Centre International de Rencontres Mathématiques (CIRM)
Theory of Gravitation and Variation in Cosmology

Theory of Gravitational Waves

and

Approximation Methods in General Relativity

Luc Blanchet

Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
Institut d'Astrophysique de Paris

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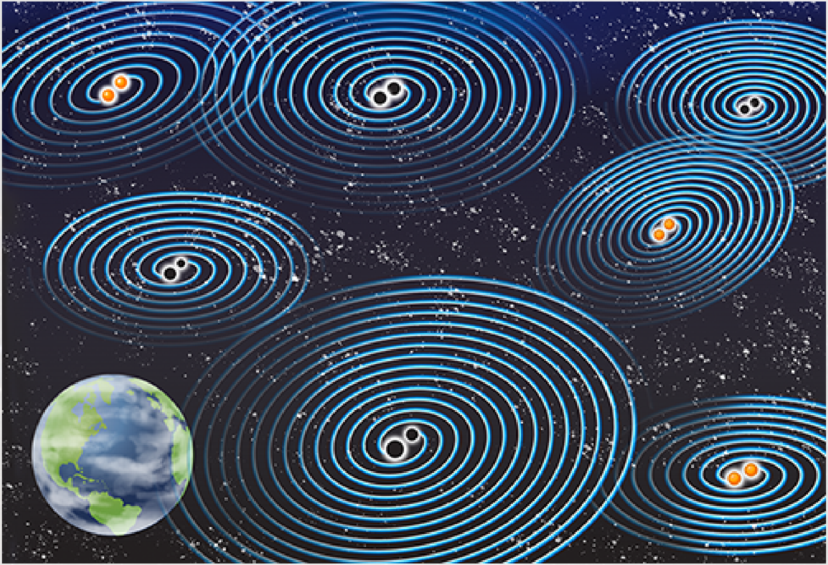
Outline of the lectures

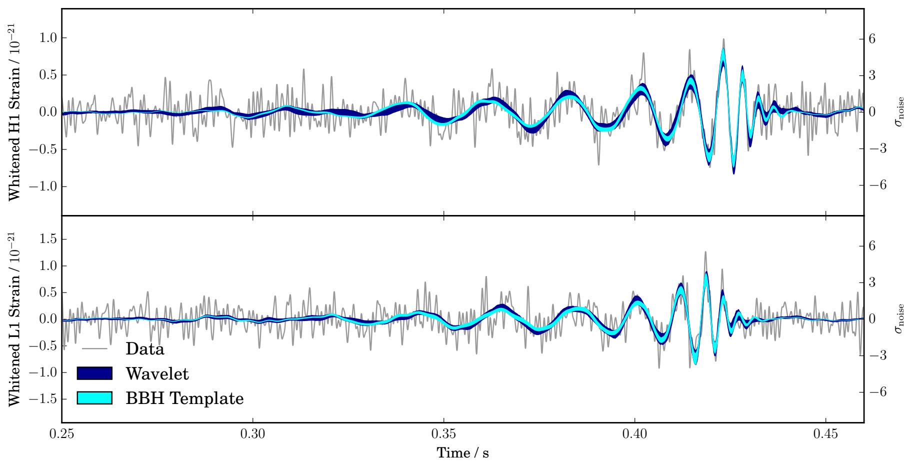
1. Gravitational wave events
2. Methods to compute gravitational waves
3. Einstein quadrupole formalism
4. Gravitational-wave generation formalism
5. Post-Newtonian parameters
6. Finite-size effects in compact binaries
7. Synergy with the effective field theory
8. Radiation reaction and balance equations

GRAVITATIONAL WAVE EVENTS

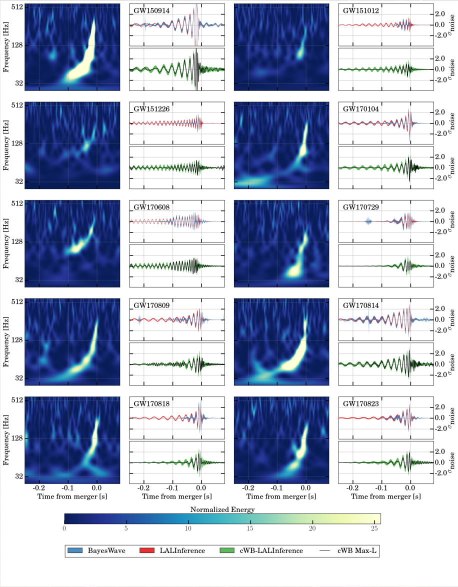
A new messenger to explore the Universe



Binary black-hole event GW150914 [LIGO/Virgo 2016]

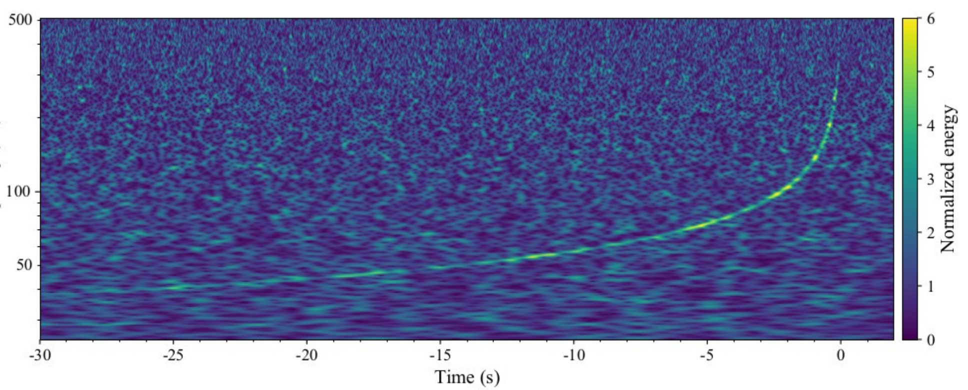


Binary black-hole events [LIGO/Virgo 2018-2020]



- For BH binaries the detectors are mostly sensitive to the merger phase
- Detected total BH masses range from $\sim 20\,M_\odot$ to $\sim 140\,M_\odot$!
- One object $\sim 2.5\,M_\odot$ is either the lightest known BH or the heaviest NS
- The signals match perfectly the waveform predicted by GR

