Geometry-of-numbers in the cusp, and class groups of orders in number fields

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Introduction

- Let $V = \{f(x,y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3\}$
- SL_2 acts on V: $(\gamma \cdot f)(x,y) = f((x,y)\cdot\gamma)$
- Ring of polynomial invariant functions of $\mathrm{SL}_2 \curvearrowright V$ is generated by a single element, namely, the discriminant Δ
- Motivation: $\mathrm{SL}_2(\mathbb{Z})$-orbits on $V(\mathbb{Z})$ naturally parametrize 3-torsion ideal classes of quadratic orders over $\mathbb{Z}$
- Recall two basic facts about orders:
  1. An order $\mathcal{O}$ in a number field K is a subring $\mathcal{O} \subset \mathcal{O}_K$ such that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O} = K$
  2. Quadratic orders are parametrized by discriminant: the map $D \mapsto \mathbb{Z}[\frac{D+\sqrt{D}}{2}]$ defines a bijection between nonsquare integers D and quadratic orders over $\mathbb{Z}$

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Parametrization of 3-torsion classes of quadratic orders

- Let $\mathcal{O}/\mathbb{Z}$ be a quadratic order of discriminant $\Delta \neq 0$
- Let I be frac. ideal of $\mathcal{O}$ such that $I^3 = (\delta)$ for some $\delta \in \mathrm{Frac}(\mathcal{O})^\times$
- Choose $\mathbb{Z}$-bases $\mathcal{O} = \mathbb{Z}\langle 1, \tau\rangle$, $I = \mathbb{Z}\langle \alpha, \beta\rangle$, and define $f_{I,\delta} \in V(\mathbb{Z})$ by

$$f_{I,\delta}(x,y) = \tau\text{-coefficient of } (\alpha x + \beta y)^3/\delta$$

Theorem (Bhargava–Varma, 2016)

Let notation be as above. The map $(I,\delta) \mapsto f_{I,\delta}$ defines a bijection

$$\frac{\left\{ \begin{array}{c} (I,\delta): \\ I^3 = (\delta) \end{array} \right\}}{\mathrm{Frac}(\mathcal{O})^\times} \longleftrightarrow \frac{\left\{ \begin{array}{c} f(x,y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3 \in V(\mathbb{Z}): \\ \Delta(f) = \Delta, \text{ and} \\ \gcd(ac - b^2, ad - bc, bd - c^2) = 1 \end{array} \right\}}{\mathrm{SL}_2(\mathbb{Z})}$$

Moreover, $f_{I,\delta}$ is reducible iff $\delta \in \mathrm{Frac}(\mathcal{O})^{\times 3}$.

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Reducibility of orbits

- Example: If $\delta = 1$, then $I^3 = (1) \Longrightarrow I \in \mathcal{I}(\mathcal{O})[3]$ (i.e., I is a 3-torsion frac. ideal of $\mathcal{O}$)
- If $\mathcal{O} = \overline{\mathcal{O}}$, then $\mathcal{I}(\mathcal{O})^{\text{tors}} = 1$. So, $I^3 = (1) \Longrightarrow I = (1)$, and parametrization implies there exists a unique reducible orbit of $\mathrm{SL}_2(\mathbb{Z})$ on $V(\mathbb{Z})$ with discriminant Δ
- If $\mathcal{O} \subsetneq \overline{\mathcal{O}}$, then $\mathcal{I}(\mathcal{O})^{\text{tors}}$ may be nontrivial — could have more than one reducible orbit per discriminant

- Objective: apply parametrization to determine the average 3-torsion in the class groups of quadratic orders
- To count 3-torsion ideal classes of quadratic *fields*, it suffices to count just the irreducible $\mathrm{SL}_2(\mathbb{Z})$-orbits on $V(\mathbb{Z})$; to do this for quadratic *orders*, one needs to count both irreducible and reducible orbits

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Average 3-torsion in the class groups of quadratic fields

Theorem (Bhargava–Varma 2016, Davenport–Heilbronn 1971)

When real (resp. imaginary) quadratic fields K are ordered by Δ,

$$\mathrm{Avg}\,\#\,\mathrm{Cl}(K)[3] = \frac{4}{3} = 1 + \frac{1}{3} \times 1 \quad (\textit{resp. } 2 = 1 + 1 \times 1)$$

- Here, red = contribution from irreducible orbits, blue = factor having to do with units, and green = contribution from reducible orbits (i.e., one 3-torsion ideal per field)
- Proof relies on parametrization + systematic methods for counting irreducible orbits of representations pioneered by Bhargava
- We generalize above result to all quadratic orders by developing new systematic methods for counting reducible orbits

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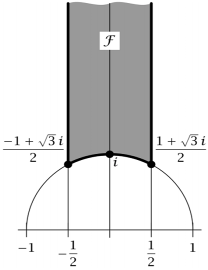
The averaging method

