Unlikely Intersections and applications to Diophantine Problems

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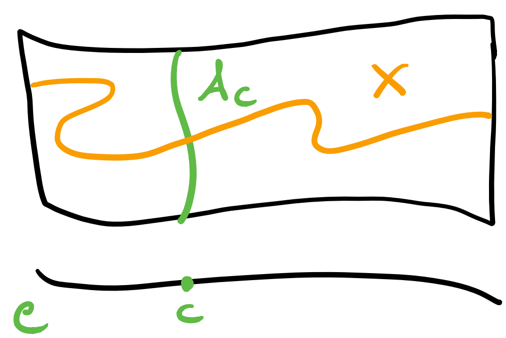
- The relative version of Zilber-Pink conjecture

$\mathcal{C}$ irreducible smooth curve $/k$ number field
$\mathcal{A} \longrightarrow \mathcal{C}$ abelian scheme of rel dim $g \geq 2$



$X \subseteq \mathcal{A}$ irreducible curve dominating $\mathcal{C}$
Denote by

$$\mathcal{A}_c^{[2]} = \bigcup H \quad \text{alg subgroup of } \mathcal{A}_c \text{ of codim} \geq 2$$

$$\boxed{\mathcal{A}^{[2]} = \bigcup_{c \in \mathcal{C}} \mathcal{A}_c^{[2]}}$$

CONJ (Pink)

Assume $X \not\subseteq$ proper alg subgroup scheme of $\mathcal{A}$, then

$$X \cap \mathcal{A}^{[2]} \quad \text{is FINITE.}$$

Regarding the conjecture of Pink, in a series of work together with F. Barroero we proved some results related to this conjecture in the setting of abelian varieties and split semi-ab. varieties.

Ex Consider again

$\mathcal{C} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

$\mathcal{E}_\lambda$ Legendre family of ell curves def by $\quad Y^2 Z = X(X-Z)(X-\lambda Z)$

$\mathcal{A} = \mathcal{E}_\lambda^3 \longrightarrow \mathcal{C}$

Consider $P_1, P_2, P_3 : \mathcal{C} \longrightarrow \mathcal{E}_\lambda$ the three sections def by

$$P_1(\lambda_0) = [\,2 : \sqrt{2(2-\lambda)} : 1\,]$$
$$P_2(\lambda_0) = [\,3 : \sqrt{6(3-\lambda)} : 1\,]$$
$$P_3(\lambda_0) = [\,5 : \sqrt{10(5-\lambda)} : 1\,]$$

Take $\sigma : \mathcal{C} \to \mathcal{E}_\lambda^3$ given by (P_1, P_2, P_3)

$X = \sigma(\mathcal{C})$ curve

Hp: No relation of the form $\sum a_i P_i(\lambda) = O$ with $a_i \in \mathbb{Z}$ not all zero holds identically on $\mathcal{C}$

Th: $\exists$ at most fin many $\lambda_0 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ s.t. $\exists$ $\underline{a}, \underline{b} \in \mathrm{End}(\mathcal{E}_{\lambda_0})^3$ lin independent s.t.

$$\begin{cases} a_1 P_1(\lambda_0) + a_2 P_2(\lambda_0) + a_3 P_3(\lambda_0) = O_{\lambda_0} \\ b_1 P_1(\lambda_0) + b_2 P_2(\lambda_0) + b_3 P_3(\lambda_0) = O_{\lambda_0} \end{cases}$$

In our work we deal only with linear relations with coefficients in $\mathbb{Z}$

Thm: (Barroero - C. '16)

Take $P_1, \ldots P_n : \mathcal{C} \to \mathcal{E}_\lambda$ n sections; assume they are generically linearly independent; then, $\exists$ at most f. many $\lambda_0 \in \mathcal{C}$ s.t.

$$\begin{cases} \sum_{i=1}^{n} a_i P_i(\lambda_0) = O_{\lambda_0} \\ \sum_{i=1}^{n} b_i P_i(\lambda_0) = O_{\lambda_0} \end{cases} \quad \text{with } \underline{a}, \underline{b} \in \mathbb{Z}^n \text{ lin indep.}$$

Notice that, in general, for every fiber $\mathcal{E}_{\lambda_0}$

we have that $\mathbb{Z} \subseteq \mathrm{End}(\mathcal{E}_{\lambda_0})$ but we can have fibers with larger endomorphism ring, and in this case we have more subgroups.

