On the prime Selmer ranks of cyclic prime twist families of elliptic curves over global function fields

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Preliminary

Let $K = \mathbb{F}_q(t)$ of characteristic $\neq 2, 3$.
Fix a non-isotrivial elliptic curve $E : y^2 = x^3 + Ax + B$ over K.
Given a polynomial $f \in \mathbb{F}_q[t]$, denote by E_f the quadratic twist of E by f:

$$E_f : y^2 = x^3 + f^2 Ax + f^3 B$$

Question A

As the degree of f grows arbitrarily large, what is the probability that $\mathrm{Rank}(E_f(K)) = r$?

$$\lim_{d \to \infty} \frac{\#\{f \in \mathbb{F}_q[t] \mid \deg f = d,\ \mathrm{Rank}(E_f(K)) = r\}}{\#\{f \in \mathbb{F}_q[t] \mid \deg f = d\}}$$

Preliminary

Strategy: Use p-Selmer groups: $\operatorname{rank}(E(K)) \leq \dim_{\mathbb{F}_p} \mathrm{Sel}_p(E/K)$.

Question A'

As the degree of f grows arbitrarily large, what is the probability that $\dim_{\mathbb{F}_p} \mathrm{Sel}_p(E_f/K) = r$?

$$\lim_{d \to \infty} \frac{\#\{f \in \mathbb{F}_q[t] \mid \deg f = d,\ \dim_{\mathbb{F}_p} \mathrm{Sel}_p(E_f/K) = r\ \}}{\#\{f \in \mathbb{F}_q[t] \mid \deg f = d\}}$$

Bhargava-Kane-Lenstra-Poonen-Rains Heuristics (2015)

Given any prime number p coprime to $\operatorname{char}(K)$, the desired probability that $\dim_{\mathbb{F}_p} \mathrm{Sel}_p(E_f/K) = r$ converges to

$$BKLPR(p,r) := \left(\prod_{j \geq 0} \frac{1}{1+p^{-j}} \right) \left(\prod_{j=1}^{r} \frac{p}{p^j - 1} \right)$$

Previous Studies

There are two parameters to control.

- q: **# Constant Field**
- d: **Degree**

Question 1: Large q, then Large d

$$\lim_{d\to\infty}\lim_{q\to\infty}\frac{\#\{f\in\mathbb{F}_q[t]\,|\,\deg f=d,\ \dim_{\mathbb{F}_p}\mathrm{Sel}_p(E_f/K)=r\ \}}{\#\{f\in\mathbb{F}_q[t]\,|\,\deg f=d\}} \tag{1}$$

Question 2: Large d, then Large q

$$\lim_{q\to\infty}\lim_{d\to\infty}\frac{\#\{f\in\mathbb{F}_q[t]\,|\,\deg f=d,\ \dim_{\mathbb{F}_p}\mathrm{Sel}_p(E_f/K)=r\ \}}{\#\{f\in\mathbb{F}_q[t]\,|\,\deg f=d\}} \tag{2}$$

Previous Studies

	Family		Large q, Large d		Large d, Large q	
	Univ.	Quad.	Average	Prob.	Average	Prob.
Landesman (2021)	✓		✓			
P., Wang (2021)		✓	✓			
Feng Landesman (2022) Rains	✓			✓		
de Jong (2002)	✓				✓ ($p = 3$)	
Hô Lê Húng (2013) Ngô	✓				✓ ($p = 2$)	

Previous studies (Optional)

Motivation (1st moments):

- A representable finite étale cover $\tau_d \to \mathcal{M}_d$ over moduli space of quad. twist families of elliptic curves E_f in $\mathcal{M}_d$ twisted by degree d polynomials f over $\mathbb{F}_q$.
- Want: $\mathbb{F}_q$-rational points of geometric fibers of $\tau_d \to \mathcal{M}_d$ to parametrize $\mathrm{Sel}_p(E_f/K)$.
- Grothendieck-Lefschetz trace formula

$$\mathbb{E}[\dim_{\mathbb{F}_p} \mathrm{Sel}_p(E_f/K) = r \mid \deg f = d] = \frac{1}{q^d} \sum_{i=0}^{2d} (-1)^i \mathrm{Tr} Frob_q | H^i_{\text{ét,c}}((\tau_d)_{\overline{\mathbb{F}_q}}, \mathbb{Q}_l)$$