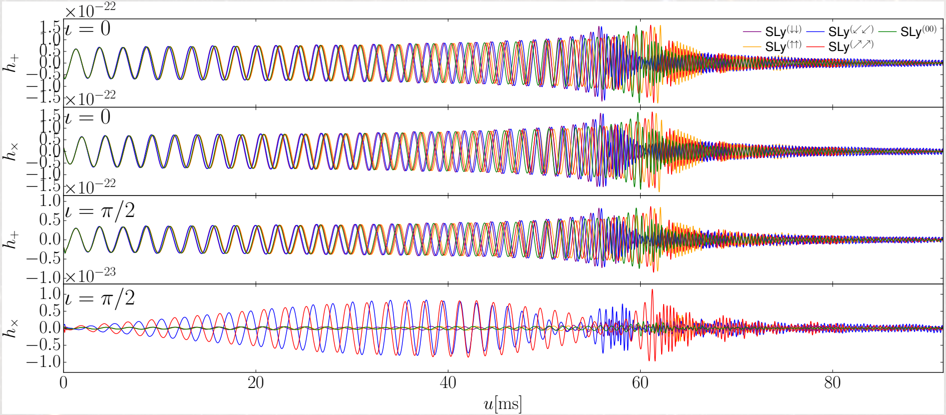
Binary neutron star event GW170817



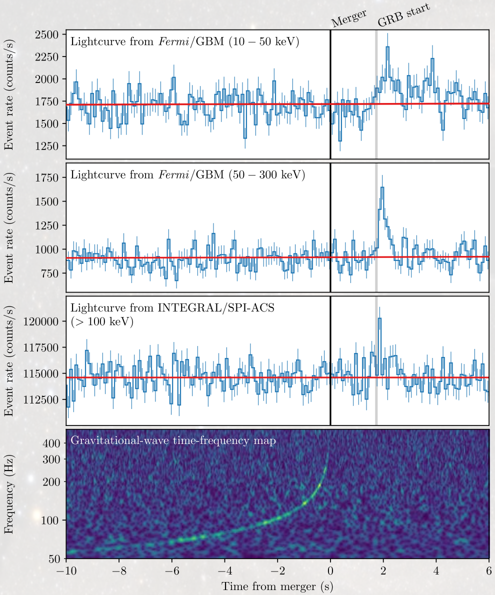
- The signal is observed during $\sim 100\,\mathrm{s}$ and ~ 3000 cycles and is the loudest gravitational-wave signal yet observed with a combined SNR of 32.4
- The chirp mass is accurately measured to $\mathcal{M} = \mu^{3/5} M^{2/5} = 1.98\,M_\odot$
- The distance is measured from the gravitational signal as $R = 40$ Mpc

Post-merger waveform of neutron star binaries

[Dietrich, Bernuzzi, Bruegmann, Ujevic & Tichy 2018]

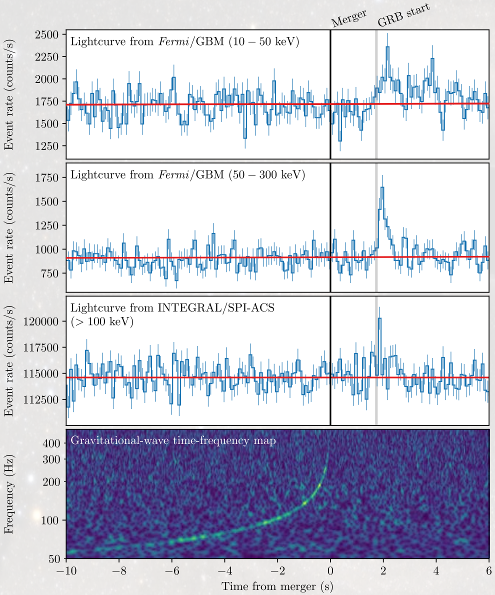


The advent of multi-messenger astronomy



- The gamma-ray burst has been detected 1.7 second after the instant of merger
- This is the closest gamma-ray burst whose distance is known and is probably seen off-axis with respect to the relativistic jet

Speed of gravitational waves versus speed of light



- The observed time delay between GW170817 and GRB170817A gives a strong constraint

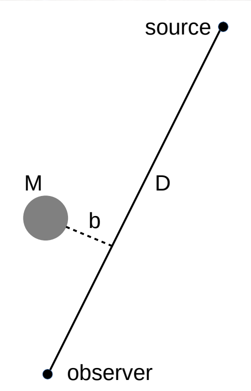
$$\boxed{|c_g - c_{em}| \lesssim 10^{-15} c}$$

- This eliminated a series of alternative theories

Test of the strong equivalence principle [Desai & Kahya 2016]

- The test involves the cumulative Shapiro time delay due to the gravitational potential of the dark matter distribution
- The violation of the equivalence principle is quantified by a PPN like parameter γ_a depending on the type of radiation $a = \text{GW}, \text{EM}$. For a spherical mass distribution

$$\Delta t^a_{\text{Shapiro}} = (1 + \gamma_a)\frac{GM}{c^3}\ln\left(\frac{D}{b}\right)$$