Ex: $\lambda_0 = -1$ $\quad \mathrm{End}\, \mathcal{E}_{-1} = \mathbb{Z}[i]$

In this case in $\mathcal{E}_{-1}^3$

$$\begin{cases} P_1 + P_2 + iP_3 = 0 \\ P_1 - iP_2 = 0 \end{cases}$$ defines an alg. subgroup of codim 2 of $\mathcal{E}_{-1}^3$

More in general, in our results we are able to deal with "flat subgroup schemes", i.e. subgroup schemes G of $\mathcal{A} \xrightarrow{\pi} \mathcal{C}$ such that $\pi|_G$ is flat (i.e. every irreducible component of G dominates $\mathcal{C}$).

The general result is the following:

Take $\mathcal{A} \to \mathcal{C}$ abelian scheme of rel. dim $g \geq 2$ over a smooth irr curve, all defined over a number field k.

Denote by $\qquad \mathcal{A}^{\{2\}} = \bigcup_{\substack{\text{flat sub} \\ \text{schemes of} \\ \text{codim} \geq 2}} H$

Thm (Barroero - C. '20)

Let $X \subseteq \mathcal{A}$ be an irred. curve $/k$; then, if $X \nsubseteq$ proper subgroup scheme, then

$$X \cap \mathcal{A}^{\{2\}} \quad \text{is FINITE.}$$

We are going to give an idea of the main ingredients of the proof of these results, with a special focus on the point counting results coming into the picture.

We first consider the case of relative Manin-Mumford. As seek of example, we consider the following formulation:

Take $\mathcal{C} = \mathbb{P}^1 \setminus \{0,1,\infty\}$ and $\mathcal{E}_\lambda \to \mathcal{C}$ where

$$\mathcal{E}_\lambda : y^2 z = x(x-z)(x-\lambda z)$$

Thm (Masser-Zannier)

Take $P_\lambda = [2 : \sqrt{2(2-\lambda)} : 1]$ and $Q_\lambda = [4, \sqrt{12(4-\lambda)} : 1]$
then, the set

$$\Lambda = \{\lambda \in \mathbb{C} \setminus \{0,1\} \mid P_{\lambda_0} \text{ and } Q_{\lambda_0} \text{ are both TORSION}\}$$

is finite.

Proof: We want to apply Pila-Zannier method, introduced in 2008 to give an alternative proof of Manin-Mumford conj. using o-minimality.

To apply Pila-Zannier method, it is CRUCIAL that $P_\lambda, Q_\lambda \in E_\lambda(\overline{\mathbb{Q}(\lambda)})$

① One can observe that $\Lambda \subseteq \overline{\mathbb{Q}}$, hence when one wants to prove finiteness results, a very powerful tool is to consider the WEIL HEIGHT of the points of the set.

• THE HEIGHT OF AN ALGEBRAIC NUMBER

The height of an algebraic number is a measure of the complexity of the number.

Ex. If $\alpha = \frac{m}{n}$ reduced fraction

$$\boxed{H(\alpha) = \max\{|m|, |n|\}}$$

In general, given a number field K, denote by M_K the set of all the places of K; then, $\forall v \in M_K$, denote by K_v the completion of K wrt v.

$\Rightarrow K_v$ is a finite estension of $\mathbb{Q}_v$

$\mathbb{R}$ if $|\cdot|_v$ is ARCHIMEDEAN

$\mathbb{Q}_p$ if $|\cdot|_v$ is NON-ARCHIM.

