Arithmetic Statistics Questions Motivated by Knot Theory

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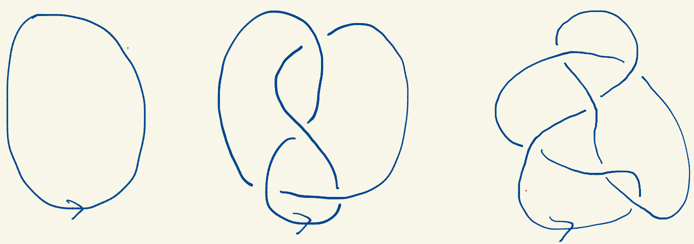
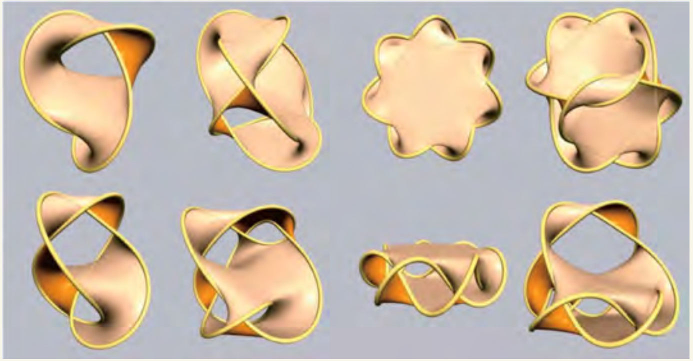
Knots & Seifert Surfaces

A knot K is an oriented S^1 embedded in S^3



[Technicalities – we are in the topological category & all embeddings are locally flat]

A Seifert surface Σ is an oriented surface embedded in S^3 with boundary a knot



[image credit: van Wijk & Cohen SeifertView]

The natural map

$$\{\text{Seifert surfaces}\} \longrightarrow \{\text{knots}\}$$

is surjective but very much not injective

The Seifert Pairing

Σ a Seifert surface of genus g

$H^1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ has a natural alternating pairing:

the intersection pairing

it also has a natural non-symmetric pairing

$$\text{Seifert}: H^1(\Sigma, \mathbb{Z}) \times H^1(\Sigma, \mathbb{Z}) \to \mathbb{Z}$$

$$(\alpha, \beta) \longmapsto \mathrm{lk}(\alpha^+, \beta)$$

the alternating part of Seifert is the intersection pairing

the symmetric part is called the quadratic form of Σ

— but it should be thought of as

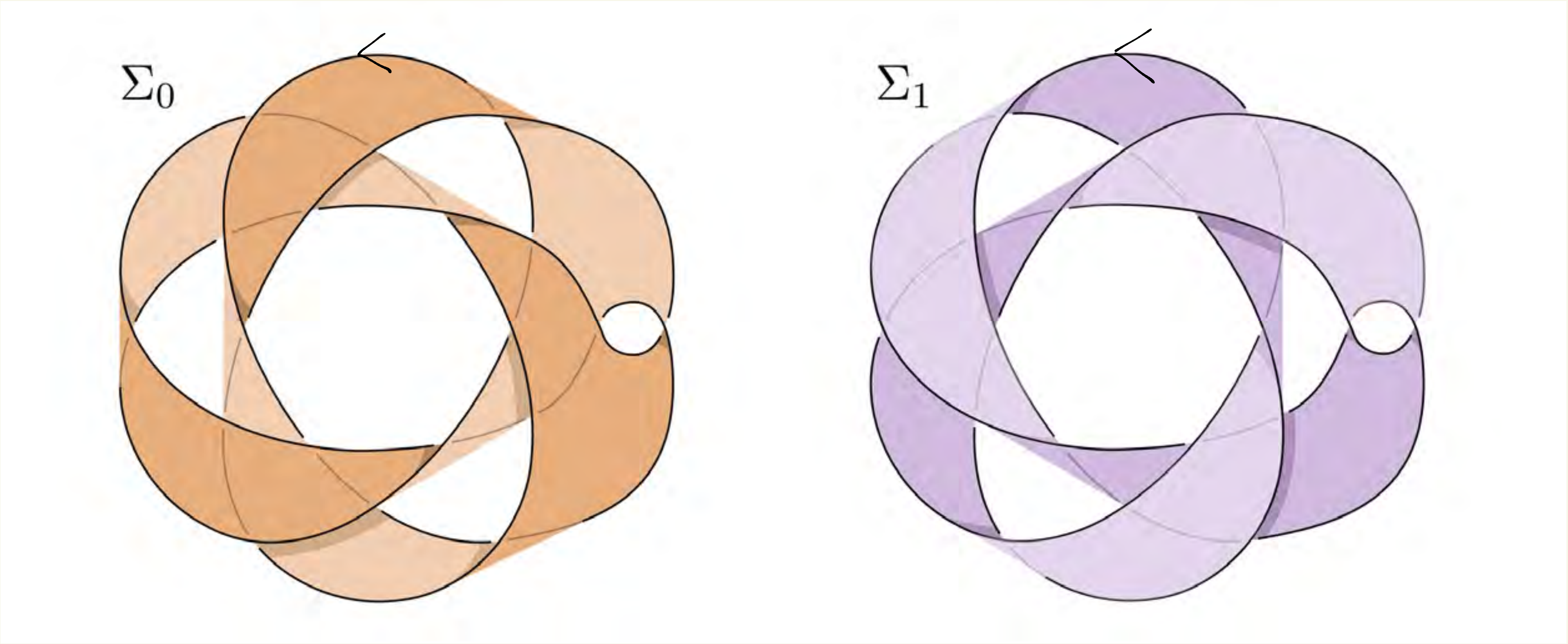
a quadratic form up to $Sp_{2g}(\mathbb{Z})$-equivalence!

