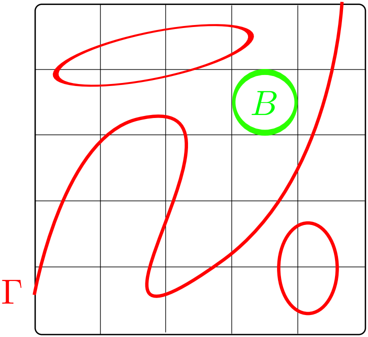
*Let $\Gamma \subset \mathbb{R}^2$ be a real algebraic curve of degree $d$. Find a ball of radius $1/\operatorname{poly}(d)$ in the unit square which is disjoint from $\Gamma$.*

- In our case, $S^1 \cdot \Gamma$ is a real-codimension-one set in $\mathbb{R}^4$.
- Essentially the same proof works.

The proof is a picture!

# Covering by Weierstrass polydiscs (end)

1. Divide $[0,1]^2$ into an $N \times N$ grid.
2. The number $V_0$ of conn. components of $\Gamma$ is at most $d^2$ (Harnack).
3. The number $V_1$ of intersections between $\Gamma$ and a line in the grid is at most $d$ (Bezout).
4. $\Gamma$ meets at most $V_0 + 2NV_1 \sim d^2 + 2Nd$ cells out of the $N^2$ cells in the grid.
5. Conclusion: once $N \gg d$ we can find a cell that doesn't meet $\Gamma$.



$N = 5$

In dimension $n$ with $\Gamma$ any algebraic (or even sub-Pfaffian) set we will have $N^n$ cells, and $\Gamma$ meets at most $N^{n-1} \cdot \text{poly}(d)$ of them.

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...and hoping to see you at the Fields in Toronto!

Thematic Program on

**Tame Geometry, Transseries and Applications to Analysis and Geometry**

(Fields Inst., January–June, 2022)