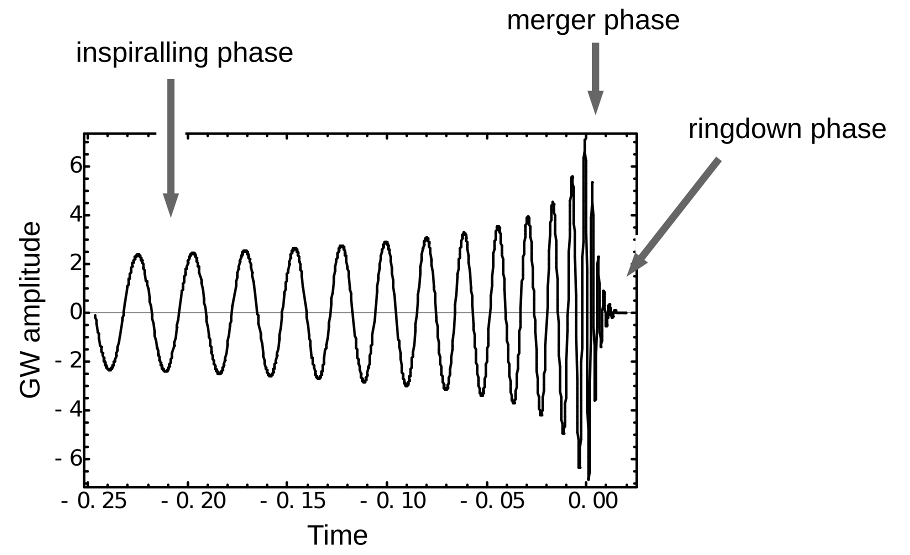


- The main contributions come from the galaxy NGC4993 and our own Galaxy with mass $M_{\text{MW}} = 5.6\,10^{11}\,M_\odot$
- Assuming an isothermal density profile for dark matter this yields about $400\,$days delay in GR
- The observed difference in arrival time $\Delta t = 1.7\,$s yields

$$\boxed{|\gamma_{\text{GW}} - \gamma_{\text{EM}}| \lesssim 10^{-7}}$$

METHODS TO COMPUTE GRAVITATIONAL WAVES

The gravitational chirp of binary black holes



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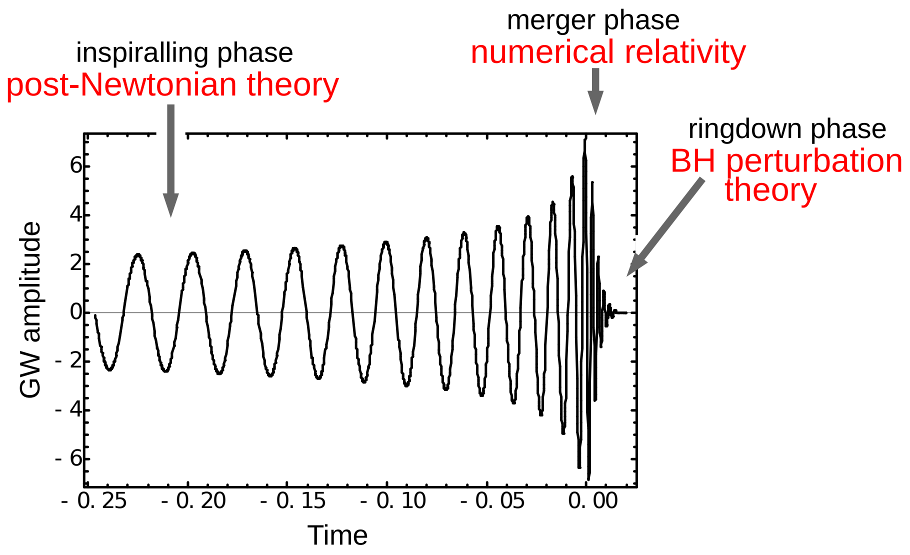
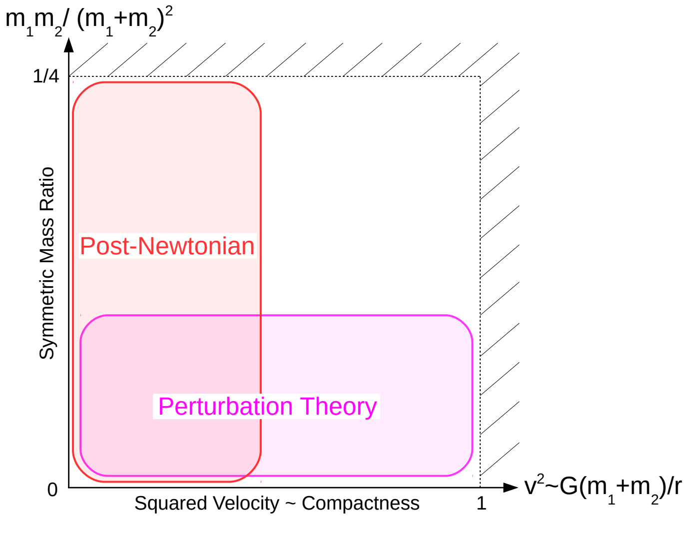


Methods to compute GW templates



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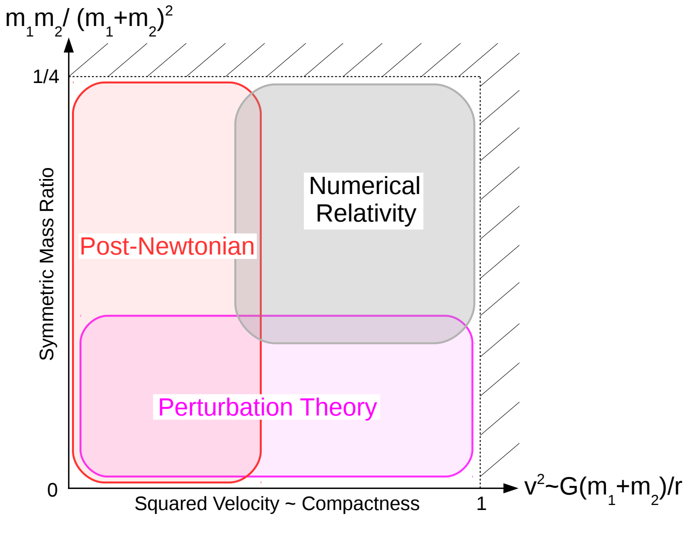