- In what follows, we restrict our consideration to the case of negative discriminant (imaginary quadratic orders) for simplicity
- Let $V(\mathbb{R})^- := \{f \in V(\mathbb{R}) : \Delta(f) < 0\}$; view $\mathrm{SL}_2(\mathbb{Z})$-orbits on $V(\mathbb{Z})$ as lattice points in a fundamental set for action of $\mathrm{SL}_2(\mathbb{Z})$ on $V(\mathbb{R})^-$
- We construct this fundamental set in two steps:
  1. $\mathrm{SL}_2(\mathbb{R}) \curvearrowright V(\mathbb{R})^-$: $\{f \in V(\mathbb{R})^- : \Delta(f) = \Delta\}$ consists of a single $\mathrm{SL}_2(\mathbb{R})$-orbit, represented by $|\Delta|^{1/4} \cdot f_0$, where $f_0(x, y) = 3x^2y - \frac{1}{4}y^3$
  2. $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}) = \mathcal{F}K$, where $K = \mathrm{SO}_2(\mathbb{R})$ and $\mathcal{F} \subset NT$ is the usual guillotine-shaped region in the upper half-plane



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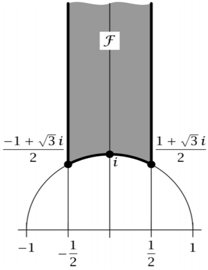

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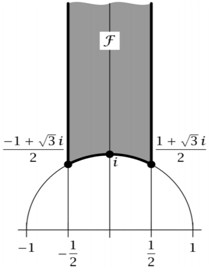
- In what follows, we restrict our consideration to the case of negative discriminant (imaginary quadratic orders) for simplicity
- Let $V(\mathbb{R})^- := \{f \in V(\mathbb{R}) : \Delta(f) < 0\}$; view $\mathrm{SL}_2(\mathbb{Z})$-orbits on $V(\mathbb{Z})$ as lattice points in a fundamental set for action of $\mathrm{SL}_2(\mathbb{Z})$ on $V(\mathbb{R})^-$
- We construct this fundamental set in two steps:
  1. $\mathrm{SL}_2(\mathbb{R}) \curvearrowright V(\mathbb{R})^-$: $\{f \in V(\mathbb{R})^- : \Delta(f) = \Delta\}$ consists of a single $\mathrm{SL}_2(\mathbb{R})$-orbit, represented by $|\Delta|^{1/4} \cdot f_0$, where $f_0(x, y) = 3x^2y - \frac{1}{4}y^3$
  2. $\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R}) = \mathcal{F}K$, where $K = \mathrm{SO}_2(\mathbb{R})$ and $\mathcal{F} \subset NT$ is the usual guillotine-shaped region in the upper half-plane



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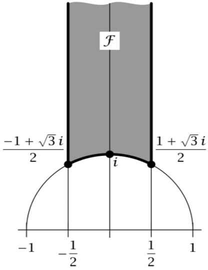

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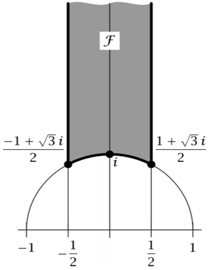

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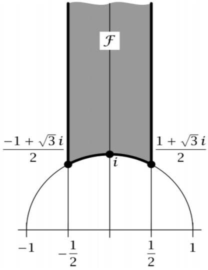

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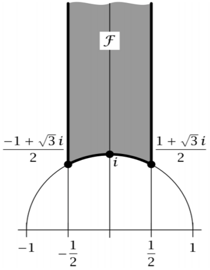

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Proposition

$\mathcal{F}K \cdot \mathbb{R}_{>0} \cdot f_0$ is a fundamental set for $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright V(\mathbb{R})^-$.

Some notation:

- Let $H \subset \mathrm{SL}_2(\mathbb{R})$ be a compact subset of nonzero volume, left $\mathrm{SO}_2(\mathbb{R})$-invariant
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- Given $n \in N, t \in T$, let

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- Turns out: almost all reducible orbits lie in the cusp of fundamental set, namely the regime where t is large
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- For $f \in ntKg \cdot \mathbb{R}_{>0} \cdot f_0$, one checks that $b(f) \ll t^{-1}$, so in the cusp, $b(f)$ is tiny; thus, to estimate $\#(\widetilde{B}(n,t,X) \cap V(\mathbb{Z}))$, we must fiber over x^2y-coefficient
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$N(X) \sim \zeta(2) \cdot X/2$.

- $\zeta(2) =$ fundamental volume of SL_2!

