Frobenius distributions of abelian varieties over finite fields[1]

Joint with Deewang Bhamidipati and Soumya Sankar

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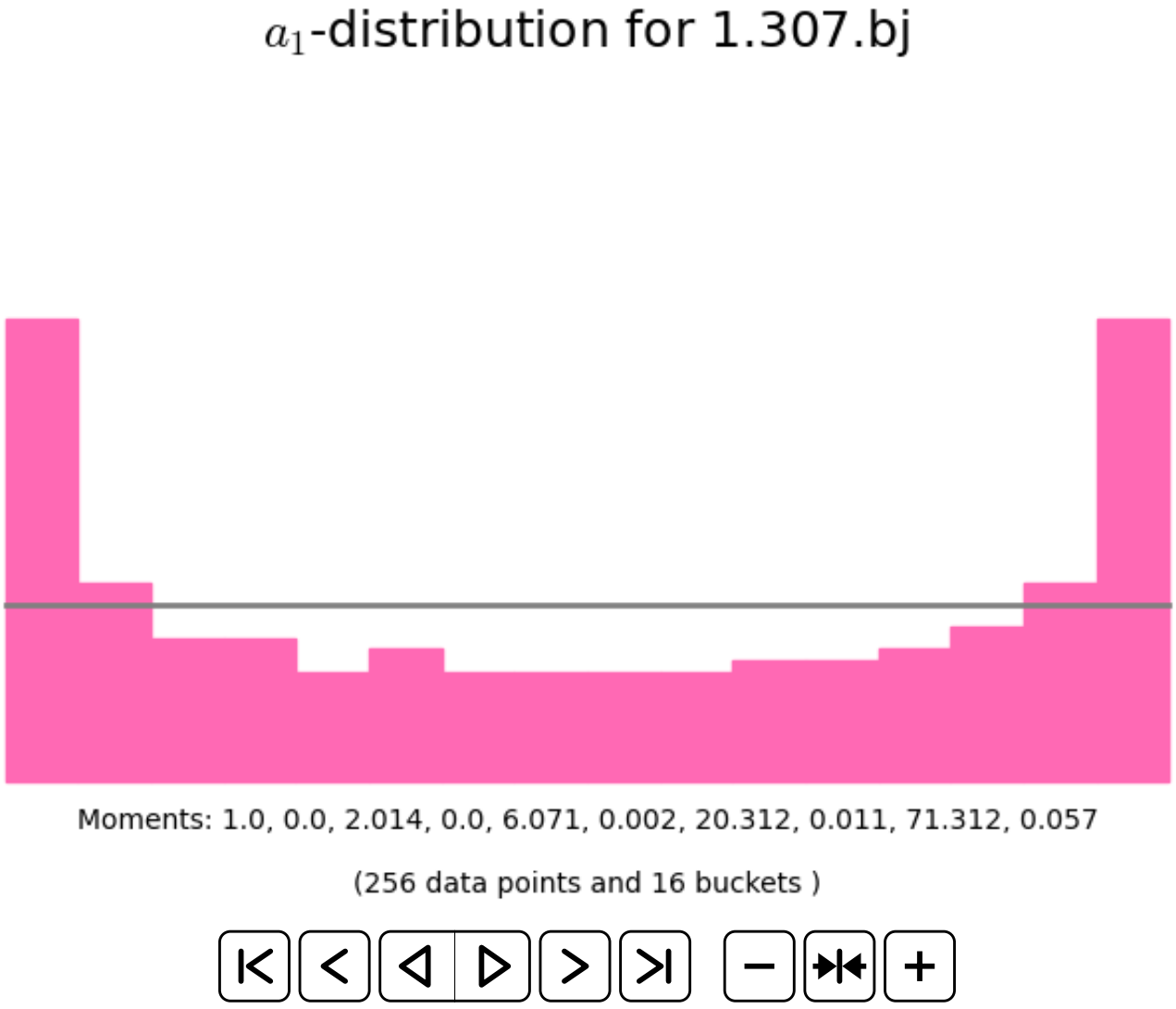
Emory University

