Regular projections and regular covers in o-minimal structures

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Abstract

We prove that for any definable subset $X \subset \mathbb{R}^n$ in a polynomially bounded o-minimal structure, with $dim(X) < n$, there is a finite set of regular projections (in the sense of Mostowski[6]). We discribe also a weak version of this theorem in any o-minimal structure proved by Nhan Nguyen, and we give a counter example in o-minimal structures that are not polynomially bounded. As an application we show that in any o-minimal structure there exists a regular covers in the sense of Parusiński[2].

Background

The regular projections were introduced by Mostowski for the proof of the existence of Lipschitz stratification in the complex analytic case, extended to the subanalytic case by A.Parusiński. Then they were used by A.Parusiński in the proof of the existence of regular subanalytic covers.

Let $\mathcal{A}$ be an o-minimal structure on $\mathbb{R}$. Let $X \subset \mathbb{R}^n$ be a definable subset of $\mathbb{R}^n$. For any $\lambda \in \mathbb{R}^{n-1}$, We denote by $\pi_{|\lambda} : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$, the projection parallel to the vector $(\lambda, 1) \in \mathbb{R}^n$. Fix ε and $p \in \mathbb{N}^\star$. For $\lambda \in \mathbb{R}^{n-1}$ and $x \in \mathbb{R}^n$ we define the cone $C_\varepsilon(x, \lambda)$ by :
$$C_\varepsilon(x, \lambda) = \{x + t(\lambda', 1) \ : \ t \in \mathbb{R}^\star \ and \ \lambda' \in B(\lambda, \varepsilon)\}$$

Definition: We say that the projection π_λ is (ε, p)- **weak regular** at a point $x \in \mathbb{R}^n$(with respect to X) if :

(1) $\pi_{|X}$ is finite.

(2) the intersection of X with the cone $C_\varepsilon(x, \lambda)$ is either empty or a disjoint finite union of sets of the form :
$$A_{f_i} = \{x + f_i(\lambda')(\lambda', 1) \ : \ \lambda' \in B(\lambda, \varepsilon)\}$$
Where f_i are non-vanishing C^p functions defined on $B(\lambda, \varepsilon)$.
We say that the projection π_λ is (ε, C, p)- **regular** at a point $x \in \mathbb{R}^n$ (with respect to X) if we have :

(3) $| grad(f_i) | \leq C \mid f_i \mid$ on $B(\lambda, \varepsilon)$ for all i.

Counter example in non-polynomially bounded o-minimal structures

Fix $\mathcal{A}$ a non-polynomially bounded o-minimal structure on $(\mathbb{R}, +, .)$. Consider the set X defined by :
$$X = X_1 \cup X_2$$
Where :
$$X_1 = \{(x, ax^{a+1}, x^{a+1}) \ : \ x > 0 \ and \ a \in \mathbb{R}\}$$
$$X_2 = \{(x, -ax^{a+1}, x^{a+1}) \ : \ x > 0 \ and \ a \in \mathbb{R}\}$$
By Miller's Dichotomy theorem[5], the graph of the exponential map $x \longrightarrow exp(x)$ is a definable set, and therefore X is a definable set.

Then we prove that there is no finite set of regular projections for the germ of the curve $x(s) = (s, 0, 0)$ at $(0, 0, 0)$.

References

References

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The weak regular projection theorem

Let $\mathcal{A}$ be an o-minimal structure on $\mathbb{R}$ (no condition on $\mathcal{A}$).

Theorem 1: Let X be a definable subset of $\mathbb{R}^n$ such that $dim(X) < n$, and $p \in \mathbb{N}^\star$. Then there is $\varepsilon > 0$ and $\{v_1, ..., v_k\} \subset \mathbb{R}^{n-1}$ such that for every $x \in \mathbb{R}^n$ there is i such that π_{v_i} is (ε, p)-weak regular at x with respect to X.

The **Proof** of this theorem is a direct consequence of the next lemma :

Lemma 1: (See [7], Lemma3]) Take C a definable subset of $\mathbb{R}^n$, and let B to be a box in $\mathbb{R}^k$. Let Δ be a definable subset of $\mathbb{R}^n \times \mathbb{R}^k$ such that $dim(\Delta_x) < k$, for all $x \in C$. Then there exists a finite definable partition $\mathcal{C}$ of C, such that for each $D \in \mathcal{C}$ there is a box $B_D \subset B$ such that we have :
$$(D \times B_D) \cap \Delta = \emptyset$$

Remark:

(1) We can easily extend the last lemma to the case of C^p-definable manifolds (we can replace $\mathbb{R}^k$ and $\mathbb{R}^n$ by definable manifolds) by choosing charts and reducing to the case of the last lemma.

(2) From the proof of Theorem 1 we can require $\{v_1, ..., v_k\}$ to be from an open subset of $(\mathbb{R}^{n-1})^k$.