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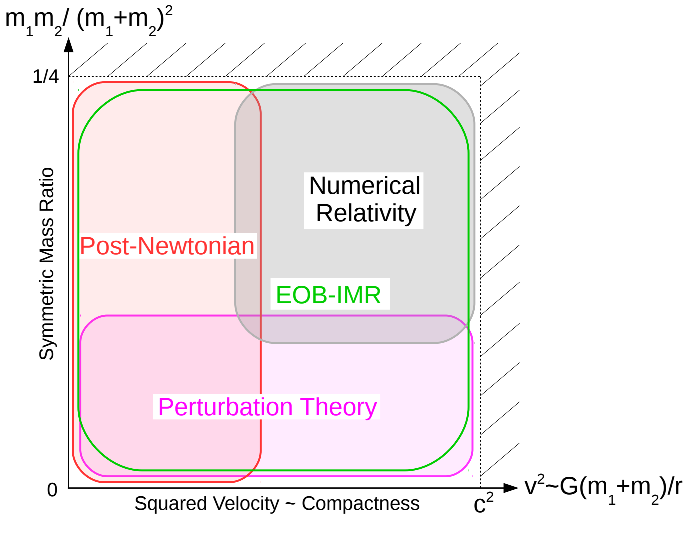
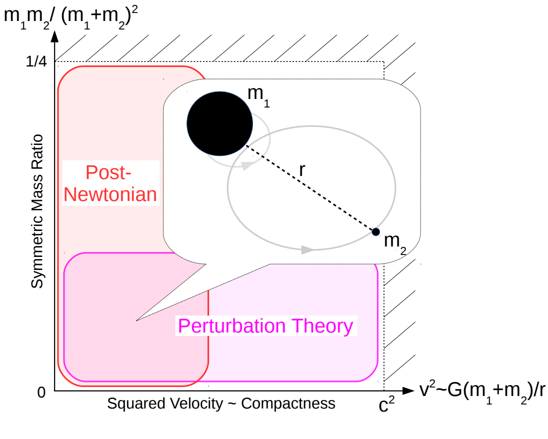
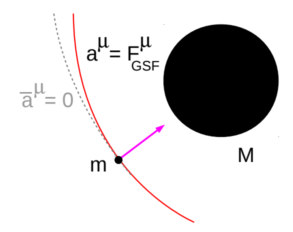


Post-Newtonian versus gravitational self-force (GSF)





$$\boxed{a^\mu = F^\mu_{\mathsf{GSF}} = \mathcal{O}\left(\frac{m}{M}\right)}$$

PN predictions for the conservative dynamics are consistent with linear GSF calculations up to high order [Detweiler 2008; Blanchet, Detweiler, Le Tiec & Whiting 2010]

General problem of linear perturbations

1. Suppose we know a solution $\overline{g}(x)$ of the second-order PDE

$$\mathcal{E}\left[\overline{g}(x)\right] = 0$$

2. Assume a one-parameter family of solutions $g(x,\lambda)$ with $g(x,0)=\overline{g}(x)$

$$\mathcal{E}\left[g(x,\lambda)\right] = 0$$

3. Defining $h(x) \equiv (\partial g/\partial\lambda)(x,0)$ we obtain the linear second-order PDE

$$\boxed{h\frac{\partial\mathcal{E}}{\partial g}[\overline{g}] + \partial h \frac{\partial\mathcal{E}}{\partial(\partial g)}[\overline{g}] + \partial^2 h \frac{\partial\mathcal{E}}{\partial(\partial^2 g)}[\overline{g}] = 0}$$

4. A good approximation to the exact solution $g(x,\lambda)$ for non-zero but small λ is

$$\boxed{g_{\text{linear}}(x) = \overline{g}(x) + \lambda\, h(x)}$$

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Reliability of the perturbative equations

- To any one-parameter family of solutions $g(x, \lambda)$ corresponds a solution $h(x)$ of the linear perturbative equations
- But the converse is not necessarily true, *i.e.* given a solution $h(x)$ there does not necessarily exist an exact solution such that $h(x) = (\partial g/\partial\lambda)(x, 0)$
- More generally, an infinite set of solutions $h_n(x)$ (with $n \in \mathbb{N}$) of the perturbation equations to all non-linear orders n does not necessarily come from the Taylor expansion of some exact solution $g(x, \lambda)$ when $\lambda \to 0$

Knowing if it does is the problem of the reliability of the perturbation equations

Einstein field equations as a "Problème bien posé"

- Start with the GR action for the metric $g_{\mu\nu}$ with the matter term

$$\boxed{S_{\mathsf{GR}} = \underbrace{\frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,R}_{\text{Einstein-Hilbert action}} + \underbrace{S_{\mathsf{m}}[g_{\mu\nu},\Psi]}_{\text{matter fields}}}$$

- Add the harmonic coordinates gauge-fixing term (where $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$)

$$\boxed{S_{\mathsf{GR}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\left(\sqrt{-g}\,R \underbrace{-\frac{1}{2}\mathfrak{g}_{\alpha\beta}\partial_\mu\mathfrak{g}^{\alpha\mu}\partial_\nu\mathfrak{g}^{\beta\nu}}_{\text{gauge-fixing term}}\right) + S_{\mathsf{m}}}$$

- Get a well-posed system of equations [Hadamard 1932; Choquet-Bruhat 1952]

$$\boxed{\begin{aligned}&\mathfrak{g}^{\mu\nu}\partial^2_{\mu\nu}\mathfrak{g}^{\alpha\beta} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta} + \overbrace{\Sigma^{\alpha\beta}[\mathfrak{g},\partial\mathfrak{g}]}^{\text{non-linear source term}}\\ &\partial_\mu\mathfrak{g}^{\alpha\mu} = 0\end{aligned}}$$

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Perturbation around Minkowski space-time

Assume space-time slightly differs from Minkowski space-time $\eta_{\alpha\beta}$

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta} \qquad \text{with} \quad |h| \ll 1$$

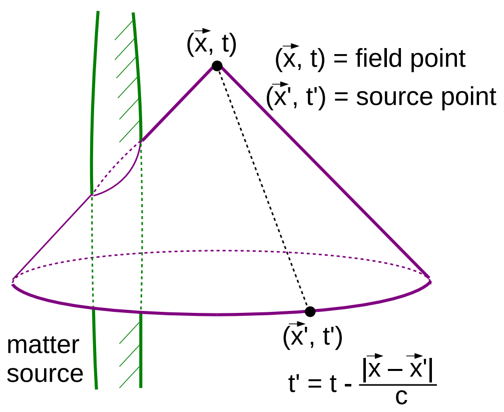
$$\boxed{\begin{aligned} &\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}|g|T^{\alpha\beta} + \overbrace{\Lambda^{\alpha\beta}[h, \partial h, \partial^2 h]}^{\text{non-linear source term}} \\ &\underbrace{\partial_\mu h^{\alpha\mu} = 0}_{\text{harmonic-gauge condition}} \end{aligned}}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat d'Alembertian operator