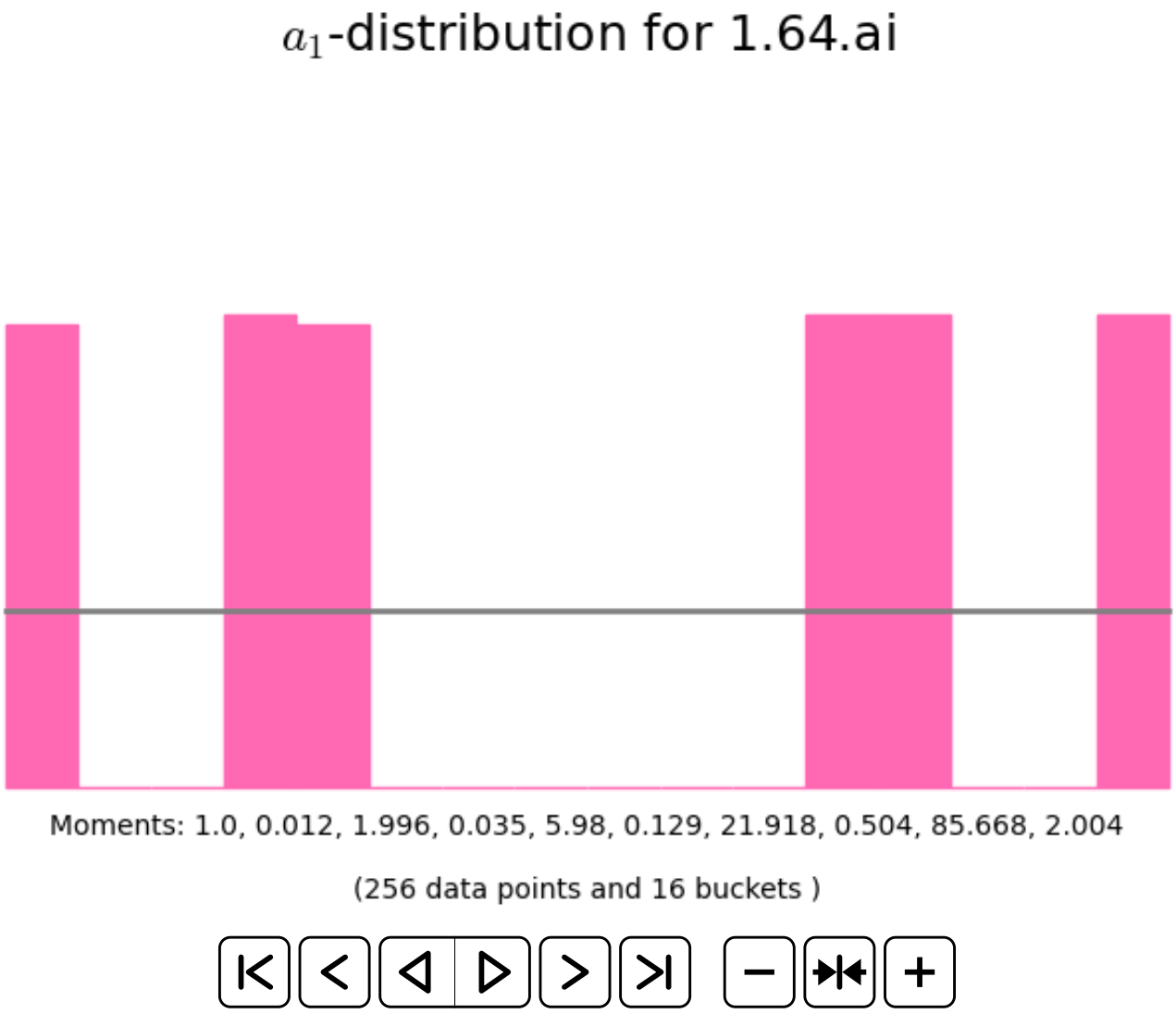
Conférence Statistiques Arithmétiques
CIRM
May 15, 2023

[1] Requires Acrobat Reader to play the animations.