Existence of Regular covers

Take $\mathcal{A}$ an o-minimal structure on $\mathbb{R}$. Let U be an open definable subset of $\mathbb{R}^n$. By a regular cover of U, we mean a finite cover (U_i) by open definable sets, such that :

(1) each U_i is homeomorphic to the open $n - dimensional$ ball of $\mathbb{R}^n$ by a definable homeomorphism

(2) there is a positive number C such that for all x in $\mathbb{R}^n$ we have :
$$d(x, \mathbb{R}^n \setminus U) \leq C max_i d(x, M \setminus U_i)$$

Theorem 3: For any open definable subset U of $\mathbb{R}^n$ there exist a regular cover.

Idea of the proof: We take $A = \partial U = \overline{U} \setminus U$. Take Take $\Lambda = \{\pi_1, ..., \pi_k\}$ a set of ε-weak regular projection with respect to A. For every projection π_i we define the sets :
$$R_i = \{x \in U \ : \ \pi_i \ is \ \varepsilon - weak \ regular \ at \ x \ with \ respect \ to \ A\}$$
By induction on the dimension, we choose a regular covers of the sets $\pi_i(R_i)$, and then we take this regular covers back to U by the projections π_i.

The regular projection theorem in polynomially bounded o-minimal structures

Let $\mathcal{A}$ be a polynomially bounded o-minimal structure on $\mathbb{R}$.

Theorem 2: Let A be definable subset of $\mathbb{R}^n$ such that $dim(A) < n$ and take $p \in \mathbb{N}^\star$. Then there is $\varepsilon > 0$, $C > 0$, and $\{v_1, ..., v_k\} \subset \mathbb{R}^{n-1}$ such that for every $x \in \mathbb{R}^n$ there is $i \in \{1, ..., k\}$ such that π_{v_i} is (ε, C, p)- regular at x with respect to A.

Idea for the proof: before talking about the idea of the proof we introduce the next notion :

Definition: Let X be a definable subset of a definable manifold N, and Ω a definable open subset of $\mathbb{R}^m$. Take $f : X \times \Omega \longrightarrow \mathbb{R}$, $(x, y) \mapsto f(x, y)$ a definable C^1 map. We say that f is X-rectifiable with respect to y, if we can find a definable partition $\mathcal{P}$ of X, and $c > 0$ such that for every $D \in \mathcal{P}$ there is a box $B_D \subset \Omega$ such that :
$$\forall x \in D \text{ and } \forall y \in B_D : \mid D_y f(x, y) \mid \leq c \mid f(x, y) \mid$$

By theorem 1, we can find a cell decomposition $\mathcal{C} = \{C_1, ..., C_k\}$ of $\mathbb{R}^n$, and $\varepsilon > 0$ such that for every $i \in \{1, ..., k\}$ there is $v_i \in \mathbb{R}^{n-1}$ such that π_{v_i} is (ε, p)-weak regular at every point $x \in C_i$ with respect to A. Hence for every $i \in \{1, ..., k\}$ there are C^p maps:
$$f_{i,l} : C_i \times B(v_i, \varepsilon) \longrightarrow \mathbb{R}^\star$$
$$(x, v) \mapsto f_{i,l}(x, v)$$
such that we have :
$$C_\varepsilon(x, v_i) \cap X = \sqcup_l \{x + f_{i,l}(x, v)(v, 1) \ : \ v \in B(v_i, \varepsilon)\}$$
Hence to finish the proof of theorem 2 it's enough to prove that the maps $f_{i,l}$ are C_i-rectifiable with respect to y, and for that we prove the next fundamental Lemma :

Lemma 2: Let X be a definable subset of $\mathbb{S}^n$, Ω be an open definable subset of $\mathbb{R}^m$, and $x_0 \in \partial X$. Let $f : X \times \Omega \longrightarrow \mathbb{R}$, $(x, y) \mapsto f(x, y)$ be a definable C^1 map. Then there is an a neighborhood U of x_0 in $\mathbb{S}^n$ such that $f_{|U \cap X \times \Omega}$ is $U \cap X$-rectifiable with respect to y , means that we can find $\alpha > 0$, $c > 0$, $\mathcal{C}$ a definable partition of X, and $\mathcal{B}$ a finite collection of boxes in Ω such that for every $C \in \mathcal{C}$ with $x_0 \in \overline{C}$, there is $B_C \in \mathcal{B}$ such that :
$$\forall x \in C \text{ with } d(x, \ x_0) < \alpha, \forall y \in B_C :$$
$$\mid D_y f(x, y) \mid \leq c \mid f(x, y) \mid$$

Idea for the proof of Lemma2 : First we reduce the proof to the case of $n = 1$, and then after that we use Proposition5.2 in [4], wing lemma, and definable choice to proof the case of $n = 1$.