Congruence conditions

- For every prime p, choose an open $\mathrm{SL}_2(\mathbb{Z}_p)$-invariant subset $\Sigma_p \subset V(\mathbb{Z}_p)$ such that $\Sigma_p = V(\mathbb{Z}_p)$ for all $p \gg 1$; write $\Sigma = (\Sigma_p)_p$
- Define $N_\Sigma(X)$ by

$$N_\Sigma(X) := \#\left(\left\{ f \in V(\mathbb{Z}) \cap \bigcap_p \Sigma_p : \begin{array}{c} \Delta(f) \in (-X, 0) \\ f \text{ is red.} \end{array} \right\} \Big/ \mathrm{SL}_2(\mathbb{Z})\right)$$

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- Let $P \subset \mathrm{SL}_2$ be lower-$\triangle$ subgroup; let $V_0 := \{f \in V : a(f) = 0\}$
- Action of SL_2 on V restricts to action of P on V_0

Corollary (SSSV, 2022)

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- Turns out: every reducible orbit has representative f with $a(f) = 0$; for almost every such $f \in V(\mathbb{Z})$, have $P(\mathbb{Z}) \cdot f = \mathrm{SL}_2(\mathbb{Z}) \cdot f \cap V_0(\mathbb{Z})$
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A local-to-global principle for reducible orbits

- Note: the average number of $P(\mathbb{Z})$-orbits on $V_0(\mathbb{Z})$ is the product over primes p of the average number of $P(\mathbb{Z}_p)$-orbits on $V_0(\mathbb{Z}_p)$; is there a local-to-global principle that explains this observation?

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- Two ingredients needed for the proof:
  1. The group P has class number 1 over $\mathbb{Q}$ (i.e., $P(\mathbb{A}_\mathbb{Q}) = P(\mathbb{A}_\mathbb{Z}) \cdot P(\mathbb{Q})$)
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Application to counting 3-torsion ideal classes

- For every prime p, choose an open subset $D_p \subset \mathbb{Z}_p$ such that $D_p = \mathbb{Z}_p$ for all $p \gg 1$

Corollary (SSSV 2022)

$$\underset{\substack{\Delta(\mathcal{O})<0 \\ \Delta(\mathcal{O})\in\bigcap_p D_p}}{\operatorname{Avg}} \#\operatorname{Cl}(\mathcal{O})[3] = 1 + 1 \times \underset{\substack{\Delta(\mathcal{O})<0 \\ \Delta(\mathcal{O})\in\bigcap_p D_p}}{\operatorname{Avg}} \#\mathcal{I}(\mathcal{O})[3]$$

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- Note: the average number of 3-torsion ideals is the product over primes p of the corresponding local averages; is there a local-to-global principle that explains this observation?

Theorem

Let $\mathcal{O}$ be an order in a number field. Then $\mathcal{I}(\mathcal{O}) \simeq \bigoplus_p \mathcal{I}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O})$, and in particular, $\mathcal{I}(\mathcal{O})^{\text{tors}} \simeq \prod_p \mathcal{I}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O})^{\text{tors}}$.

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Theorem (Bhargava–Varma 2016, SSSV 2022)

When real (resp. imaginary) quadratic orders $\mathcal{O}$ are ordered by Δ,

$$\mathrm{Avg}\,\#\,\mathrm{Cl}(\mathcal{O})[3] = 1 + \frac{1}{3} \times \frac{\zeta(2)}{\zeta(3)} \quad \left(\textit{resp. } 1 + 1 \times \frac{\zeta(2)}{\zeta(3)}\right)$$

- Similar methods work to count reducible orbits of many other representations that occur in arithmetic statistics!
- Integral orbits of such representations can be used to parametrize class groups in various natural families of orders in number fields
- As a consequence, we can prove analogues for *orders* of theorems on average sizes of class groups of *fields*

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Further applications

Theorem (Bhargava 2005)

When totally real (resp., complex) cubic fields K are ordered by Δ,

$$\mathrm{Avg}\,\#\,\mathrm{Cl}(K)[2] = \frac{5}{4} = 1 + \frac{1}{4} \times 1 \quad \left(\textit{resp.}\ \frac{3}{2} = 1 + \frac{1}{2} \times 1\right)$$

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- An order $\mathcal{O}$ is *monogenic* if $\mathcal{O} = \mathbb{Z}[\theta]$ for some algebraic integer θ

Theorem (Bhargava–Hanke–Shankar 2019)

When totally real (resp., complex) cubic fields K with monogenic ring of integers are ordered by Δ,

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Theorem (Bhargava–Hanke–Shankar 2019)

When totally real (resp., complex) cubic fields K with monogenic ring of integers are ordered by Δ*,*

$$\mathrm{Avg}\,\#\,\mathrm{Cl}(K)[2] = \frac{3}{2} = 1 + \frac{1}{2} \times 1 \quad (\textit{resp. } 2 = 1 + 1 \times 1)$$

Theorem (SSSV 2022)

When totally real (resp., complex) cubic monogenic orders $\mathcal{O}$ *are ordered by* Δ*,*

$$\mathrm{Avg}\,\#\,\mathrm{Cl}(\mathcal{O})[2] = 1 + \frac{1}{2} \times \frac{\zeta(2)}{\zeta(3)} \quad \left(\textit{resp. } 1 + 1 \times \frac{\zeta(2)}{\zeta(3)}\right)$$

Thank You!!