Amalgamating Gamma and zeta

(Joint work with Tamara Servi and Jean-Philippe Rolin)

Γ : Euler's Gamma function restricted to $(0,\infty)$

ζ : Riemann zeta function restricted to $(1,\infty)$.

The o-minimal structure $\mathbb{R}_{an,exp}$ is the expansion of the real field $\overline{\mathbb{R}}$ by all restricted analytic functions and the exponential function.

Γ and ζ are meromorphic

$\Longrightarrow$ their restriction to any bounded subinterval of $(0,\infty)$ are definable in $\mathbb{R}_{an}$.

Fact (van den Dries, Macintyre, Marker)

$\Gamma|_{(0,\infty)}$ and $\zeta|_{(1,\infty)}$ are not definable in $\mathbb{R}_{an,exp}$.

Argument for Γ: Binet's second formula

$$\log \Gamma(z) = \left(z-\frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \varphi(z)$$

for $z \notin (-\infty, 0]$

$\varphi\left(\frac{1}{x}\right)$ has a divergent asymptotic expansion at 0^+

Theorem [DMM]

Let f be the germ at 0^+ of a function definable in $\mathbb{R}_{an,exp}$, and assume that f has an asymptotic expansion $F \in \mathbb{R}[[X]]$ at 0^+. Then F converges.

$\Rightarrow$ $\varphi\left(\frac{1}{x}\right)$ is not definable in $\mathbb{R}_{an,exp}$

Argument for ζ:

$$\zeta(-\log x) = \sum_{n=1}^{\infty} x^{\log n}$$

Definition: A generalized power series is a series $F(X) = \sum_{r \geq 0} a_r X^r$ such that the support

$$\mathrm{supp}(F) := \{r \geq 0 : a_r \neq 0\}$$

is well-ordered. $\Leftarrow$ wrt. to $<$

$\mathbb{R}[\![X^*]\!]$ = ring of all gen. p. series

Theorem [DMM]

Let f be the germ at 0^+ of a function definable in $\mathbb{R}_{an,exp}$, and assume that f has an asymptotic expansion $F \in \mathbb{R}[\![X^*]\!]$ at 0^+. Then $\mathrm{supp}(F)$ is finitely generated.

i.e. $\mathrm{supp}(F) \subseteq r_1\mathbb{N} + \dots + r_n\mathbb{N}$

$r_i > 0$

Back to Γ

$$\log \Gamma(z) = \left(z - \tfrac{1}{2}\right)\log z - z + \tfrac{1}{2}\log(2\pi) + \varphi(z)$$

$\varphi\left(\frac{1}{x}\right)$ is Gevrey of order 1 at 0^+ and Borel summable, i.e.,

if $F(X) = \sum_{i=0}^{\infty} a_i X^i$ is the Taylor series of $\varphi\left(\frac{1}{x}\right)$, then $\mathcal{B}F(X) := \sum \frac{a_i}{i!} X^i$ converges to $\tilde{\varphi}(z)$, which extends to $(0,\infty)$ with exp. bounds, and $\varphi\left(\frac{1}{x}\right) =$ Laplace tr. of $\tilde{\varphi}$.

$$\boxed{\begin{minipage}{0.9\textwidth}
\textbf{Theorem} [van den Dries, S]

(1) There is an o-minimal expansion $\mathbb{R}_g$ of $\mathbb{R}_{an}$ in which all Borel summable germs at 0^+ are definable, and $\mathbb{R}_g$ is polynomially bounded.

(2) If $\tilde{\mathbb{R}}$ is an o-minimal exp. of $\mathbb{R}_{an}$, then the expansion $\tilde{\mathbb{R}}_{exp}$ of $\tilde{\mathbb{R}}$ by exp is o-minimal.
\end{minipage}}$$

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$\Rightarrow \Gamma\restriction_{(0,\infty)}$ definable in $\mathbb{R}_{g,exp}$.

Back to ζ

$$\zeta(-\log z) = \sum_{n=1}^{\infty} z^{\log n}$$

Definition: for $F(X) = \sum a_r X^r \in \mathbb{R}[[X^*]]$ and $\rho > 0$, set

$$\|F\|_\rho := \sum |a_r| \rho^r \in [0,\infty]$$

and

$$\mathbb{R}\{X^*\}_\rho := \{F \in \mathbb{R}[[X^*]] : \|F\|_\rho < \infty\}.$$

Also,

$$\mathbb{R}\{X^*\} := \bigcup_{\rho>0} \mathbb{R}\{X^*\}_\rho$$

is the set of convergent generalized power series.

