Holonomy and equivalence of singular analytic foliations

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Abstract

We show a generalization of the well-known result obtained by Mattei-Moussu [4] about the relation between the conjugation of holonomies along a separatrix and the analytic classification of the germs of singular foliations.

Introduction

We treat the class of germs of singular one dimensional analytic foliations in $\mathbb{C}^{n+1}$ called crossing type. A **crossing type** foliation in $(\mathbb{C}^{n+1}, 0)$ is a triple $(\mathcal{F}, H, \Gamma)$ such that:

1. $\mathcal{F}$ is a germ of 1-dimensional analytic foliation.
2. H is a smooth hyper-surface and Γ is a smooth invariant curve such that:
   (a) H and Γ are transverse at the origin.
   (b) Both are invariant by the foliation $\mathcal{F}$.
3. Each local generator of $\mathcal{F}$ has a nonzero eigenvalue in the Γ-direction.

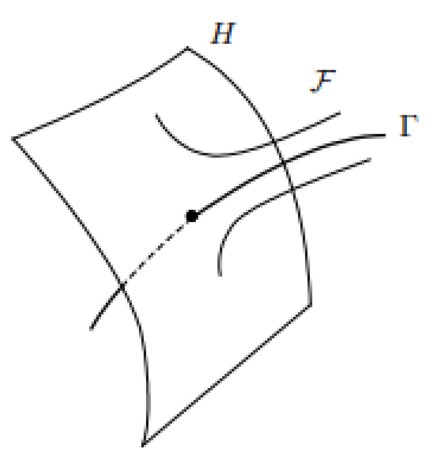

Figure 1: *Transversality of H and Γ at origin*

We say that two crossing type foliations $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ are **analytically equivalent** if there exists an analytic change of coordinates mapping the leaves of $\mathcal{F}$ to the leaves of $\mathcal{G}$ and the pair (H, Γ) to (L, Ω).

The following theorem is the main result in this poster. To prove it, we adapted and generalized an idea originally introduced by Arame Diaw [2, 3]. Below, we denote by **Γ-holonomy** and **Ω-holonomy** the respective local holonomies along the curves Γ of $\mathcal{F}$ and Ω of $\mathcal{G}$.

Theorem 1 (Main Theorem) *Let $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ be two crossing type foliations such that:*

(a) The linear part of the local generators of $\mathcal{F}$ and $\mathcal{G}$ are conjugated.

(b) The local generators of $\mathcal{F}$ (and therefore that of $\mathcal{G}$) have no transverse negative resonance (see definition below).

(c) The respective Γ-holonomy and Ω-holonomy are analytically conjugated.

Then, $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ are analytically equivalent.

We say that a vector field, with $1, \mu_1 \dots, \mu_n$ as the eigenvalues of its linear part, has **no transverse negative resonance** if no element in the positive cone $C = \{\sum_{i=1}^{n} p_i\mu_i; p_1 + \cdots + p_n \geq 1\}$, where $p_i \in \mathbb{Z}_{\geq 0}$, can be written in the form $\mu_j + q$, with $q \in \mathbb{Z}_{\geq 1}$, for any $1 \leq j \leq n$.

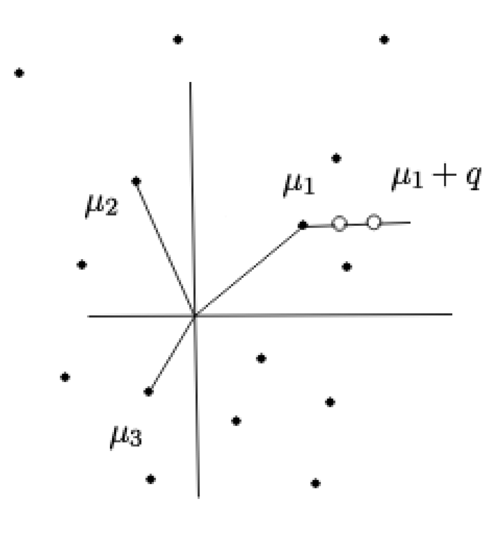

Figure 2: *No transverse negative resonance*

If the local generators of $(\mathcal{F}, H, \Gamma)$ have no transverse negative resonance, we can choose a local generator which has the form