Then, if $\alpha \in K$, we denote by

$$\boxed{H(\alpha) = \prod_{v \in M_K} \max\{1, |\alpha|_v\}^{d_v/d}}$$

MULT. WEIL HEIGHT

where $d = [K : \mathbb{Q}]$ and $d_v = [K_v : \mathbb{Q}_v]$

Rmk: The exponent makes the definition INDEPENDENT on the NUMBER field.

Sometimes it is useful for applications to consider the LOGARITHMIC WEIL HEIGHT:

$$\boxed{h(\alpha) = \log H(\alpha)}$$

[NORTHCOTT PROPERTY: $\exists$ only finitely many algebraic numbers of bounded height and bounded degree.]

The height can be defined also on $\mathbb{P}^N$, and then on projective varieties.

If $P = [x_0 : \ldots : x_n] \in \mathbb{P}^N(K)$,

$$\boxed{H(P) = \prod_{v \in M_K} \max\{|x_0|_v, \ldots, |x_N|_v\}^{\frac{d_v}{d}}}$$

Applying a theorem of Silverman we have a bound on the height of the points in Λ.

Thm (SILVERMAN) Take $\mathcal{A} \to \mathcal{C}$ abelian scheme over a curve $/k$ number field; take the SPECIALIZATION MAP

$$\begin{array}{rccc} \sigma_c : & \mathcal{A}(\mathcal{C}) & \longrightarrow & \mathcal{A}_c(\bar{k}) \\ & P & \longmapsto & P_c \end{array}$$

homo from the group of sections to the gp of points on the fiber

Assume that $\mathcal{A}$ has no constant part; then

$$\{ c \in \mathcal{C}(\bar{k}) \mid \sigma_c \text{ is not injective} \}$$

is a set of bounded height.

In our case, the points $\lambda_0 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ s.t. $P_{\lambda_0} = O_{\lambda_0}$ are exactly points in which the specialization map is not injective

Indeed if P_{λ_0} is torsion, i.e. $nP_{\lambda_0} = O_{\lambda_0}$ for some n, then σ_{λ_0} is not injective, indeed

$$\sigma_{\lambda_0}(nP) = \sigma_{\lambda_0}(O) = O_{\lambda_0}$$

$$\Rightarrow \exists\, c_1 > 0 \text{ s.t. } \forall \lambda_0 \in \Lambda \quad H(\lambda_0) \leq c_1.$$

This implies that, in order to get finiteness, we need a bound on the degree of the points of Λ.

Take $\lambda_0 \in \Lambda$ and denote by $d_0 = [\mathbb{Q}(\lambda_0) : \mathbb{Q}]$ which we suppose to be large.

GOAL BOUND for d_0

Steps:

(I) Taking conjugates: If $\lambda_0 \in \Lambda$, then $\forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ $\sigma(\lambda_0) \in \Lambda$

$$\Rightarrow \#\Lambda \geq d_0$$

(II) To apply point-counting we need to consider a small disk $D_{\lambda_0} \subseteq \mathbb{C}$ containing "many conjugates".
This can be done thank to the bound on the height.

CLAIM: $\exists\, c_2 > 0$ s.t. $\# \Lambda \cap D_{\lambda_0} \geq \frac{1}{c_2} d_0$.

(III) POINT COUNTING:

We consider the exponential function

$$\exp_\lambda : \mathbb{C}/L_\lambda \longrightarrow \mathcal{E}_\lambda$$

$$z(\lambda) \longrightarrow P(\lambda)$$

$$u(\lambda) \longrightarrow Q(\lambda)$$

where L_λ is a local system of periods
$\Rightarrow$ lattice of rk 2

$\exists\, \omega_1, \omega_2 : D_{\lambda_0} \to \mathbb{C}$ such that $\omega_1(\lambda), \omega_2(\lambda)$ basis of L_λ.

