Vanishing of twisted L-functions of elliptic curves over function fields

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Let E be an elliptic curve over $\mathbb{Q}$ with conductor N_E and L-function

$$L(E,s) = \prod_{p\nmid N_E} \left(1-\alpha_p p^{-s}\right)^{-1}\left(1-\overline{\alpha}_p p^{-s}\right)^{-1} \prod_{p\mid N_E}\left(1-a_p p^{-s}\right)^{-1}$$
$$= \prod_{p\nmid N_E}\left(1-a_p p^{-s}+p^{1-2s}\right)^{-1}\prod_{p\mid N_E}\left(1-a_p p^{-s}\right)^{-1},$$

where for $p \nmid N_E$, $E/\mathbb{F}_p$ is an elliptic curve, $|E(\mathbb{F}_p)| = p+1-(\alpha_p+\overline{\alpha}_p)$.

Functional equation (Wiles 1995)

$$\Lambda(E,s) = \left(\frac{\sqrt{N_E}}{2\pi}\right)^s \Gamma(s)L(E,s) = \omega_E\,\Lambda(E,2-s),\ \ \omega_E = \pm 1.$$

Vanishing at $s=1$ is related to the rational solutions of $E/\mathbb{Q}$ via the Birch and Swinnerton-Dyer conjecture:

$$\operatorname{ord}_{s=1}L(E,s) \stackrel{?}{=} \operatorname{rank}(E(\mathbb{Q})).$$

The order of vanishing of $L(E,s)$ at $s=1$ is the analytic rank of E.

Twisted L-functions of elliptic curves

Let χ be a Dirichlet character over $\mathbb{Q}$, the twisted L-function is

$$L(E,\chi,s) = \prod_{p\nmid N_E} \left(1-\chi(p)\alpha_p p^{-s}\right)^{-1} \left(1-\chi(p)\overline{\alpha}_p p^{-s}\right)^{-1} \times \prod_{p|N_E} \left(1-\chi(p)a_p p^{-s}\right)^{-1} = \sum_n a_n \chi(n) n^{-s}.$$

Let χ of order ℓ prime, and $K/\mathbb{Q}$ is the cyclic extension associated to χ.

$$\zeta_K(s) = \zeta(s) \prod_{\substack{\psi \in \widehat{\mathrm{Gal}(K/\mathbb{Q})} \\ \psi \neq \psi_0}} L(s,\psi) = \zeta(s) \prod_{j=1}^{\ell-1} L(s,\chi^j).$$

Also,

$$L(E/K,s) = L(E,s) \prod_{j=1}^{\ell-1} L(E,\chi^j,s).$$

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Rank growth in quadratic extensions

For quadratic twists, $L(E, \chi_D, s) = L(E_D, s)$ where

$$E : y^2 = x^3 + Ax + B, \quad E_D : Dy^2 = x^3 + Ax + B.$$

If $D \nmid N_E$,

$$\omega_{E_D} = \omega_E \, \chi_D(N_E).$$

- Goldfeld (1974) conjectured that half of the twists $E_D/\mathbb{Q}$ have analytic rank zero, and half have analytic rank one (asymptotically).
- Heath-Brown (2004): a positive proportion of the twists have analytic ranks 0 and 1 (assuming GRH).
- Smith (2022): almost all elliptic curves satisfy Goldfeld conjecture (assuming BSD).
- Gouvea and Mazur (1991): the analytic rank of E_D is at least 2 for $\geq X^{1/2-\epsilon}$ discriminants $|D| \leq X$.

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Rank growth in cyclic extensions of prime order $\ell \geq 3$

- David-Fearnley-Kisilevsky (2007) and Mazur-Rubin (2021): predict that $L(E, \chi, 1) = 0$ is very rare for χ of order $\ell \geq 3$ (heuristics on distribution of modular symbols and random matrix theory.)
- Mazur-Rubin (2021): conjectured that if $K/\mathbb{Q}$ abelian such that K contains only finitely many subfields of degree $2, 3, 5$ over $\mathbb{Q}$, then $E(K)$ is finitely generated.
- Larsen-Mazur-Rubin (2018): proved for a positive proportion of primes ℓ, infinitely many ℓ-cyclic extensions $K/\mathbb{Q}$ of order ℓ such that $E(K) = E(\mathbb{Q})$.
- Fearnley-Kisilevsky-Kuwata (2012): For any $E/\mathbb{Q}$ and $\ell = 3$, if there exists one χ such that $L(E, \chi, 1) = 0$, then there are infinitely many.