$$x\partial_x + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}z_j\partial_{z_i} + \sum_{i=1}^{n} b_i(x, \mathbf{z})\partial_{z_i}, \quad (1)$$

where $(a_{ij})_{n\times n}$ is a constant matrix, and $b_i(x, 0) = \partial b_i/\partial z_j(x, 0) = 0$ for all $i, j \in \{1, \dots, n\}$. We say that a vector field of this form is an **x-normalized** vector field.

1. $D_{r,R}$-transversely theory

The basic tool used to prove the Main Theorem is the concept of $D_{r,R}$-transversely formal series. A $\mathbf{D_{r,R}}$**-transversely formal series** is a formal series of the form $\sum_{k_i \in \mathbb{N}} f_K(x)\mathbf{z}^K$, where $\mathbf{z}^K = z_1^{k_1} \dots z_n^{k_n}$, and each coefficient $f_K(x)$ is convergent in an annulus $D_{r,R} := \{x \in \mathbb{C}; r < |x| < R\}$, where $r, R > 0$. We denote by $\mathcal{O}_{r,R}[[\mathbf{z}]]$ the ring of $D_{r,R}$-transversely formal series.

We denote the $\mathcal{O}_{r,R}[[\mathbf{z}]]$-module of derivations over the ring of $D_{r,R}$-transversely formal series by $\mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$. In particular, we say that a $D_{r,R}$-transversely formal vector field (derivation in $\mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$) is **x-normalized** if it has the form (1), and $b_1, \dots, b_n$ all lie in the ideal generated by monomials of the form z_iz_j for $i, j \in \{1, \dots, n\}$ in the ring of $D_{r,R}$-transversely formal series. We denote by $\mathcal{D}_{norm}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ the set of such derivations.

If the components f_K of a $D_{r,R}$-transversely formal vector field converge in the disk $D_R := \{z \in \mathbb{C}; |z| < R\}$, where $R > 0$, we say that the vector field is a $\mathbf{D_R}$**-transversely formal vector field**.

Theorem 2 *Let X be a vector field in $\mathcal{D}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$ which has no transverse negative resonance and Y be a vector field in $\mathcal{D}_{norm}(\mathcal{O}_{r,R}[[\mathbf{z}]])$. If $[X, Y] = 0$, then $Y \in \mathcal{D}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$.*

A change of coordinates in $\mathcal{O}_{r,R}[[\mathbf{z}]]$ can be seen as a linear automorphism that preserves the maximal ideal of the ring of $D_{r,R}$-transversely formal series. We call such automorphisms by $\mathbf{D_{r,R}}$**-transversely formal automorphism**, and we denote such ring by $\mathcal{A}(\mathcal{O}_{r,R}[[\mathbf{z}]])$.

Given a derivation X, a special kind of change of coordinates are the symmetries for X. A $\mathbf{D_{r,R}}$**-transversely formal symmetry** for $X \in \mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ is an automorphism $\Phi \in \mathcal{A}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ such that

$$\Phi \circ X \circ \Phi^{-1} = X.$$

Proposition 3 *The map $\exp Z$ is a $D_{r,R}$-transversely formal symmetry for X if only if X commutes with Z, i.e. $[X, Z] = 0$.*

2. $D_{r,R}$-transversely equivalent foliations

A $D_{r,R}$-transversely formal automorphism Φ is said to be **x-normalized** if it is $\mathcal{O}_{r,R}$-linear and has the form

$$\left(x, \sum_{i=1}^{n} a_{1i}z_i + \phi_1(x, \mathbf{z}), \dots, \sum_{i=1}^{n} a_{ni}z_i + \phi_n(x, \mathbf{z})\right),$$

where $(a_{ij})_{n\times n}$ is an invertible constant matrix and $\phi_1, \dots, \phi_n \in \mathfrak{m}^2 \subset \mathcal{O}_{r,R}[[\mathbf{z}]]$. We denote by $\mathcal{A}_{norm}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ the set of such automorphisms.