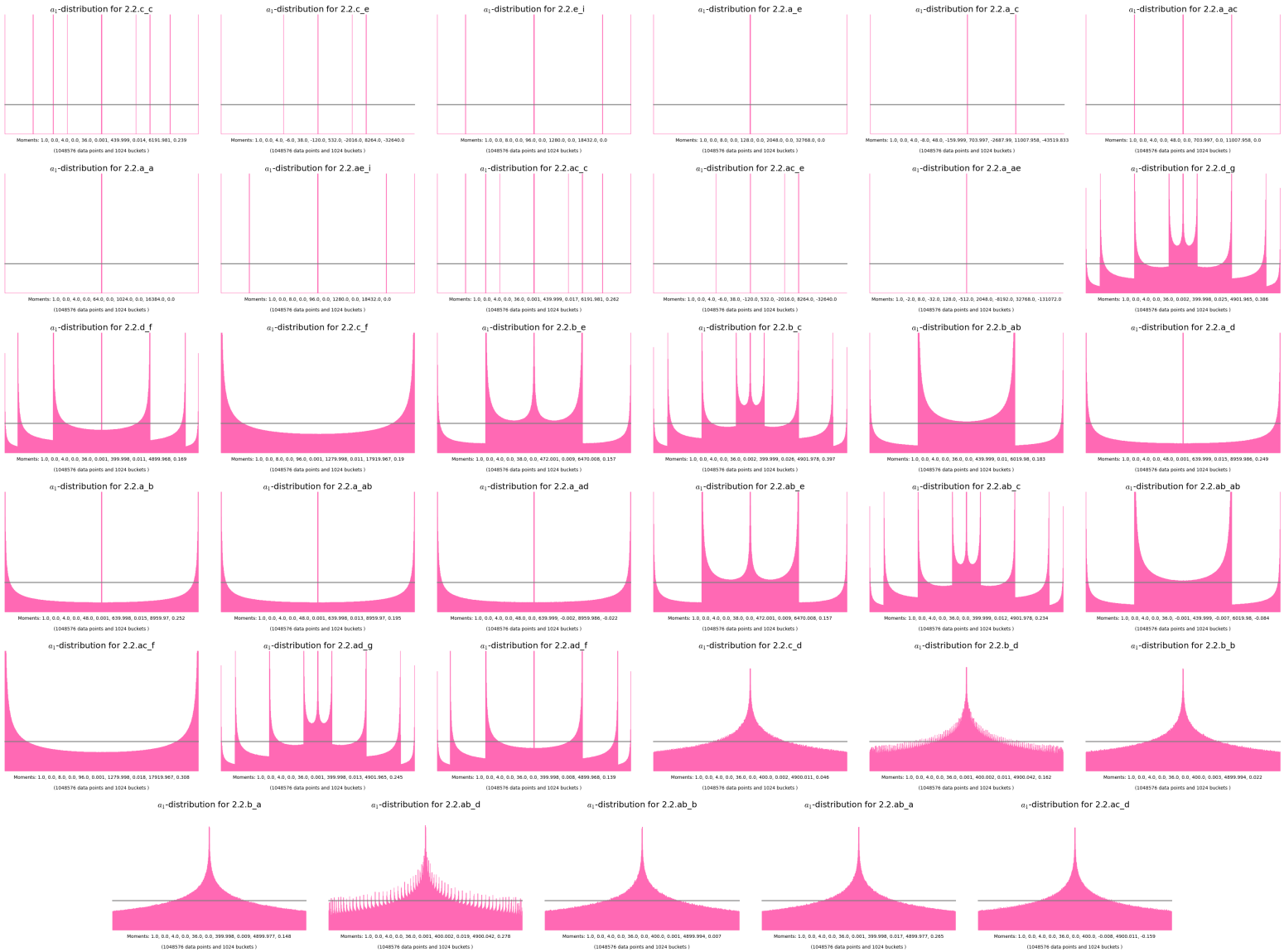
Ordinary elliptic curve 1.307.bj



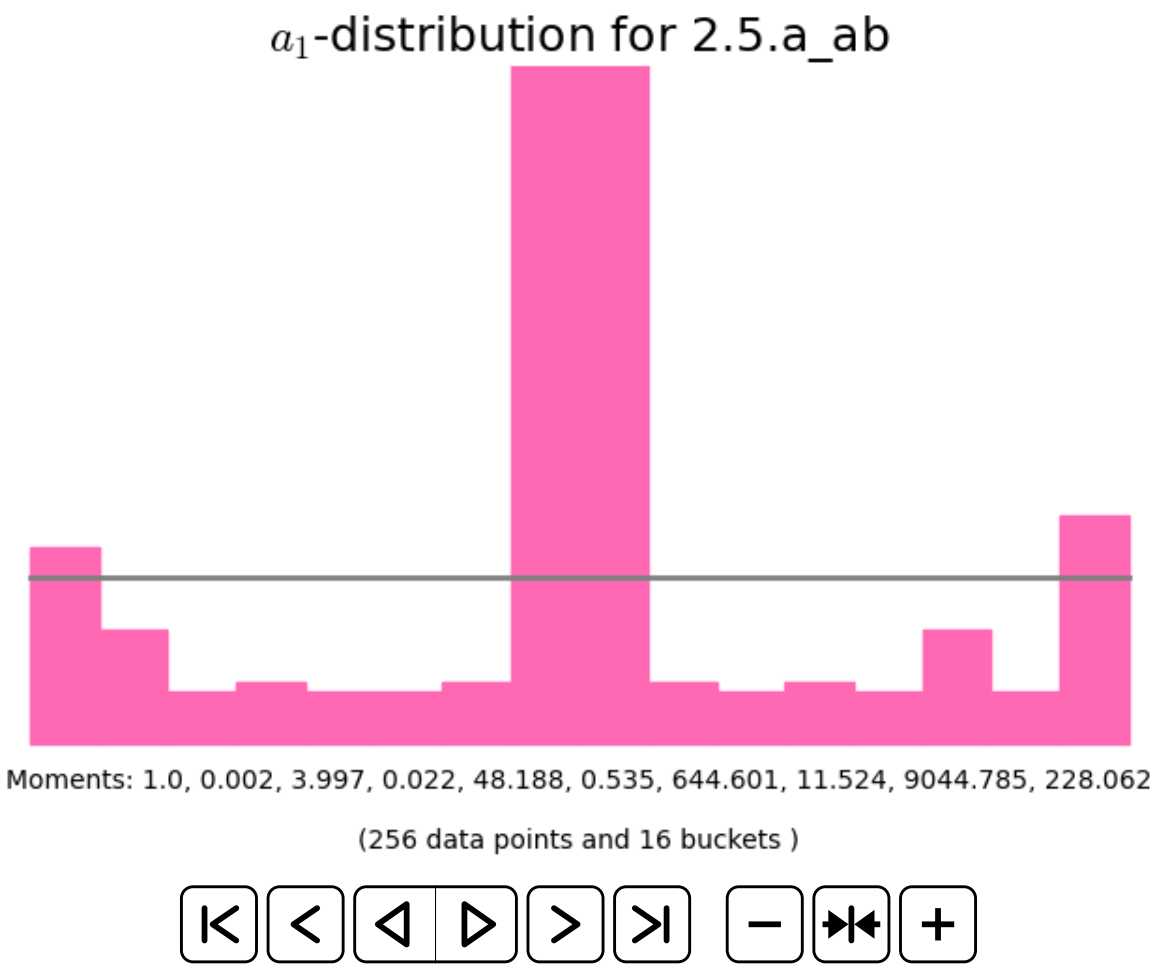
Supersingular elliptic curve 1.64.ai



The 35 isogeny classes of abelian surfaces over $\mathbf{F}_2$

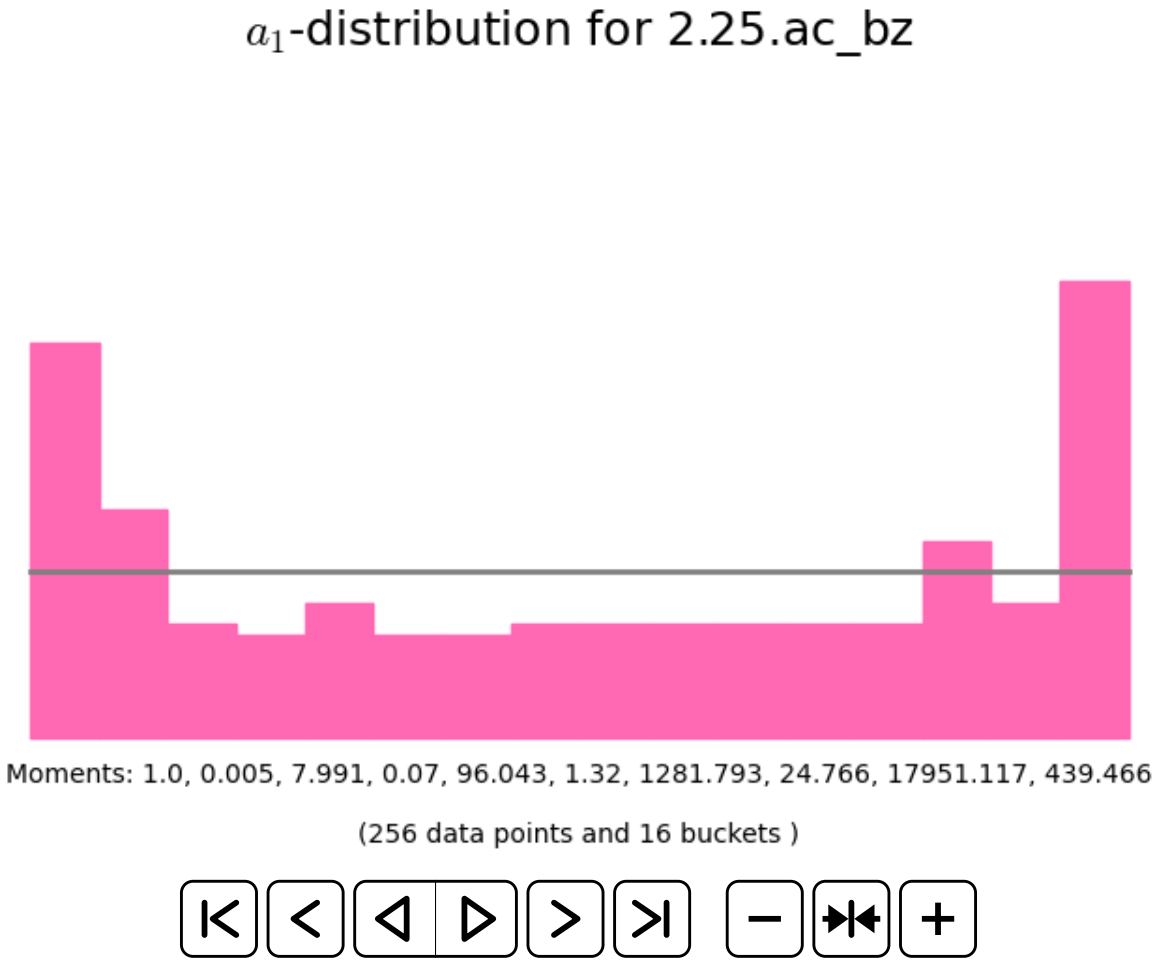


Simple ordinary surface 2.5.a_ab.



Let $A/\mathbf{F}_5$ be a simple ordinary abelian variety in this isogeny class.