Theorem [DS]
There is an o-minimal expansion $\mathbb{R}_{an^*}$ of $\mathbb{R}_{an}$ in which the germs at 0^+ of all convergent generalized power series are definable.

$\Rightarrow \zeta\restriction_{(1,\infty)}$ definable in $\mathbb{R}_{an^*,\exp}$

What about Γ and ζ together?

Does Riemann's functional equation

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\, \zeta(1-z)$$

imply that Γ and ζ cannot be definable in the same o-minimal structure?

NO: can only recover $\sin\restriction_{(0,\frac{\pi}{2})}$

Theorem (Rolin, Servi, S)
There is an o-minimal expansion $\mathbb{R}_{\mathcal{G}^*}$ of $\mathbb{R}_{an}$ that is pol. bounded, and in which both $\varphi\restriction_{(0,\infty)}$ and $\zeta(-\log x)\restriction_{(0,1)}$ are definable.

$\Rightarrow$ $\Gamma\restriction_{(0,\infty)}$ and $\zeta\restriction_{(1,\infty)}$ dfbl in $\mathbb{R}_{\mathcal{G}^*,\exp}$

Idea of proof.

Mimic the proof of o-minimality of $\mathbb{R}_g$ over $\mathbb{R}_{an}$, but starting with $\mathbb{R}_{an^*}$ instead.

$$
\begin{tikzcd}
\mathbb{R}_{an} \arrow[rr, bend left, "\text{extend support}"] \arrow[r, phantom, "\subseteq"] \arrow[d, phantom, sloped, "\subseteq"] \arrow[d, bend right, "\text{multi summation}"'] & {} & \mathbb{R}_{an^*} \arrow[d, phantom, sloped, "\subseteq"] \arrow[d, bend left, "\text{gen. multi summation}"] \\
\mathbb{R}_g \arrow[rr, phantom, "\subseteq"] & {} & \mathbb{R}_{g^*}
\end{tikzcd}
$$

So we need to redevelop multisummability for generalized power series.

Why "multisummability"?

The algebra of Borel summable germs does not have enough closure properties for the proof of o-minimality to work.

So what is "multisummable"?

- Borel summable = 1-summable
- if $\varphi(x)$ is 1-summable, then $\varphi(x^2)$ is 2-summable
- but $\varphi(x) + \varphi(x^2)$ is neither 1-summable nor 2-summable; it is $(1,2)$-summable
- same for $\varphi(x)\varphi(x^2)$
- iteration leads to $(k_1, \dots, k_\ell)$-summable functions, where $\ell \in \mathbb{N}$ and $0 \le k_1 < \dots < k_\ell$
- 0-summable = convergent.

Back to idea of proof

Note: we only work with generalized power series with natural support.

Definition: ① $S \subseteq \mathbb{R}$ is *natural* if $S \cap (-\infty, a)$ is finite for every $a \in \mathbb{R}$.

② $S \subseteq \mathbb{R}^n$ is *natural* if its projections on each coordinate are natural.

Theorem (Rolin, Servi, S)

The theory of multisummability extends to generalized power series with natural support. The corresponding rings of germs have all the closure properties required by [Rolin, Servi] to yield o-minimality of $\mathbb{R}_{\mathcal{G}^*}$.

Starting point: Tougeron's characterization of multisummability in the "classical" case

Fact (Tougeron)

A function $f : [0,R] \to \mathbb{C}$ is 1-summable iff there exist a sector $S \subseteq \mathbb{C}$, symmetric wrt. $\mathbb{R}$ and of angular opening $> \pi$, and for each $p \in \mathbb{N}$ there exist holomorphic