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Rank growth in non-abelian extensions

- Lemke Oliver and Thorne (2021) There are infinitely many $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{S}_d$ with $\mathrm{rank}(E(K)) > \mathrm{rank}(E(\mathbb{Q}))$, for each $d \geq 2$.
- Fornea (2019) for some curves $E/\mathbb{Q}$, the analytic rank of E increases for a positive proportion of the quintic fields with Galois group $\mathbb{S}_5$.
- Keliher (2022) (under certain conditions on E) infinitely many $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{S}_4$ such that the rank does not increase.

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Function fields

Let q power of a prime, $\mathbb{F}_q$ finite field with q elements.

Number Fields		Function Fields
$\mathbb{Q}$	$\leftrightarrow$	$\mathbb{F}_q(t)$
$\mathbb{Z}$	$\leftrightarrow$	$\mathbb{F}_q[t]$
p positive prime	$\leftrightarrow$	$P(t)$ monic irreducible polynomial or P_∞
$\lvert n\rvert = \lvert\mathbb{Z}/n\mathbb{Z}\rvert$	$\leftrightarrow$	$\lvert f(t)\rvert = \lvert\mathbb{F}_q[t]/(f(t))\rvert = q^{\deg f}$
$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$	$\leftrightarrow$	$\zeta_q(s) = \sum_{\substack{f\in\mathbb{F}_q[t] \\ f \text{ monic}}} \frac{1}{\lvert f\rvert^s} = \frac{1}{1-q^{1-s}}$
Riemann Hypothesis ☹	$\leftrightarrow$	Riemann Hypothesis ☺

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Dirichlet L-functions over function fields

Let χ be a Dirichlet character of order ℓ over $\mathbb{F}_q(t)$ with conductor $F_\chi \in \mathbb{F}_q[t]$ and

$$\mathcal{L}(\chi, u) = \prod_P (1 - \chi(P) u^{\deg P})^{-1} \ \ (P \in \mathbb{F}_q[t] \text{ or } P = P_\infty). \qquad (u = q^{-s}).$$

Weil: $\mathcal{L}(\chi, u)$ is a polynomial of degree $\deg F_\chi + \delta_\chi - 2$,

$$\mathcal{L}(\chi, u) = \prod_{j=1}^{\deg F_\chi + \delta_\chi - 2} \left(1 - u q^{1/2} e^{i\theta_j}\right).$$

$$\mathcal{L}(\chi, u) = \omega_\chi \left(\sqrt{q} u\right)^{\deg F_\chi + \delta_\chi - 2} \mathcal{L}(\overline{\chi}, 1/qu),$$

where $\delta_\chi = 0, 1$ if χ is even, odd respectively.

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Vanishing of Dirichlet L-functions over function fields

The vanishing of $\mathcal{L}(\chi, u)$ at $u = q^{-\frac{1}{2}}$ corresponds to vanishing of $L(s, \chi)$ at $s = \frac{1}{2}$. Over $\mathbb{Q}$, Chowla conjectured that $L(\frac{1}{2}, \chi) \neq 0$.

Theorem (Li (2018), Donepudi-Li (2021))

- $\gg q^{\frac{n}{3}-\varepsilon}$ *of the* Cq^n $\square$ *characters of* $\deg(\mathrm{cond}(\chi)) \leq n$ *such that* $\mathcal{L}(\chi, q^{-\frac{1}{2}}) = 0$.
- $\gg q^{\frac{n}{2}-\varepsilon}$ *when* $q = p^{2e}$.
- $\gg q^{\frac{2n}{3}-\varepsilon}$ *of the* Cq^n *characters of* $\deg(\mathrm{cond}(\chi)) \leq n$ *such that* $\mathcal{L}(\chi, q^{-\frac{1}{2}}) = 0$ *when* $q = p^{4e}$ $(p \equiv 2 \pmod 3)$.
- $\gg q^{\frac{2n}{3}-\varepsilon}$ *of the* Cq^n *characters of order* ℓ *of* $\deg(\mathrm{cond}(\chi)) \leq n$ *such that* $\mathcal{L}(\chi, q^{-\frac{1}{2}}) = 0$ *when* $q = p^e$ $(p \equiv -1 \pmod \ell)$.