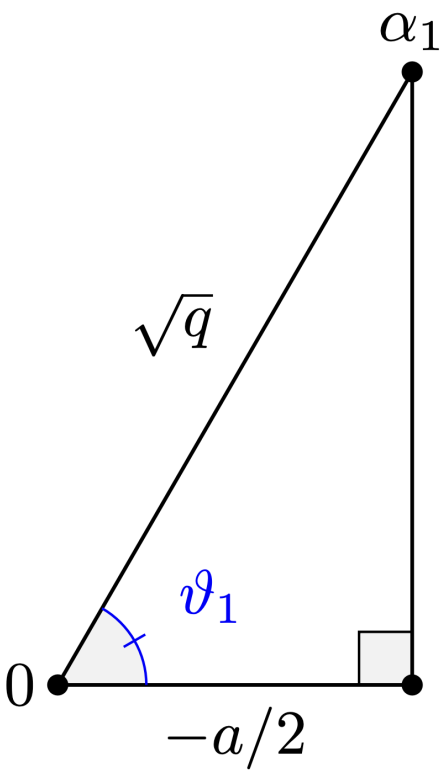
Ordinary surface 2.25.ac_bz $= 1.25.\mathrm{ab}^2$.



Let $A_{(2)}/\mathbf{F}_{25}$ the quadratic base extension of A. $A_{(2)} \sim E^2$.

Classification for $g = 1$

Waterhouse [Wat69], building on work of Deuring [Deu41], classified the possible Frobenius polynomials of elliptic curves.



Theorem (Elliptic curves)