- **Question 1**: Suffice to understand $H^{2d}_{\text{ét,c}}((\tau_d)_{\overline{\mathbb{F}_q}}, \mathbb{Q}_l)$.
- **Question 2**: Understand $H^i_{\text{ét,c}}((\tau_d)_{\overline{\mathbb{F}_q}}, \mathbb{Q}_l)$ for all $0 \leq i \leq 2d$.
- **Large q**: $O(\frac{1}{\sqrt{q}})$, originating from $H^i_{\text{ét,c}}((\tau_d)_{\overline{\mathbb{F}_q}}, \mathbb{Q}_l)$ for $i \leq 2d-1$.

Previous Studies

Let's return back to the original question.

$$\lim_{d\to\infty} \frac{\#\{f \in \mathbb{F}_q[t] \mid \deg f = d,\ \mathrm{Sel}_p(E_f/K) = r\ \}}{\#\{f \in \mathbb{F}_q[t] \mid \deg f = d\}}$$

Meanwhile...

Let's use the generalized Riemann hypothesis over $\mathbb{F}_q(t)$.
Here are some notable consequences:

1. Effective Chebotarev density theorem
2. Effective Erdös-Kac theorem

Previous Studies

	Family		Large q, Large d		Large d, Large q	
	Univ.	Quad.	Average	Prob.	Average	Prob.
Landesman (2021)	✓		✓			
P., Wang (2021)		✓	✓			
Feng Landesman (2022) Rains	✓			✓		
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Previous Studies

	Family		Large q, Large d		Large d, Large q		**Large d**
	Univ.	Quad.	Average	Prob.	Average	Prob.	**Prob.**
Landesman (2021)	✓		✓				
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Previous Studies

	Family		Large q, Large d		Large d, Large q		**Large d**
	Univ.	Quad.	Average	Prob.	Average	Prob.	**Prob.**
Landesman (2021)	✓		✓				
P., Wang (2021)		✓	✓				
Feng Landesman (2022) Rains	✓			✓			
de Jong (2002)	✓				✓ ($p = 3$)		
Hồ Lê Húng (2013) Ngô	✓				✓ ($p = 2$)		
P. (2022)		✓					✓ ($p = 2$)

Main Theorem

Theorem (P. (2022))

Suppose E/K satisfies the following three conditions.

1. E is non-isotrivial.
2. $\mathrm{Gal}(K(E[2])/K) \cong S_3$.
3. E has a place of split multiplicative reduction.

Then for sufficiently large d, there exists a fixed constant $B > 0$ independent of d and q such that

$$|\mathbb{P}[\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E_f) = r \mid \deg f = d] - BKLPR(2, r)| < B \cdot d^{-1/3}.$$

Motivation

Key Tools:

1. Effective Erdös-Kac Theorem
2. Effective Chebotarev Density Theorem
3. Ergodicity of Markov chains

Klagsbrun, Mazur, Rubin (2014): (Number fields) Loosely speaking...

1. Variations of **non-canonically ordered** $\mathrm{Sel}_2(E_{p_1 p_2 \cdots p_k})$ twisted by consecutive **k** prime twists $p_1, p_2, \cdots, p_k$.
2. Apply Markov Chain by **k** iterations.

Motivation

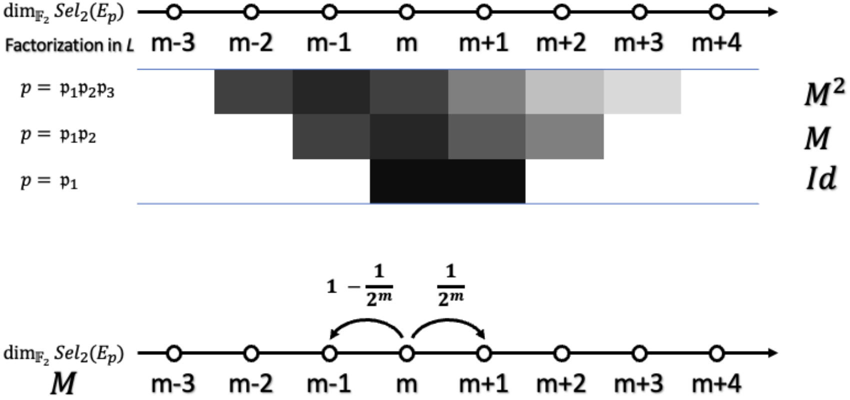
$E_p^{[1]}: py^2 = x^3 + x - 1$
$0 < p < 500{,}000$
$\dim_{\mathbb{F}_2} Sel_2(E^{[1]}) = 1$
$L = \mathbb{Q}[x]/(x^3 + x - 1)$