Definition 1 *We say that $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ are* $\mathbf{D_{r,R}}$***-transversely equivalent*** *if there exist $0 < r < 1 < R$, respective adapted coordinates $(x, \mathbf{z})$ and $(y, \mathbf{w})$, and a biananalytic map Ψ between two open neighborhoods $U, V \subset \mathbb{C}^{n+1}$ of $D_{r,R} \times \{\mathbf{0}\}$ such that Ψ conjugates the x-normalized and y-normalized local generators restricted to U and V, and we can write Ψ in the form*

$$\left(x, \sum_{i=1}^{n} a_{1i}z_i + \psi_1(x, \mathbf{z}), \dots, \sum_{i=1}^{n} a_{ni}z_i + \psi_n(x, \mathbf{z})\right),$$

where $(a_{ij})_{n\times n}$ is an invertible constant matrix and $\psi_1, \dots, \psi_n \in \mathfrak{m}^2 \subset \mathcal{O}_{r,R}[[\mathbf{z}]]$.

We observe that Ψ in the definition 1 is an x-normalized $D_{r,R}$-transversely automorphism.

It is well known that two regular analytic foliations are equivalent if only if the respective holonomies relative to the respective separatrices are analytically conjugated. As a consequence of this result, we can enunciate the following.

Proposition 4 *Let $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ be two crossing type foliations with respective local generators X and Y verifying (a), (b), and (c). Then, the crossing type foliations are $D_{r,R}$-transversely equivalent.*

3. The idea of the proof of the Main Theorem

By Proposition (4), two crossing type foliations satisfying (a), (b), and (c) are $D_{r,R}$-transversely equivalent. To fix the notation, let X_i be the respective local generator of $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$, $\Phi_i^* X_i = Y_i$ be their respective normal forms, and Ψ be the x-normalized $D_{r,R}$-transversely automorphism given by Proposition (4).

Since the local generators have conjugated linear parts, the map $\Phi_2 \circ \Psi \circ \Phi_1^{-1}$ is a symmetry for the semisimple part Y_s of the local generators. In addition, there exist an $D_{r,R}$-transversely formal vector field W and a linear change of coordinates A, such that $\Phi_2 \circ \Psi \circ \Phi_1^{-1} = A \exp W$. Since $\Phi_2 \circ \Psi \circ \Phi_1^{-1}$ is a symmetry for the linear parts of the local generator, so is $\exp W$.

By Proposition (3), $\exp W$ is a symmetry for Y_s if only if $[W, Y_s] = 0$. By Theorem (2), we have that $W \in \mathcal{D}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$. As consequence, the automorphism Ψ lies in the intersection $\mathcal{A}(\mathcal{O}_R[[\mathbf{z}]]) \cap \mathcal{A}(\mathcal{O}_{r,R}\{\mathbf{z}\})$. To finish the proof, we show that the ring in the last row is the intersection of the two in the middle line.

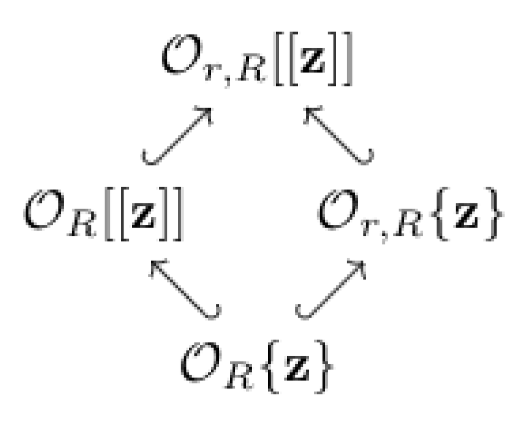

Figure 3: *Diagram of inclusions*

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