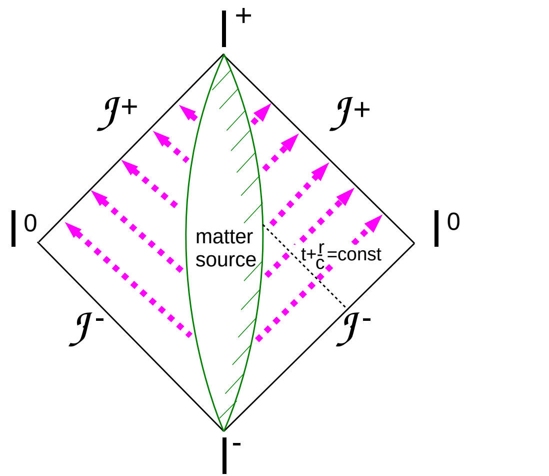
Kirchhoff's formula

For an homogeneous solution of the wave equation $\Box h_{\text{hom}} = 0$

$$\boxed{h_{\text{hom}}(\mathbf{x},t) = \lim_{|\mathbf{x}'|\to+\infty} \iint \frac{\mathrm{d}\Omega'}{4\pi} \left(\frac{\partial}{\partial r} + \frac{\partial}{c\partial t}\right)(r h_{\text{hom}}) \left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}$$



No-incoming radiation condition



$$\boxed{\lim_{\substack{r\to+\infty\\ t+\frac{r}{c}=\text{const}}}\left(\frac{\partial}{\partial r}+\frac{\partial}{c\partial t}\right)\left(rh^{\alpha\beta}\right)=0}$$

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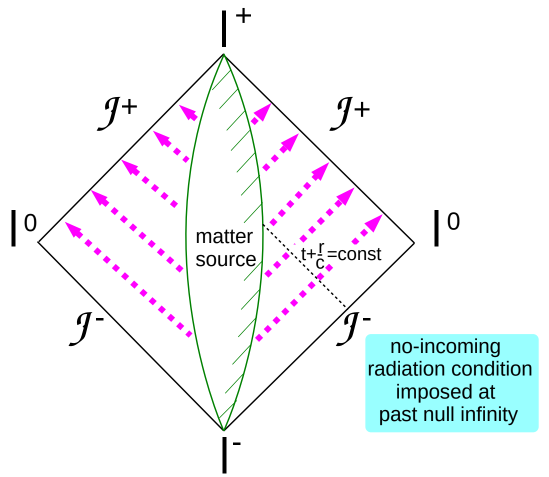

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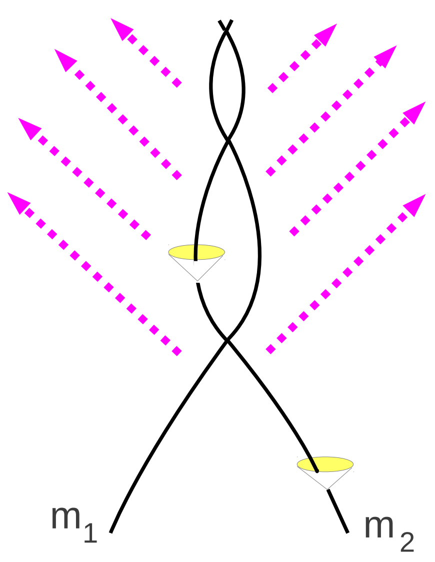
Two-body system formed from freely falling particles

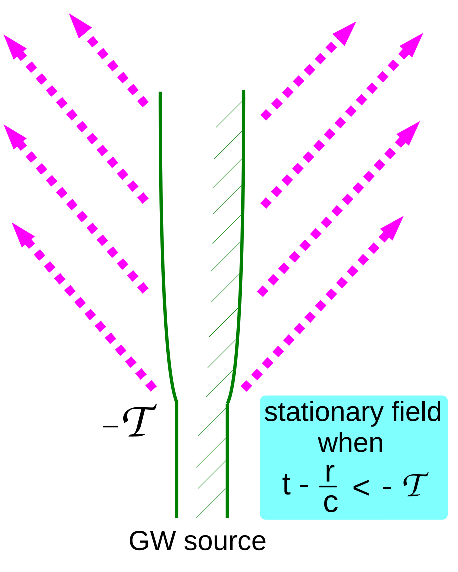


Gravitational motion of initially free particles when $t \to -\infty$ [Eder 1989]

$$\boldsymbol{x}(t) = \boldsymbol{V}\,t + \boldsymbol{W}\log(-t) + \boldsymbol{X} + o(t^0)$$

where $\boldsymbol{V}$ and $\boldsymbol{X}$ are constant vectors, and $\boldsymbol{W} = GM\boldsymbol{V}/V^3$

Hypothesis of stationarity in the remote past



In practice all GW sources observed in astronomy (*e.g.* a compact binary system) will have been formed and started to emit GWs only from a finite instant in the past $-\mathcal{T}$

The post-Minkowskian approximation

[Bertotti 1956; Bertotti & Plebanski 1960; Westpfahl *et al.* 1980, 1985; Bel *et al.* 1981; Bern *et al.* 2019]

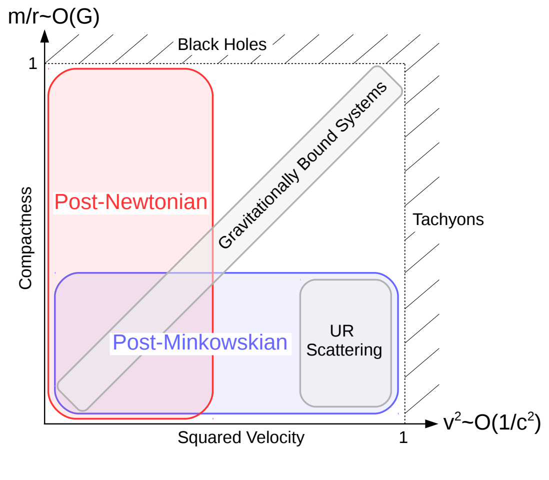
Appropriate for weakly self-gravitating isolated matter sources

$$\gamma_{\text{PM}} \equiv \frac{GM}{c^2 a} \ll 1 \quad \left\{ \begin{array}{l} M \text{ mass of source} \\ a \text{ size of source} \end{array} \right.$$