$\dim_{\mathbb{F}_2} Sel_2(E_p^{[1]})$ / Factorization in L	0	1	2	3	4	5	6	7
$p = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$	12.56	65.95	12.49	9.00	0	0	0	0
$p = \mathfrak{p}_1\mathfrak{p}_2$	37.41	25.01	37.58	0	0	0	0	0
$p = \mathfrak{p}_1$	25.04	74.96	0	0	0	0	0	0

$E_p^{[2]}: py^2 = x^3 + 4x + 1$
$0 < p < 500{,}000$
$\dim_{\mathbb{F}_2} Sel_2(E^{[2]}) = 1$
$L = \mathbb{Q}[x]/(x^3 + 4x + 1)$

$\dim_{\mathbb{F}_2} Sel_2(E_p^{[2]})$ / Factorization in L	0	1	2	3	4	5	6	7
$p = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$	18.63	43.80	29.94	6.04	1.59	0	0	0
$p = \mathfrak{p}_1\mathfrak{p}_2$	25.14	37.68	24.68	12.50	0	0	0	0
$p = \mathfrak{p}_1$	0	50.04	49.96	0	0	0	0	0

Motivation



$\dim_{\mathbb{F}_2} Sel_2(E_p)$
Factorization in L
m-3 m-2 m-1 m m+1 m+2 m+3 m+4
$p = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$
$p = \mathfrak{p}_1\mathfrak{p}_2$
$p = \mathfrak{p}_1$
M^2
M
Id
$1 - \frac{1}{2^m}$
$\frac{1}{2^m}$
$\dim_{\mathbb{F}_2} Sel_2(E_p)$
M
m-3 m-2 m-1 m m+1 m+2 m+3 m+4

Idea of Proof

1. Markov Chain of our interest:

$$\frac{1}{3} + \frac{1}{2}M + \frac{1}{6}M^2 \tag{3}$$

2. **Use Chebotarev density theorem** over L^*/K:

$$L^* := \text{Fixed field of} \bigcap_{c \in \mathrm{Sel}_2(E_{p_1 p_2 \cdots p_k})} \mathrm{Ker}\left(c : \mathrm{Gal}(\overline{K}/K(E[2])) \to E[2]\right)$$

(Require that $\mathrm{Gal}(K(E[2])/K) \cong S_3$.)

Yes: Consecutive iteration of $M_L \iff$ Consecutive prime twist of E.

But: Assume $0 < |p_1| \ll |p_2| \ll \cdots \ll |p_k|$.

Non-canonical ordering of E_f's.

Unknown explicit error terms.

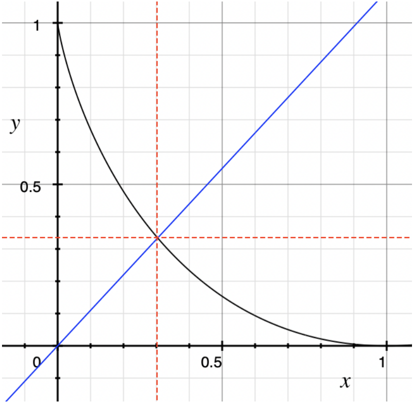
Idea of Proof

Fix a constant $0 < \rho < 1$.