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L-functions of elliptic curves over function fields

Let E be an elliptic curve

$$E : y^2 = x^3 + A(t)x + B(t), \;\; A(t), B(t) \in \mathbb{F}_q[t].$$

Let P be a prime of $\mathbb{F}_q(t)$ (including $P = P_\infty$), and

$$\mathbb{F}_P = \mathbb{F}_q[t]/(P) \cong \mathbb{F}_{q^{\deg P}}.$$

If P is a prime of good reduction ($P \nmid N_E$)

$$\#E(\mathbb{F}_P) = q^{\deg P} + 1 - a_P, \;\; a_P = \alpha_P + \overline{\alpha}_P, \;\; |\alpha_P| = \sqrt{q^{\deg P}}.$$

Let

$$\mathcal{L}_P(E, u) = 1 - a_P u + q^{\deg P} u^2 = (1 - \alpha_P u)(1 - \overline{\alpha}_P u)$$

be the L-function of $E/\mathbb{F}_P$.

L-functions of elliptic curves over function fields

The L-function of $E/\mathbb{F}_q(t)$ is

$$\mathcal{L}(E,u) = \prod_{P\nmid N_E} \mathcal{L}_P(E,u^{\deg P})^{-1} \prod_{P|N_E} \mathcal{L}_P(E,u^{\deg P})^{-1}.$$

Weil (1971): if E a non-constant elliptic curve over $\mathbb{F}_q(t)$, $\mathcal{L}(E,u)$ is a polynomial of degree $\deg N_E - 4$

$$\mathcal{L}(E,u) = \prod_{j=1}^{\deg N_E - 4} \left(1 - que^{i\theta_j}\right).$$

Then, $\mathcal{L}(E,u)$ satisfies the functional equation

$$\mathcal{L}(E,u) = \omega_E\ (qu)^{\deg(N_E)-4}\mathcal{L}(E,1/(q^2u)),\ \ \omega_E = \pm 1.$$

This comes from seeing $E/\mathbb{F}_q(t)$ as a surface over $\mathbb{F}_q$.

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Elliptic curves L-functions twisted by Dirichlet characters

The twisted L-function $\mathcal{L}(E,\chi,u)$ is

$$\prod_{P\nmid N_E}(1-\chi(P)\alpha_P u^{\deg(P)})^{-1}(1-\chi(P)\overline{\alpha}_P u^{\deg(P)})^{-1} \times \prod_{P|N_E}(1-\chi(P)a_P u^{\deg(P)})^{-1}.$$

Suppose $(N_E,F_\chi)=1$. $\mathcal{L}(E,\chi,u)$ is a polynomial of degree

$$N:=\deg N_E+2\deg F_\chi+2\delta_\chi-4$$

$$\mathcal{L}(E,\chi,u)=\omega_{E\otimes\chi}\,(qu)^N\,\mathcal{L}(E,\overline{\chi},1/(q^2u)),\quad \omega_{E\otimes\chi}=\omega_\chi^2\,\omega_E\,\chi(N_E).$$

If $K/\mathbb{F}_q(t)$ is the cyclic extension of order ℓ associated to χ,

$$\mathcal{L}(E/K,u)=\mathcal{L}(E,u)\prod_{i=1}^{\ell-1}\mathcal{L}(E,\chi^i,u).$$

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Characters, extensions $K/\mathbb{F}_q(t)$, and curves over $\mathbb{F}_q$

$\chi/\mathbb{F}_q(t)$ are associated to abelian extensions $K/\mathbb{F}_q(t)$.
$K/\mathbb{F}_q(t)$ are associated to curves $C/\mathbb{F}_q$ by $K = \mathbb{F}_q(C)$.

Example $C : y^2 = f(t)$, $f(t) \in \mathbb{F}_q(t)$. Then,

$$\mathbb{F}_q(C) = \mathbb{F}_q[t, y]/(y^2 - f(t)) = \mathbb{F}_q(t)(\sqrt{f(t)})$$

is a quadratic extension.

Characters of order ℓ are associated with ℓ-cyclic extensions K, which are associated with ℓ-cyclic covers C.