Diagram

$$\begin{tikzcd}
\begin{matrix}\text{genus } g \\ \text{Seifert surfaces}\end{matrix} \arrow[r, two heads] \arrow[d, two heads] & \begin{matrix}Sp_{2g}(\mathbb{Z})\text{-equivalence classes} \\ \text{of } 2g\text{-ary quadratic forms} \\ \text{(with mod 2 conditions)}\end{matrix} \arrow[d, two heads] \\
\begin{matrix}\text{genus} \leq g \\ \text{knots}\end{matrix} \arrow[r, two heads] & \begin{matrix}S\text{-equivalence classes of } g \times g \\ \text{Seifert pairing matrices}\end{matrix}
\end{tikzcd}$$

The Hayden-Kim-Miller-Park-Sundberg example

Kyle Hayden, Seungwon Kim, Maggie Miller, JungHwan Park, and Isaac Sundberg, Seifert surfaces in the 4-ball arXiv:2205.15283 [math.GT]

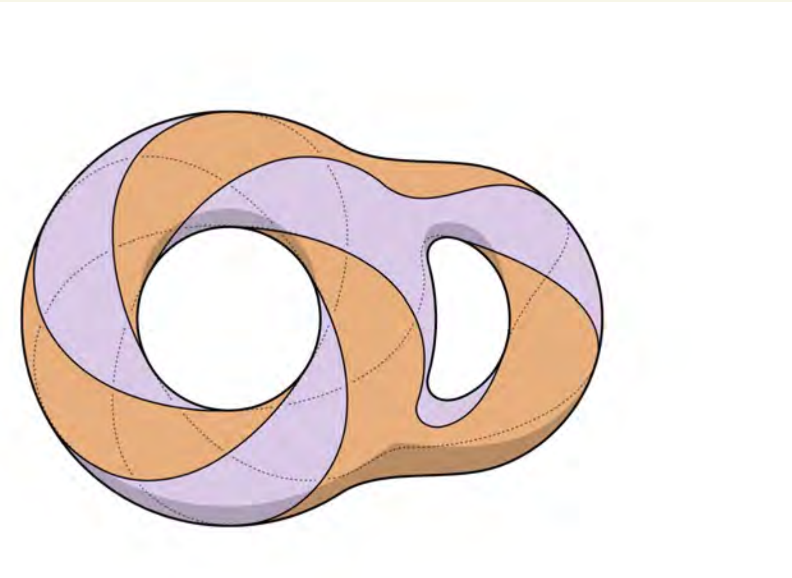


$2x^2+xy+3y^2$ Disc $= -23$ $x^2+xy+6y^2$

Not $Sp_2(\mathbb{Z}) = SL_2(\mathbb{Z})$ -equivalent

Not even $GL_2(\mathbb{Z})$ - equivalent $\Rightarrow$ surfaces still different when pushed into D^4

Where the HKMPS example comes from:



Any separating curve on a genus 2 surface gives a knot with two genus 1 Seifert surfaces.

Work in progress with Menny Aka, Peter Feller, Andreas Wieser characterizing which pairs of quadratic forms are attained this way

Sp_{2g}-orbits of quadratic forms

For $g=1$ $Sp_2 = SL_2$ and this is SL_2 acting on binary quadratic forms!