Take z, u holomorphic function such that
$$\exp_\lambda(z(\lambda)) = P_\lambda, \quad \exp_\lambda(u(\lambda)) = Q_\lambda$$

Now we can write
$$\begin{cases} z(\lambda) = z_1(\lambda)\,\omega_1(\lambda) + z_2(\lambda)\,\omega_2(\lambda) \\ u(\lambda) = u_1(\lambda)\,\omega_1(\lambda) + u_2(\lambda)\,\omega_2(\lambda) \end{cases}$$

where $z_i, u_i : D_{\lambda_0} \to \mathbb{R}$, and take

$$S = \theta(D_{\lambda_0}) \quad \text{where } \theta : D_{\lambda_0} \longrightarrow \mathbb{R}^4, \quad \lambda \longmapsto (z_1, z_2, u_1, u_2)$$

Now P_{λ_0} and Q_{λ_0} are simultaneously TORSION of order dividing n $\Leftrightarrow$

$$(z_1(\lambda_0), z_2(\lambda_0), u_1(\lambda_0), u_2(\lambda_0)) \in \frac{1}{n}\mathbb{Z}^4$$

So we are interested in studying RATIONAL POINTS of S of BOUNDED DENOMINATOR.

Notice that S is a SUBANALYTIC SURFACE $\Rightarrow$ it is definable in the o-minimal structure $\mathbb{R}_{an}$.

We want to apply PILA-WILKIE'S THEOREM

Thm (Pila-Wilkie '06)
Let S be a definable set in an o-minimal structure and denote by

$$S^{alg} = \bigcup \text{alg arcs}$$

and by

$$S^{trans} = S \setminus S^{alg}.$$

Then, $\forall \varepsilon > 0$ $\exists$ $c(S, \varepsilon)$ such that, $\forall T > 0$,

$$\#\{\underline{x} \in S^{trans}(\mathbb{Q}) \mid H(x) \leq T\} \leq c \cdot T^{\varepsilon}.$$

To apply Pila-Wilkie to our set, we first need to show that $S^{alg} = \emptyset$, but this follows under the hypothesis that P_λ, Q_λ are not linearly related by a theorem of Bertrand.

$$\boxed{\Rightarrow \#\{\lambda_0 \in \mathbb{C}\setminus\{0,1\} \mid nP_{\lambda_0} = nQ_{\lambda_0} = 0\} \leq c_3 n^{\varepsilon}}$$

(IV) DIOPHANTINE ESTIMATES

We finally want to relate the order of torsion to the degree of λ_0; if this can be done "polynomially", then we are done.

But indeed the following fact holds:

PROP: $\exists$ an absolute constant c with the following property: suppose for some $\lambda_0 \neq 0,1$ that P_{λ_0} and Q_{λ_0} are both torsion; then

$$n \leq c\, d^2 (1 + \log H(\lambda))$$

+ BOUND on the height $\Rightarrow$ $\boxed{n \leq c_4 d^2}$

(IV) CONCLUSION:

Take $\lambda_0 \in \{ \lambda \in \mathbb{C} \setminus \{0,1\} \mid P_{\lambda_0}, Q_{\lambda_0}$ simultaneously torsion$\} = \Lambda \subseteq \overline{\mathbb{Q}}$

① $H(\lambda_0) \leq c_1$ [by SILVERMAN SP. THEOREM]

② Take $d_0 = [\mathbb{Q}(\lambda_0) : \mathbb{Q}]$; then $\exists\, D_{\lambda_0} \subseteq \mathbb{C}$ closed disc containing $\frac{1}{c_2} d_0$ conjugates of λ_0.

This implies that

$$\frac{1}{c_2} d_0 \leq \# \Lambda \cap D_{\lambda_0} \underset{\substack{\uparrow \\ \text{POINT} \\ \text{COUNTING}}}{\leq} c_3 n^{\varepsilon} \underset{\substack{\uparrow \\ \text{DIOPHANTINE} \\ \text{ESTIMATES}}}{\leq} c_3 c_4 d_0^{2\varepsilon}$$

$\Rightarrow$ for $\varepsilon = \frac{1}{4}$ we get a bound on the degree

$\square$