Example

$$C : y^\ell = f(t) \quad [q \equiv 1 \pmod{\ell}],$$

and we have

$$\mathcal{L}(C, u) = \prod_{i=1}^{\ell-1} \mathcal{L}(\chi^i, u), \qquad \mathcal{Z}(C, u) = \mathcal{Z}(u)\mathcal{L}(C, u).$$

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Constant elliptic curves over $\mathbb{F}_q(t)$

Let C be an ℓ-cyclic cover over $\mathbb{F}_q$ of genus g and L-function

$$\mathcal{L}(C,u) = \prod_{j=1}^{2g}(1-\beta_j u), \quad |\beta_j| = q^{1/2}.$$

Let E_0 be an elliptic curve over $\mathbb{F}_q$ with L-function

$$\mathcal{L}(E_0,u) = (1-\alpha_1 u)(1-\alpha_2 u), \quad |\alpha_1|, |\alpha_2| = q^{1/2}.$$

Then $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(t)$ is a constant elliptic curve over $\mathbb{F}_q(t)$.

Theorem (Milne (1968))

$$\mathcal{L}(E/K_C,u) = \mathcal{Z}(C,\alpha_1 u)\mathcal{Z}(C,\alpha_2 u) = \frac{\displaystyle\prod_{\substack{1\leq i\leq 2\\ 1\leq j\leq 2g}}(1-\alpha_i\beta_j u)}{\displaystyle\prod_{1\leq i\leq 2}(1-\alpha_i u)(1-\alpha_i q u)}.$$

L-functions of constant elliptic curves

Corollary

For $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(t)$, χ associated to C_χ and function field K_C. Then,

$$\mathcal{L}(E/K_C, q^{-1}) = 0 \quad \text{if and only if} \quad \mathcal{L}(C, \alpha_1^{-1}) = \mathcal{L}(C, \alpha_2^{-1}) = 0.$$

$$\mathcal{L}(C, u) = \prod_{j=1}^{2g}(1 - \beta_j u) = \prod_{i=1}^{\ell-1} \mathcal{L}(\chi^i, u),$$

$$\mathcal{L}(E/K_C, u) = \mathcal{L}(E, u) \prod_{i=1}^{\ell-1} \mathcal{L}(E, \chi^i, u).$$

The vanishing of $\mathcal{L}(E, \chi, u)$ at $u = q^{-1}$ reduces to: The vanishing of $\mathcal{L}(\chi, u)$ at $u = \alpha^{-1}$ where $\mathcal{L}(E_0, u) = (1 - \alpha u)(1 - \overline{\alpha} u)$.

Vanishing for constant elliptic curves

Goal: generalize the work of Donepudi-Li to general ℓ-cyclic covers $C/\mathbb{F}_q$ (and not only the Kummer ones where $q \equiv 1 \mod \ell$).

Theorem (C-L,D,L,L (2022))

If $\exists$ *ℓ-cyclic cover* $C_0/\mathbb{F}_q$ *with* $\deg(\mathrm{cond}) = d_0$ *such that* $\mathcal{L}(C_0, u_0) = 0$,
$\Rightarrow \gg q^{2n/d_0}$ *ℓ-cyclic covers* $C/\mathbb{F}_q$ *with* $\deg(\mathrm{cond}) \leq n$ *and* $\mathcal{L}(C, u_0) = 0$.

Theorem (C-L,D,L,L (2022))

Let E_0 *be an elliptic curve over* $\mathbb{F}_q$, *and let* $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(t)$. *If* $\exists\ \chi_0$ *of order* ℓ *over* $\mathbb{F}_q(t)$ *with* $\deg(\mathrm{cond}(\chi_0)) = d_0$ *and* $\mathcal{L}(E, \chi_0, q^{-1}) = 0$,
$\Rightarrow \gg q^{2n/d_0}$ *Dirichlet characters of order* ℓ *over* $\mathbb{F}_q(t)$ *with* $\deg(\mathrm{cond}(\chi)) \leq n$ *and* $\mathcal{L}(E, \chi, q^{-1}) = 0$.

Geometric vanishing criterion

Tate-Honda theory: conjugacy classes of q-Weil numbers $\leftrightarrow$ isogeny classes of simple abelian varieties over $\mathbb{F}_q$.

B is isogenous to a subabelian variety of $A \Leftrightarrow P_B(x) \mid P_A(x)$.

Theorem (Li (2018))

Let u_0 be a q-Weil number and let A_0 be the corresponding (isogeny class) abelian variety.
View A_0 as a subabelian variety of $Jac(C_0)$ for some curve $C_0/\mathbb{F}_q$.
Let C be a curve over $\mathbb{F}_q$.
Then, $\mathcal{L}(C, u_0^{-1}) = 0$ if and only if there exists a non-trivial map $C \to C_0$ if and only if $\mathcal{L}(C_0, u) \mid \mathcal{L}(C, u)$.