Let E be an elliptic curve defined over $\mathbf{F}_q$. Then

(1) *E is ordinary if and only if $\mathrm{SF}(E) = \mathrm{U}(1)$.*

(2) *E is supersingular if and only if $\mathrm{SF}(E) \in \{C_1, C_3, C_4, C_6, C_8, C_{12}\}$.*

Moreover, each one of these groups is realized for some prime power q.

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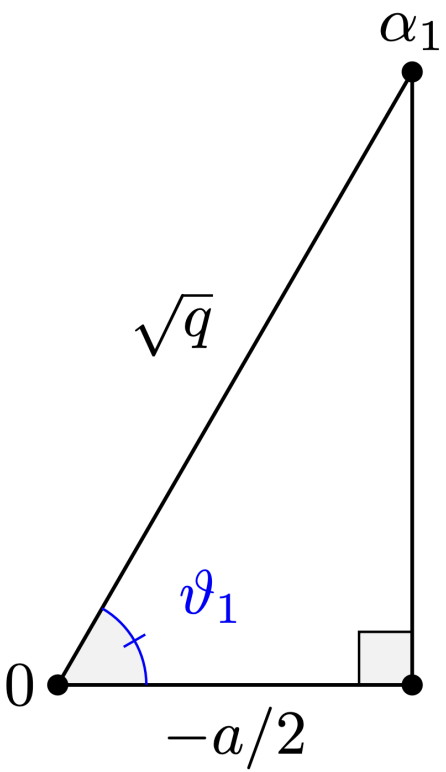