1. **Erdös-Kac**: # iterations of M_L
   - Distinct # prime factors: At least $\rho \ln d$ factors.
   - **Error term**: $\epsilon_{EK} \sim O(d^{-\rho \ln \rho + \rho - 1})$.
2. **Chebotarev**: Deviation from M_L under **canonical order**
   - Growth of prime factors: $\deg p_i > \frac{4(\ln d)^2}{\ln q}$ (for every i)
   - Reordering: $\deg p^* > \sqrt[3]{d}$ for at least one inert prime factor p^*.
   - **Error term**: $\epsilon_{Cheb} \sim O(d^{-\ln d})$.
3. **Ergodicity**: Limiting behavior of M_L
   - Stationary Distribution: BKLPR probability distribution.
   - Convergence rate: Geometric convergence at a rate $\gamma > \frac{1}{3}$.
   - **Error term**: $\epsilon_{Markov} \sim O(d^{-\rho \ln 3})$.

Idea of Proof

Total Error: $B(\epsilon_{EK} + \epsilon_{Cheb} + \epsilon_{Markov}) \sim B \cdot \left(\frac{1}{d^{\rho \ln \rho + 1 - \rho}} + d^{-\ln d} + \frac{1}{d^{\rho \ln 3}}\right)$.



Hence, we can choose $\rho \sim 0.304$ to obtain the desired error bounds

$$B \cdot \left(\frac{1}{d^{\rho \ln \rho + 1 - \rho}} + d^{-\ln d} + \frac{1}{d^{\rho \ln 3}}\right) \sim B \cdot d^{-1/3}.$$

Ending Remarks (Optional)

Mazur, Rubin (2007): Assume $\mu_p \subset K$.

$$E_{p,f} := \mathrm{Ker}\left(\mathbb{N} : \mathrm{Res}_K^{K(\sqrt[p]{f})} E \to E\right) \tag{4}$$

Note $\mathrm{Gal}(K(\sqrt[p]{f})/K) \cong \mathbb{Z}/p\mathbb{Z}$.
Choose a cyclic generator $\sigma_{p,f} \in \mathrm{Gal}(K(\sqrt[p]{f})/K)$.
Similar to quadratic twist families, assume $\mathrm{Gal}(K(E[p])/K) \supset \mathrm{SL}_2(\mathbb{F}_p)$,

$$\left|\mathbb{P}[\dim_{\mathbb{F}_p} \mathrm{Sel}_{1-\sigma_{p,f}}(E_f) = r \mid \deg f = d] - BKLPR(p,r)\right| < B \cdot \frac{1}{d^{\alpha(p)}} \tag{5}$$

If Alex Smith's work is applicable to $K = \mathbb{F}_q(t)$, then we can hope:

$$\lim_{n\to\infty} \mathbb{P}\left[\mathrm{Corank}_{\mathbb{Z}_p} \mathrm{Sel}_{(1-\sigma_{p,f})^\infty}(E_{p,f}/K) = r \mid \deg f = d\right] = \begin{cases} \frac{1}{2} & \text{if } r = 0 \\ \frac{1}{2} & \text{if } r = 1 \\ 0 & \text{otherwise} \end{cases}$$

Ending Remarks (Optional)

Future Directions: Non-abelian twist families over global fields

$$E_n/\mathbb{Q} : y^2 = x^3 - 432n^2 \tag{6}$$

Consider the following 4-dimensional abelian variety over $\mathbb{Q}$:

$$B_n := \mathrm{Ker}\left(\mathbb{N} : \mathrm{Res}_{\mathbb{Q}}^{\mathbb{Q}(\zeta_3, \sqrt[3]{n})} E_1 \to \mathrm{Res}_{\mathbb{Q}}^{\mathbb{Q}(\zeta_3)} E_1\right) \tag{7}$$

Pick a generator of the cyclic subgroup $\sigma_{3,n} \in \mathbb{Z}/3\mathbb{Z} \subset S_3$.

$$\begin{gathered} B_n[1 - \sigma_{3,n}] \cong E_1[3]^{\oplus 2}. \\ \mathrm{Rank}(E_n(\mathbb{Q})) + \mathrm{Rank}(E_{n^2}(\mathbb{Q})) \leq \dim_{\mathbb{F}_3} \mathrm{Sel}_{1-\sigma_{3,n}}(B_n/\mathbb{Q}). \end{gathered} \tag{8}$$

In progress: Compute upper / lower bounds on the probability of the addition of ranks of E_n and E_{n^2} over $\mathbb{Q}$, assuming GRH.

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