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Kummer ℓ-cyclic covers

If $q \equiv 1 \mod \ell$, let C_0 be a ℓ-cyclic cover

$$C_0 : y^\ell = f(t),\ f(t) = f_1 f_2^2 \cdots f_{\ell-1}^{\ell-1},$$

where $F_0 = f_1 f_2 \cdots f_{\ell-1}$ is $\square$-free and $d_0 := \deg(f_1 \cdots f_{\ell-1})$.
Let $h(t) \in \mathbb{F}_q[t]$, and let

$$C : y^\ell = f(h(t)),$$

where $F(t) = F_0(h(t)) = f_1(h(t)) f_2(h(t)) \dots f_{\ell-1}(h(t))$ is $\square$-free, and $\deg F = d_0 \cdot \deg h$.

There is a non-trivial map of curves

$$\begin{array}{rccc} \phi : & C & \to & C_0 \\ & (t,y) & \mapsto & (h(t), y) \end{array}$$

and we have to count the $\square$-free values $(f_1 \dots f_{\ell-1})(h(t))$.

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Square-free values of polynomials over $\mathbb{F}_q[t]$

Proposition (Poonen (2003))

Let $f \in \mathbb{F}_q[t][x_1, \dots, x_m]$ be $\square$-free in $\mathbb{F}_q(t)[x_1, \dots, x_m]$. Let

$$S_f := \{a \in \mathbb{F}_q[t]^m : f(a) \text{ is } \square\text{-free}\} \quad \text{and} \quad ||a|| := \max_{1 \leq i \leq m} |a_i|$$

For P prime of $\mathbb{F}_q[t]$, let $c_P =$ be the number of $x \in (\mathbb{F}_q[t]/P^2)^m$ that satisfy $f(x) = 0$ in A/P^2. Then

$$\mu(S_f) := \lim_{N \to \infty} \frac{|\{a \in S_f : ||a|| < N\}|}{|\{a \in \mathbb{F}_q[t]^m : ||a|| < N\}|} = \prod_P \left(1 - \frac{c_P}{|P|^{2m}}\right).$$

General ℓ-cyclic covers

Let n_q be the multiplicative order of $q \mod \ell$ and χ of order ℓ.
$\mathrm{cond}(\chi) = F \in \mathbb{F}_q[t]$ $\square$-free,

$$P \mid F \Rightarrow n_q \mid \deg P$$

(the P split completely in $\mathbb{F}_{q^{n_q}}(t)/\mathbb{F}_q(t)$).

Write F as a product of n_q conjugates in $\mathbb{F}_{q^{n_q}}(t)$,

$$F = \mathfrak{F}_1 \dots \mathfrak{F}_{n_q}, \ \phi_q(\mathfrak{F}_i) = \mathfrak{F}_{i+1} \Rightarrow N_q(\mathfrak{F}_i) = F,$$

one can write the equation of C in terms of $\mathfrak{F}_1, \dots, \mathfrak{F}_{n_q}$ (Bary-Soroker and Meisner, 2019).

$$C_F \ : \ \prod_{j=0}^{\ell-1} \left(y - \sum_{k=0}^{n_q-1} \xi_\ell^{jq^k} \sqrt[\ell]{F_{\mathbf{v}_k}} \right) = 0.$$

Example: If $\ell = 3$ and $q \equiv 2 \mod 3$ (and $n_q = 2$)

$$C_F \ : \ y^3 - 3\mathfrak{F}_1\mathfrak{F}_2 y - \mathfrak{F}_1\mathfrak{F}_2(\mathfrak{F}_1 + \mathfrak{F}_2) = 0.$$

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General ℓ-cyclic covers

For general ℓ-cyclic covers, using the result of Poonen:

Theorem (C-L,D,L,L (2022))

Let E_0 be an elliptic curve over $\mathbb{F}_q$, and let $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(t)$. If $\exists\ \chi_0$ of order ℓ over $\mathbb{F}_q(t)$ with $\deg(\mathrm{cond}(\chi_0)) = d_0$ and $\mathcal{L}(E, \chi_0, q^{-1}) = 0$, there are $\gg q^{2n/d_0}$ Dirichlet characters of order ℓ over $\mathbb{F}_q(t)$ with $\deg(\mathrm{cond}(\chi)) \leq n$ such that $\mathcal{L}(E, \chi, q^{-1}) = 0$.