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} + \underbrace{\sum_{n=1}^{+\infty} G^n \, h_n^{\alpha\beta}}_{G \text{ labels the PM expansion}}$$

$$\boxed{\begin{aligned} &\Box h_n^{\alpha\beta} = \frac{16\pi G}{c^4} |g| T_n^{\alpha\beta} \; + \; \overbrace{\Lambda_n^{\alpha\beta}[h_1, \cdots, h_{n-1}]}^{\text{known from previous iterations}} \\ &\partial_\mu h_n^{\alpha\mu} = 0 \end{aligned}}$$

Post-Newtonian versus post-Minkowskian



The post-Minkowskian 3PM two-body Hamiltonian [Bern, Cheung, Solon *et al.* 2019] has been checked with the post-Newtonian 4PN two-body equations of motion

Multipolar expansion

[Pirani 1964; Geroch 1970; Hansen 1974; Thorne 1980; Simon & Beig 1983; Blanchet 1998]

Valid in the exterior of any possibly strong field isolated source

$$\frac{a}{r} < 1 \qquad \begin{cases} a \text{ size of source} \\ r \text{ distance to source} \\ \lambda \sim cP \text{ wavelength of radiation} \end{cases}$$

$$\boxed{\underbrace{I_L \sim M a^\ell}_{\text{mass-type multipole moment}} \qquad \underbrace{J_L \sim M a^\ell v}_{\text{current-type multipole moment}}} \qquad (L = i_1 \cdots i_\ell)$$

Split space-time into near zone $r \ll \lambda$ and wave zone $r \gg \lambda$

$$\underbrace{h_{\text{NZ}} \sim \frac{G}{c^2} \sum_\ell \left[\frac{I_L}{r^{\ell+1}} + \frac{J_L}{c r^{\ell+1}} \right]}_{r \ll \lambda} \qquad \underbrace{h_{\text{WZ}} \sim \frac{G}{c^2 r} \sum_\ell \left[\frac{I_L^{(\ell)}}{c^\ell} + \frac{J_L^{(\ell)}}{c^{\ell+1}} \right]}_{r \gg \lambda}$$

Multipolar expansion

[Pirani 1964; Geroch 1970; Hansen 1974; Thorne 1980; Simon & Beig 1983; Blanchet 1998]

- The radiative multipolar field in the wave zone

$$h_{\text{WZ}} \sim \frac{G}{c^2 r} \sum_{\ell} \left[\frac{I_L^{(\ell)}}{c^\ell} + \frac{J_L^{(\ell)}}{c^{\ell+1}} \right]$$

is actually a PN expansion in the case of a PN source

$$\boxed{\frac{I_L^{(\ell)}}{c^\ell} \sim \frac{M a^\ell}{\lambda^\ell} \sim M\,\varepsilon_{\text{PN}}^\ell}$$

- The quadrupole moment formalism gives the lowest order PN contribution to the radiation field due to the mass type quadrupole moment ($\ell = 2$)

$$I_{ij} = Q_{ij} + \mathcal{O}(\varepsilon_{\text{PN}}^2)$$

$$Q_{ij}(t) = \int_{\text{PN source}} \mathrm{d}^3\mathbf{x}\, \underbrace{\rho_{\text{N}}(\mathbf{x}, t)}_{\substack{\text{Newtonian}\\ \text{mass density}}} \left(x_i x_j - \frac{1}{3}\delta_{ij} r^2 \right)$$

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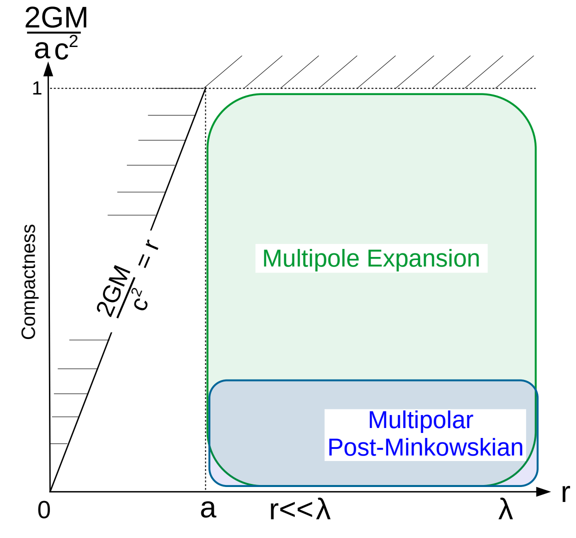
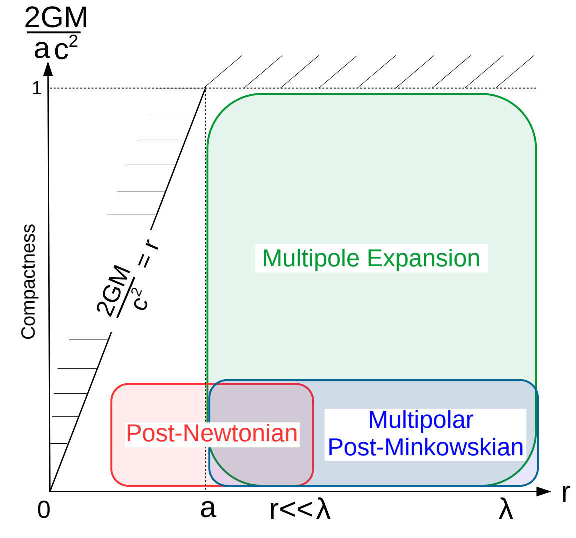
Multipolar-Post-Minkowskian expansion

[Bonnor 1959; Blanchet & Damour 1986]



Multipolar-Post-Minkowskian expansion

[Bonnor 1959; Blanchet & Damour 1986]



EINSTEIN QUADRUPOLE FORMALISM

Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1945]

$$4\pi R^2 \mathfrak{J} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^{\,2} - \frac{1}{3}\left(\sum_{\mu} \dddot{J}_{\mu\mu}\right)^2\right].$$

