Hodge theory and tame geometry

C = smooth projective curve of genus g

$$\left(\begin{array}{l} H_{\mathbb{Z}} = H^1(C,\mathbb{Z}) \simeq \mathbb{Z}^{2g} \\ Q : H_{\mathbb{Z}} \otimes_{\mathbb{Z}} H_{\mathbb{Z}} \to H^2(C,\mathbb{Z}) = \mathbb{Z} \quad \text{intersection form} \\ F = H^0(C,\Omega^1) \subset H^1_{dR}(X,\mathbb{C}) = H^1_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \end{array}\right.$$

<u>Thm</u> (Torelli) $(H_{\mathbb{Z}}, Q, F)$ determines C up to iso

$$\begin{tikzcd} \mathcal{M}_g \arrow[r, "P"] & \left\{ \begin{array}{l} \text{polarized} \\ \mathbb{Z}\text{-HS} \\ \text{of weight 1} \end{array} \right\} / \text{iso} = A_g \\ {[C]} \arrow[r, mapsto] & {[(H_{\mathbb{Z}}, Q, F)]} \end{tikzcd}$$

<u>Facts</u>: A_g is an $\mathbb{C}$-algebraic variety and the period map p is algebraic.

<u>General situation</u>: $f : X \to S$ algebraic map. Assume that f is topological fiber bundle (e.g. f is smooth proper). Fix $k \geq 0$.

$$\begin{tikzcd} S \arrow[r, "P"] & \mathcal{D} = \{\mathbb{Z}\text{-}\Pi HS^{gP}\}/\text{iso} \leftarrow \text{period space} \\ s \arrow[r, mapsto] & {[H^k(X_s,\mathbb{Z})]} \end{tikzcd}$$

<u>Facts</u>
- * $\mathcal{D}$ is a $\mathbb{C}$-analytic space and p is holomorphic
- * But in general, $\mathcal{D}$ is not algebraic!
- * However, $\mathcal{D} = \Gamma \backslash \Omega$ with

Ω = open semialgebraic subset of smooth projective variety $\check{\Omega}$, homogeneous under $G(\mathbb{C})$.

$\Gamma = G(\mathbb{Z})$ = arithmetic group acting by semialgebraic biholomorphisms on Ω.

<u>Ex</u>: period space of weight 1 $\mathbb{Z}$-HS of dim 2

$$\mathcal{D} = SL(2,\mathbb{Z}) \backslash \mathbb{H}$$

<u>Thm 1</u> (Bakker – Klingler – Tsimerman, + B.)

$\mathcal{D}$ has a canonical $\mathbb{R}_{alg}$-structure such that any period map $S \to \mathcal{D}$ coming from geometry is definable in $\mathbb{R}_{an,exp}$.

<u>Thm</u> (Fortuna – Lojasiewicz, Peterzil – Starchenko)

Let X algebraic variety and $Y \hookrightarrow X^{an}$ a closed analytic subset. Then:

Y algebraic $\Leftrightarrow$ Y is definable in o-minimal structure

(Cattani – Deligne – Kaplan, Brosnan – Pearlstein, ...)

<u>Cor</u>: Hodge loci are algebraic.

<u>Examples</u>

<u>Prop</u>: If $F \subset \Omega$ is a fundamental set, for $\Gamma \curvearrowright \Omega$ i.e. an open definable subset such that

$\Gamma . F = \Omega$ and $\{\gamma \in \Gamma \mid \gamma F \cap F \neq \emptyset\}$ is finite,

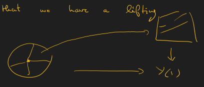
then there exists a unique definable structure on $\mathcal{D} = \Gamma \backslash \Omega$ such that $F \to \Gamma \backslash \Omega$ is definable.

<u>Ex</u> ① Modular curve

$$\begin{tikzcd} & \Omega = \mathbb{H} \circlearrowleft SL(2,\mathbb{Z}) \arrow[d] \\ \Delta^* \arrow[r] & Y(1) \overset{j}{=} \mathbb{C} \end{tikzcd}$$

↑ is definable ?

a) Δ^* is covered by finitely many sectors such that we have a lifting

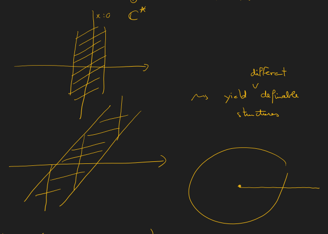


b) the coordinate functions are definable.

<u>Ex</u> ②

$$\begin{tikzcd} \Omega = \mathbb{C} \circlearrowleft \mathbb{Z} \arrow[d, "e"] \\ \mathbb{C}^* \end{tikzcd}$$

$e(z) = \exp(2\pi i z)$



⇝ yield $\overset{\text{different}}{\vee}$ definable structures

<u>Thm 2</u> (Bakker – B. – Tsimerman)

Any geometric period map factorizes

$$\begin{tikzcd} S \arrow[rr, "P"] \arrow[dr] & & \mathcal{D} \\ & T \arrow[ur, "f"] & \end{tikzcd}$$

algebraic and dominant → (on $S \to T$); ← closed immersion (on f); ↑ algebraic variety (on T)

<u>Thm 3</u> (BBT)

X = algebraic variety, $\mathcal{Y}$ = definable analytic space

$f : X^{def} \to \mathcal{Y}$ a definable analytic proper map

then

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"] & & \mathcal{Y} \\ & Z \arrow[ur, "f"] & \end{tikzcd}$$

algebraic and dominant → (on g); ← closed immersion (on f); ↖ algebraic space (on Z)

<u>Rem</u>: The equivalence relation induced by f,

$$R = X \times_{\mathcal{Y}} X \hookrightarrow X \times X,$$

is closed analytic definable, hence algebraic.

⇝ need to develop a theory of possibly non-reduced definable analytic spaces and coherent sheaves on them.

<u>Thm</u> (BBT, definable GAGA)

X = $\mathbb{C}$-algebraic space

The functor $Coh(X) \to Coh(X^{def})$ is exact, fully-faithfull and its essential image is stable under subobjects and quotients.