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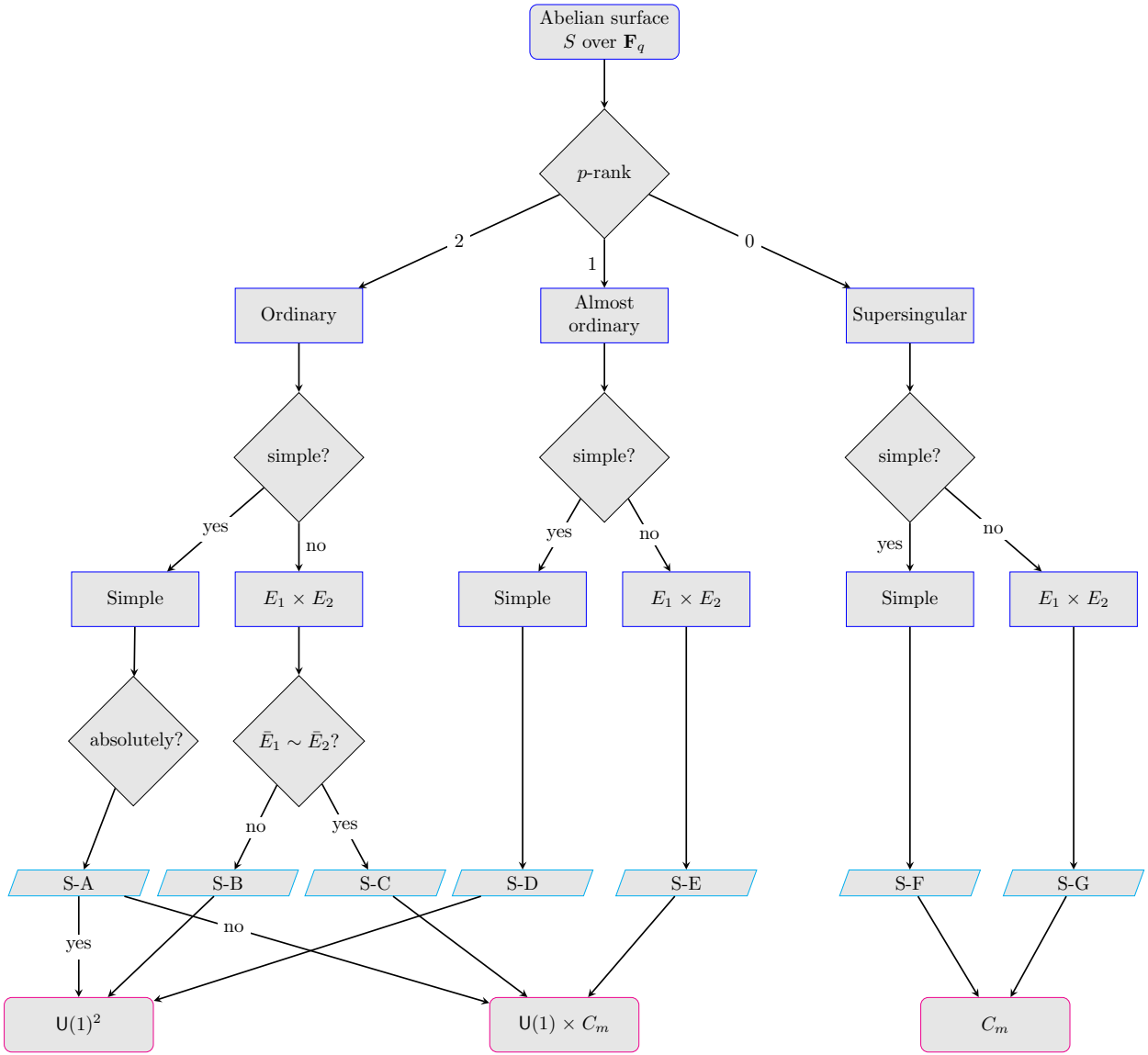
Classification for $g = 2$