Remark: The d_0 comes from C_0. The 2 comes using $h(t) = u(t)/v(t)$ in the map from C to C_0 and Poonen's sieve with $m = 2$ for the tuples $(u(t), v(t))$.

Let $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(t)$. How do you produce an ℓ-cyclic cover C_0 over $\mathbb{F}_q$ such that $\mathcal{L}(E/K_{C_0}, q^{-1}) = 0$?

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Examples for constant curves

Experimentally, for any prime p and ℓ such that $p \equiv -1 \bmod \ell$, the theorem applies to any supersingular $E_0/\mathbb{F}_p$.

Example Over $\mathbb{F}_{13}$, the 7-cyclic cover C_0 ($n_q = 2$)

$$y^7 + (6t^4 + 6t^3 + 6t^2 + 12t + 1)y^5 +$$
$$(t^8 + 2t^7 + 3t^6 + 6t^5 + t^4 + 5t + 4)y^3 +$$
$$(6t^{12} + 5t^{11} + 10t^{10} + 7t^8 + 2t^7 + \cdots + 2t^3 + 6t^2 + t + 4)y +$$
$$11t^{14} + 6t^{13} + 12t^{12} + 10t^{11} + \cdots + 7t^4 + 12t^3 + 3t^2 + 3t + 9 = 0$$

has $\mathcal{L}(C_0, u) = (1 + 13u^2)^6 = \mathcal{L}(E_0, u)^6$ where E_0 is a supersingular elliptic curve over $\mathbb{F}_{13}$. Then, $\mathcal{L}(C_0, \alpha_0^{-1}) = 0$ and $\mathcal{L}(E/K_{C_0}, q^{-1}) = 0$ for $E = E_0 \times_{\mathbb{F}_q} \mathbb{F}_q(t)$, and there are infinitely many such 7-cyclic covers C.

χ of order ℓ over $\mathbb{F}_p$ such that $\mathcal{L}(\chi, u) = (1 + p^2 u)$

ℓ	p	n_p	$\mathcal{L}(\chi, u) = 1 + a_p u + pu^2$
13	103	2	0
17	67	2	0
	101	2	0
19	37	2	0
31	61	2	0
37	73	2	0

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Numerical data for non-constant curves

Our method to obtain results for constant elliptic curves does not apply to non-constant elliptic curves.

Experimentally, we observe that for any $\ell > 5$, the vanishing for $L(E, \chi, 1)$ is extremely rare. So for non-constant elliptic curves, we expect similar heuristics as David–Fearnley–Kisilevsky and Mazur–Rubin for number fields.

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Twists of order 3 for the Legendre curve $y^2 = x(x-1)(x-t)$

twist order	p	n_p	deg conductor d	rank 0	rank 1	rank 2
3	5	2	2	6	4	0
			4	205	32	3
			6	5784	260	16
			8	302640	116	4
	7	1	1	5	0	0
			2	37	4	0
			3	324	37	1
			4	2935	73	0

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Twists of order 5 for the Legendre curve $y^2 = x(x-1)(x-t)$

twist order	p	n_p	deg conductor d	rank 0	rank 1
5	7	4	4	585	3
	11	1	1	9	0
			2	199	0
			3	3759	5
			4	65143	11
	19	2	2	170	1

We have not found any character of order $7, 11, 13$ with the twisted L-function vanishing for the Legendre curve.

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Twists of order 3 for $y^2 = (x-1)(x-2t^2-1)(x-t^2)$

p	n_p	deg(conductor χ)	rank 0	rank 1	rank 2	rank 3
5	2	2	8	2	0	0
		4	214	26	0	0
		6	5780	280	0	0
		8	149222	2136	20	2
7	1	1	4	0	0	0
		2	30	2	0	0
		3	264	22	2	0
		4	2299	49	4	0
		5	18670	240	2	0
		6	148537	1343	32	0

Food for thought

- Let p and ℓ be such that $p \equiv -1 \pmod{\ell}$ and $E_0/\mathbb{F}_p$ supersingular. Can we always find a χ of order ℓ such that $\mathcal{L}(E, \chi, q^{-1}) = 0$?

- For a non-constant elliptic curve $E/\mathbb{F}_q(t)$ and some $\ell > 2$, if there exists χ with $L(E, \chi, 1) = 0$, does that imply the existence of more such characters?

Thanks very much for your attention!!!

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