1. Einstein quadrupole formula

$$\boxed{\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\text{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}}$$

2. Amplitude quadrupole formula

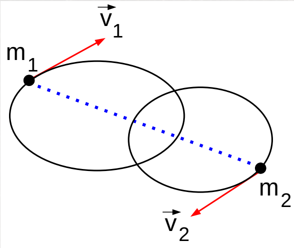
$$\boxed{h_{ij}^{\text{TT}} = \frac{2G}{c^4 R}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t - \frac{R}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\text{TT}} + \mathcal{O}\left(\frac{1}{R^2}\right)}$$

3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$\boxed{F_i^{\text{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7}$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

Application to compact binaries [Peters & Mathews 1963; Peters 1964]



$$\begin{cases} a \text{ semi-major axis of relative orbit} \\ e \text{ eccentricity of relative orbit} \\ \omega = \frac{2\pi}{P} \text{ orbital frequency} \end{cases}$$

$$M = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{M} \qquad \boxed{\nu = \frac{\mu}{M} \quad 0 < \nu \leqslant \frac{1}{4}}$$

Averaged energy and angular momentum balance equations

$$\langle \frac{\mathrm{d}E}{\mathrm{d}t} \rangle = -\langle \mathcal{F}^{\mathrm{GW}} \rangle \qquad \langle \frac{\mathrm{d}J_i}{\mathrm{d}t} \rangle = -\langle \mathcal{G}_i^{\mathrm{GW}} \rangle$$

are applied to a Keplerian orbit (using Kepler's law $GM = \omega^2 a^3$)

$$\boxed{\begin{aligned} \langle \frac{\mathrm{d}P}{\mathrm{d}t} \rangle &= -\frac{192\pi}{5c^5}\nu \left(\frac{2\pi GM}{P}\right)^{5/3} \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}} \\ \langle \frac{\mathrm{d}e}{\mathrm{d}t} \rangle &= -\frac{608\pi}{15c^5}\nu \frac{e}{P} \left(\frac{2\pi GM}{P}\right)^{5/3} \frac{1 + \frac{121}{304}e^2}{(1-e^2)^{5/2}} \end{aligned}}$$

Orbital phase evolution of compact binaries

[Dyson 1969; Esposito & Harrison 1975; Wagoner 1975]

1. Compact binaries are circularized when they enter the detector's bandwidth

$$\boxed{E = -\frac{Mc^2}{2}\nu\, x \qquad \mathcal{F}^{\mathsf{GW}} = \frac{32}{5}\frac{c^5}{G}\nu^2 x^5}$$

where $x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$ denotes a small PN parameter defined with ω

2. Equating $\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}^{\mathsf{GW}}$ gives a differential equation for x

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{64}{5}\frac{c^3\nu}{GM}x^5 \qquad \Longleftrightarrow \qquad \frac{\dot{\omega}}{\omega^2} = \frac{96\nu}{5}\nu\left(\frac{GM\omega}{c^3}\right)^{5/3}}$$

3. This permits to solve for the orbital phase

$$\phi = \int \omega\,\mathrm{d}t = \int \frac{\omega}{\dot{\omega}}\,\mathrm{d}\omega$$

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[Dyson 1969; Esposito & Harrison 1975; Wagoner 1975]

1. The amplitude and phase evolution follow an adiabatic chirp in time

$$\boxed{\begin{aligned} a(t) &= \left(\frac{256}{5}\frac{G^3M^3\nu}{c^5}(t_c - t)\right)^{1/4} \\ \phi(t) &= \phi_c - \frac{1}{32\nu}\left(\frac{256}{5}\frac{c^3\nu}{GM}(t_c - t)\right)^{5/8} \end{aligned}}$$

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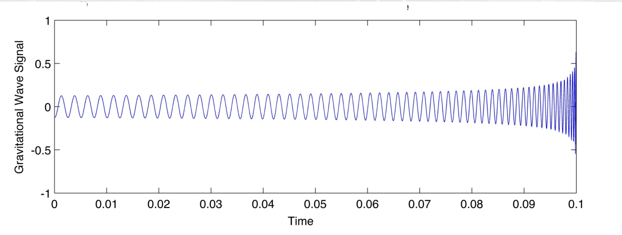

The quadrupole formula works for GW150914

- The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi}\left[\frac{256}{5}\frac{G\mathcal{M}^{5/3}}{c^5}(t_\mathsf{c} - t)\right]^{-3/8}$$

- Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[\frac{5}{96}\frac{c^5}{G\pi^{8/3}} f^{-11/3} \dot{f}\right]^{3/5}}$$

  which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

- The GW amplitude is predicted to be[1]

$$h_\mathsf{eff} \sim 4.1 \times 10^{-22} \left(\frac{\mathcal{M}}{M_\odot}\right)^{5/6} \left(\frac{100\,\mathrm{Mpc}}{R}\right) \left(\frac{100\,\mathrm{Hz}}{f_\mathsf{merger}}\right)^{-1/6} \sim 1.6 \times 10^{-21}$$

- The distance $R = 400\,\mathrm{Mpc}$ is measured from the signal itself [Schutz 1986]

[1] $h_\mathsf{eff} \sim h\sqrt{N}$ where $N \sim \omega^2/\dot{\omega}$ is the number of cycles around frequency ω

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Total energy radiated away by GW150914

1. The ADM energy of space-time is constant and reads (at any time t)

$$E_{\mathrm{ADM}} = (m_1+m_2)c^2 - \frac{G m_1 m_2}{2r} + \frac{G}{5c^5}\int_{-\infty}^{t} \mathrm{d}t' \left(Q_{ij}{}^{(3)}\right)^2(t')$$

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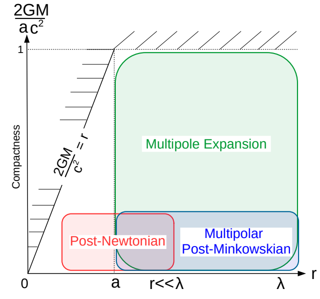
GRAVITATIONAL-WAVE GENERATION FORMALISM