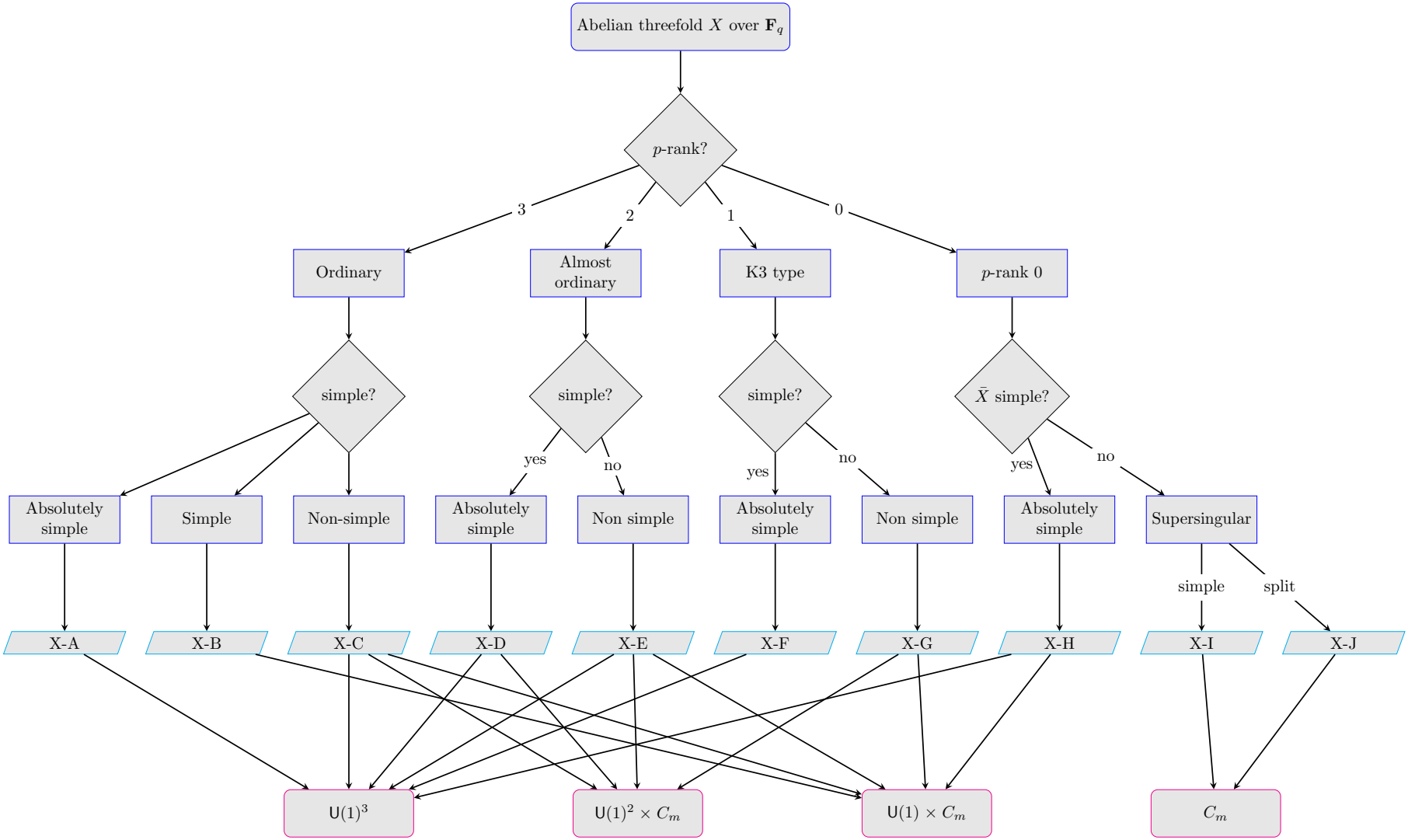
Classification for $g = 2$

Theorem (A-P, Bhamidipati, Sankar)

Let S be an abelian surface over $\mathbf{F}_q$. Then, S has Serre–Frobenius group according to the following table. Further, each one of these groups is realized for some prime power q.

$\mathsf{SF}(S)^\circ$	$\#\mathsf{SF}(S)/\mathsf{SF}(S)^\circ$
1	1,2,3,4,5,6,8,10,12,24
$\mathsf{U}(1)$	1,2,3,4,6,8,12
$\mathsf{U}(1)^2$	1

Classification for $g = 3$



Classification for $g = 3$

Theorem (A-P, Bhamidipati, Sankar)

Let X be an abelian threefold over $\mathbf{F}_q$. Then, X has Serre–Frobenius group according to the following table. Further, each one of these groups is realized for some prime power q.

$\mathsf{SF}(X)^\circ$	$\#\mathsf{SF}(X)/\mathsf{SF}(X)^\circ$
1	1,2,3,4,5,6,7,8,9,10,12,14,15,18,20,24,28,30,36
$\mathsf{U}(1)$	1,2,3,4,5,6,7,8,10,12,24
$\mathsf{U}(1)^2$	1,2,3,4,6,8,12,24
$\mathsf{U}(1)^3$	1

Previous work

We use previous work on the classification of Frobenius polynomials and the multiplicative relations between their roots. Most notably:

- Zarhin [Zar93; Zar91; Zar92; Zar15].
- Lenstra and Zarhin [LZ93].
- Zhu [Zhu01].
- Howe and Zhu [HZ02].
- Rück [Rüc90].
- Maisner and Nart [MN02].
- Nart and Ritzenthaler [NR08].
- Xing [Xin94; Xin96].
- Haloui [Hal10].
- Singh, McGuire, and Zaytsev [SMZ14].
- Dupuy, Kedlaya, Roe, Vincent [Dup+21].
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Sophie Germain primes

Theorem (A-P, Bhamidipati, Sankar)

Let A be a simple ordinary abelian variety defined over $\mathbf{F}_q$ of prime dimension $g > 2$. Then, exactly one of the following conditions holds.