Invariant theory

- Invariants of a quadratic form with matrix Q given by coeffs of $\det(t \cdot J - Q)$ ← related to Alexander polynomial
- orbits contain extra arithmetic info related to Alexander Module + Blanchfield pairing

Asymptotic counting of Sp_{2g}-equivalence classes of quadratic forms with bounded invariants

applying Bhargava's averaging method to get analogous results

Genus one knots

$D = 1-4m$ squarefree
$m \neq 0$.

oriented class group $Cl^+(R)$

$$= \left\{ \begin{matrix} \text{fractional ideal } \mathfrak{a} \text{ of } R \\ + \text{ generator of } \Lambda^2 \mathfrak{a} \end{matrix} \right\} \Big/ \sim$$

The diagram:

$$\begin{tikzcd}
\begin{matrix}\text{genus } 1 \\ \text{Seifert surfaces} \\ \text{of disc } D\end{matrix} \arrow[r, two heads] \arrow[d, two heads] & \begin{matrix} SL_2(\mathbb{Z})\text{-equivalence classes} \\ \text{of binary quad forms} \\ \text{are } \mathbb{Z} \text{ of det } D \end{matrix} \arrow[r, leftrightarrow] \arrow[d, two heads] & Cl^+\left(\mathbb{Z}\left[\tfrac{1+\sqrt{D}}{2}\right]\right) \arrow[d, two heads, "i_*"] \\
\begin{matrix}\text{genus } 1 \\ \text{Knots of disc } D \\ (= \det -D)\end{matrix} \arrow[r, two heads] & \begin{matrix} SL_2(\mathbb{Z}[\tfrac{1}{m}])\text{-equivalence classes} \\ \text{of binary quad forms} \\ \text{over } \mathbb{Z}[\tfrac{1}{m}] \text{ of det } D \end{matrix} \arrow[r, leftrightarrow] & Cl^+\left(\mathbb{Z}[\tfrac{1}{m}]\left[\tfrac{1+\sqrt{D}}{2}\right]\right)
\end{tikzcd}$$

The kernel

$$0 \longrightarrow \ker i_* \longrightarrow Cl^+\left(\mathbb{Z}\left[\tfrac{1+\sqrt{D}}{2}\right]\right) \xrightarrow{i_*} Cl^+\left(\mathbb{Z}\left[\tfrac{1}{m}\right]\left[\tfrac{1+\sqrt{D}}{2}\right]\right) \longrightarrow 0$$

$\ker i_*$ is generated by $g_p = \left[\left(p, \frac{1+\sqrt{D}}{2}\right)\right]^2$ $\quad p \mid m = \frac{1-D}{4}$

they satisfy $\prod_p g_p^{v_p(m)} = [1]$

$\ker i_* = \{1\}$ if $m = p$ prime

Cohen-Lenstra lets us model $Cl^+\left(\mathbb{Z}\left[\frac{1+\sqrt{D}}{2}\right]\right)$

can extend to a model of $Cl^+\left(\mathbb{Z}\left[\frac{1}{m}\right]\left[\frac{1+\sqrt{D}}{2}\right]\right)$

by assuming g_p random subject to $\prod_p g_p^{v_p(m)} = [1]$

Predictions of the model:

- with probability 1 $\quad \ker i_* = Cl^+(\mathbb{Z}[\tfrac{1+\sqrt{D}}{2}])^2$

$$Cl^+(\mathbb{Z}[\tfrac{1}{m}][\tfrac{1+\sqrt{D}}{2}]) \simeq Cl^+(\mathbb{Z}[\tfrac{1+\sqrt{D}}{2}]) / Cl^+(\mathbb{Z}[\tfrac{1+\sqrt{D}}{2}])^2$$

for $D<0$ expected size $\log D$

- however in the family $m = p > 0$ prime

we know $\ker i = \{1\}$ and $Cl^+(\mathbb{Z}[\tfrac{1}{m}][\tfrac{1+\sqrt{D}}{2}]) \simeq Cl^+(\mathbb{Z}[\tfrac{1+\sqrt{D}}{2}])$

for $D<0$ expected size $\sqrt{D}$

combining, predict

$$\sum_{\substack{D\in[-X,0] \\ D = 1-4m \\ D \text{ sq free}}} |Cl^+(\mathbb{Z}[\tfrac{1}{m}][\tfrac{1+\sqrt{D}}{2}])| \asymp \frac{X^{\frac{3}{2}}}{\log X}$$

with main term coming from $D = 1-4p$

Back to Seifert Matrices – predictions and results

removing the square free condition we expect the same asymptotics

Theorem: [M-Xiao]

$$\#\{S\text{-equivalence classes of } 2\times 2 \text{ Seifert matrices with discriminant } 1-4p \in [-X,0]\} \simeq \frac{X^{3/2}}{\log X}$$

Conjecture:

$$\#\{S\text{-equivalence classes of } 2\times 2 \text{ Seifert matrices with discriminant } 1-4m \in [-X,0],\ m \text{ not prime}\} \simeq X \log X$$

Thm: [M]

This count is $o(X^{3/2})$ – can we do better?