$$f_p : S \cup D\left(\frac{R}{1+p}\right) \to \mathbb{C}$$

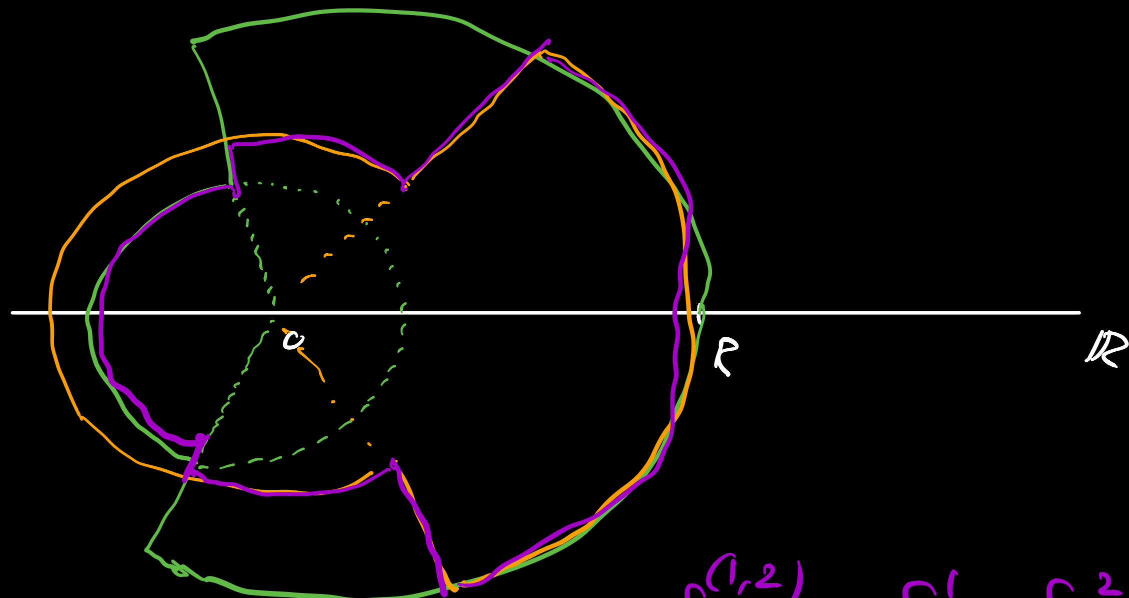
such that

(i) $\sum_p \|f_p\|_{S \cup D(\frac{R}{1+p})} \cdot r^p < \infty$ for some $r > 1$

(ii) $f(x) = \sum_p f_p(x)$ for $x \in [0,R]$.

$$S_p^1 := S \cup D\left(\frac{R}{1+p}\right)$$

$$S_p^2 := T \cup D\left(\frac{R}{(1+p)^{\frac{1}{2}}}\right)$$



$$S_p^{(1,2)} := S_p^1 \cap S_p^2$$

Note: a corresponding characterization is proved for all multisummable functions.

Note: Since even generalized monomials have a branch point at 0, we need to work on $\mathbb{L}$, the Riemann surface of log.

Set $\widetilde{S_p'}$ = lifting of S_p' to $\mathbb{L}$

Definition: A function $f : [0, R] \to \mathbb{C}$ is generalized 1-summable if there exist a natural set $\Delta \subseteq [0, \infty)$, and for each $p \in \mathbb{N}$ there exist a holom. function $f_p : \widetilde{S_p'} \to \mathbb{C}$ and a gen. power serie F_p with support in Δ, such that:

(i) $\sum_p \|f_p\|_{\widetilde{S_p'}} \, r^p < \infty$ for some $r > 1$

(ii) $f(x) = \sum_p f_p(x)$ for $x \in [0, R]$

(iii) $\sum_p \|F_p\|_{R/(1+p)} \cdot r^p < \infty$ for some $r > 1$

(iv) $f_p(z) = F_p(z)$ for $|z| < \frac{R}{1+p}$.

In the classical situation, (iii) and (iv) follow from the holomorphy of the f_p at 0. In particular, this definition generalizes the classical one.

Questions

① Is every gen. $(1,2)$-summable function the sum of a gen. 1-sum. function and a gen. 2-summable function?

This is true in the classical situation, but the proof uses Cauchy integrals that do not work in $\mathbb{L}$.

② The choice of the above definition being "centered" around the positive real direction is due to our focus on Γ and ζ. One could instead work in any direction on the Riemann surface $\mathbb{L}$.

In the classical setting, this leads to the notion "multisummable", defined as multisummable in all but finitely many directions (modulo 2π).

But generalized multisummable functions are not periodic in the argument of the input in general.

So how should one define "generalized multisummable"?

(3) Question 2 may be related to this question: is there a generalized Ramis-Sibuya theorem, say for generalized meromorphic ODEs? What would the corresponding Stokes phenomena look like?