(1) *A is absolutely simple.*

(2) *A splits over a degree g extension of $\mathbf{F}_q$ as a power of an elliptic curve, and $\mathrm{SF}(A) \cong \mathrm{U}(1) \times C_g$.*

(3) *$2g+1$ is prime and A splits over a degree $2g+1$ extension of $\mathbf{F}_q$ as a power of an elliptic curve, and $\mathrm{SF}(A) \cong \mathrm{U}(1) \times C_{2g+1}$.*

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Thank you!

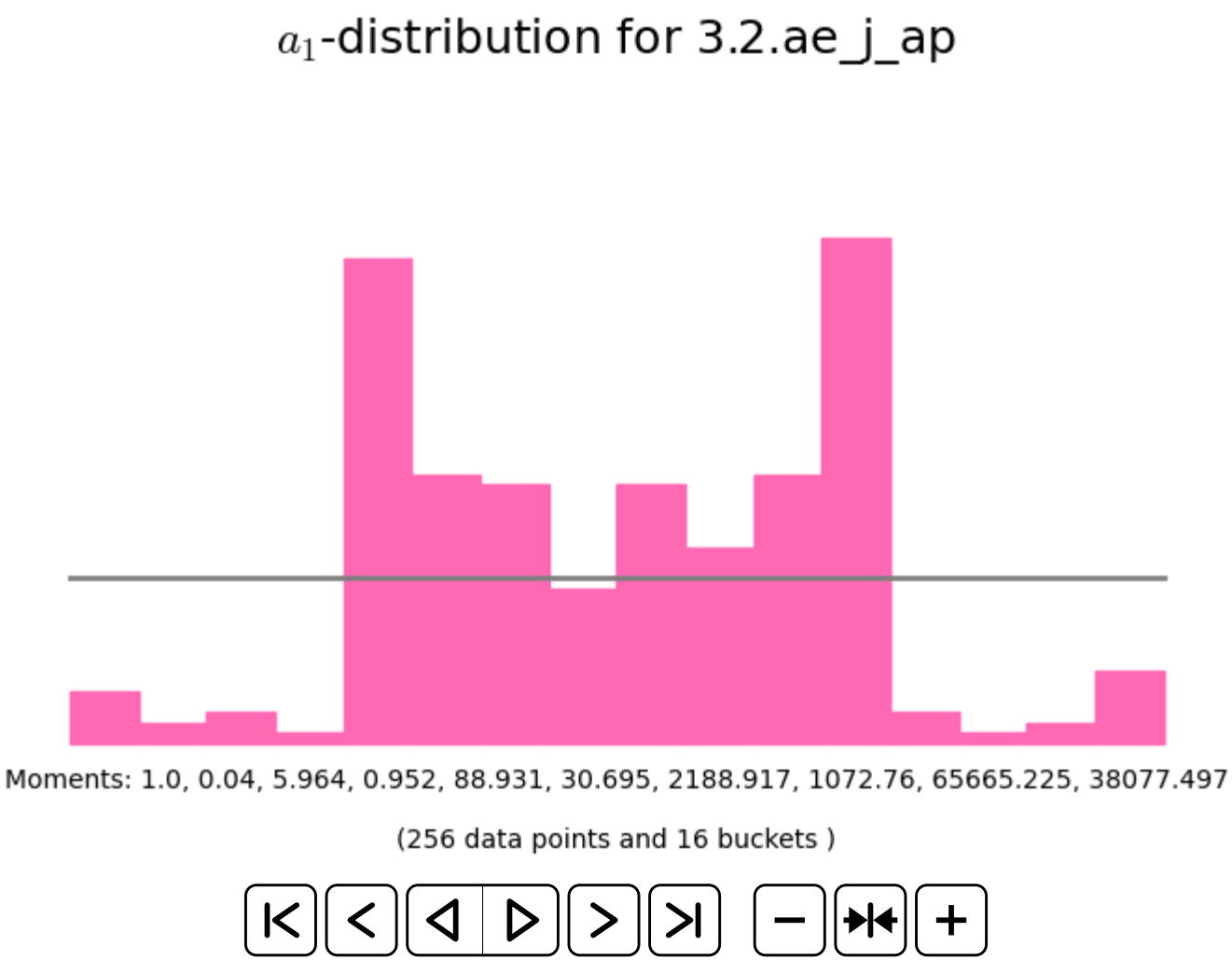

Figure: Isogeny class 3.2.ae_j_ap.

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