PN-matched Multipolar-post-Minkowskian



1. Construct the most general multipolar expansion outside the source in the form of a PM expansion

$$\overbrace{\mathcal{M}(h)}^{\text{multipole expansion}} = \underbrace{G\,h_1 + G^2\,h_2 + \cdots + G^n\,h_n + \cdots}_{\text{post-Minkowskian expansion}}$$

2. Match the MPM solution to the PN expansion of the field inside the source

Linearized multipolar vacuum solution [Pirani 1964; Thorne 1980]

Solution of linearized vacuum field equations in harmonic coordinates

$$\boxed{\Box h_1^{\alpha\beta} = \partial_\mu h_1^{\alpha\mu} = 0}$$

$$h_1^{00} = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}I_L\right)$$

$$h_1^{0i} = \frac{4}{c^3}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}I^{(1)}_{iL-1}\right) + \frac{\ell}{\ell+1}\varepsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}J_{bL-1}\right)\right\}$$

$$h_1^{ij} = -\frac{4}{c^4}\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left(\frac{1}{r}I^{(2)}_{ijL-2}\right) + \frac{2\ell}{\ell+1}\partial_{aL-2}\left(\frac{1}{r}\varepsilon_{ab(i}J^{(1)}_{j)bL-2}\right)\right\}$$

- multipole moments $I_L(u)$ and $J_L(u)$ are arbitrary functions of $u = t - r/c$
- mass $M \equiv I = \mathrm{const}$, center-of-mass position $G_i \equiv I_i = \mathrm{const}$
  linear momentum $P_i \equiv I_i^{(1)} = 0$, angular momentum $J_i = \mathrm{const}$

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The plug-and-grind MPM algorithm

1. At n-th post-Minkowskian order we need to solve

$$\partial_\nu h_n^{\alpha\beta} = 0$$
$$\Box h_n^{\mu\nu} = \Lambda^{\mu\nu}\big(\underbrace{h_1, \cdots h_{n-1}}_{\text{known from previous iterations}}\big)$$

2. A particular solution with the required multipole structure reads

$$u_n^{\alpha\beta} = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{Ret}} \left[\left(\frac{r}{r_0}\right)^B \Lambda_n^{\alpha\beta} \right]$$

3. In order to guarantee that the harmonic gauge condition $\partial_\mu h_n^{\alpha\mu} = 0$ is satisfied we add an homogeneous solution $v_n^{\alpha\beta}$ hence

$$\boxed{h_n^{\alpha\beta} = u_n^{\alpha\beta} + v_n^{\alpha\beta}}$$

4. The MPM solution is generated as a functional of $I_L(u)$ and $J_L(u)$

The plug-and-grind MPM algorithm

Theorem 1

The MPM solution is the most general solution of Einstein's vacuum equations outside an isolated matter system

Theorem 2

When expanded in the near zone ($r \to 0$) the MPM solution yields the general structure of the PN expansion as

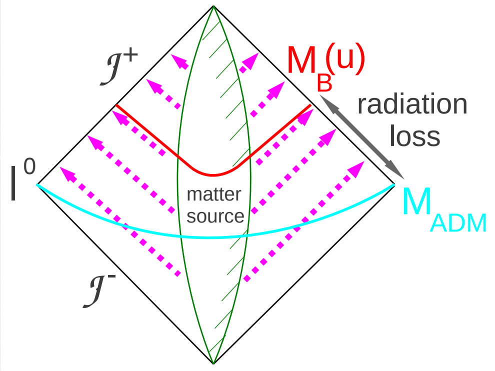
$$\boxed{h^{\alpha\beta}_{\mathrm{PN}}(\mathbf{x}, t, c) = \sum_{p\geqslant 2}\sum_{q\geqslant 0} \frac{(\ln c)^q}{c^p} h^{\alpha\beta}_{(p,q)}(\mathbf{x}, t)}$$

Theorem 3

When expanded in the far zone ($r \to \infty$, $u = \mathrm{const}$) the MPM solution becomes asymptotically flat in Penrose's sense and recovers the Bondi-Sachs formalism

Asymptotic structure of radiating space-time

[Bondi *et al.* 1962; Sachs 1962; Penrose 1963, 1965]



$$M_{\mathsf{B}}(u) = M_{\mathsf{ADM}} - \overbrace{\frac{G}{5c^7}\int_{-\infty}^{u} \mathrm{d}u'\, U_{ij}^{(1)}(u')U_{ij}^{(1)}(u')}^{\text{mass-energy emitted in GW}} + \text{higher multipolar contributions}$$

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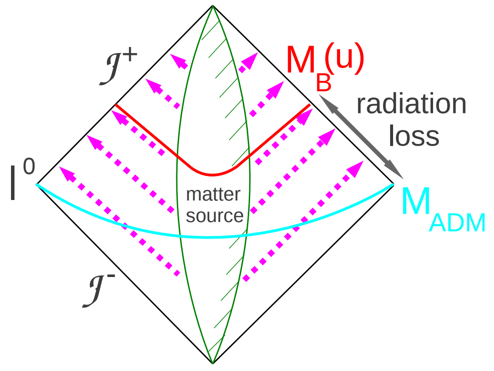

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where $$\boxed{U_{ij}(u) = I^{(2)}_{ij}(u) + \mathcal{O}(G)}$$

Problem of the matching

[Lagerström *et al.* 1967; Burke & Thorne 1971; Kates 1980; Anderson *et al.* 1982; Blanchet 1998]

1. Most general multipolar(-post-Minkowskian) solution in the source's exterior

$$\mathcal{M}(h) = \underset{B=0}{\mathrm{FP}} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \mathcal{M}(\Lambda) \right] + \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{M_L(t-r/c)}{r} \right\}$$

where the homogeneous solution is parametrized by multipole moments

2. Most general PN solution in the source's near zone

$$\bar{h} = \underset{B=0}{\mathrm{FP}} \Box^{-1}_{\text{sym}} \left[\left(\frac{r}{r_0}\right)^B \bar{\tau} \right] + \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{A_L(t-r/c) - A_L(t+r/c)}{r} \right\}$$

where the homogeneous solution (regular when $r \to 0$) is parametrized by "radiation reaction" multipole moments

Problem of the matching

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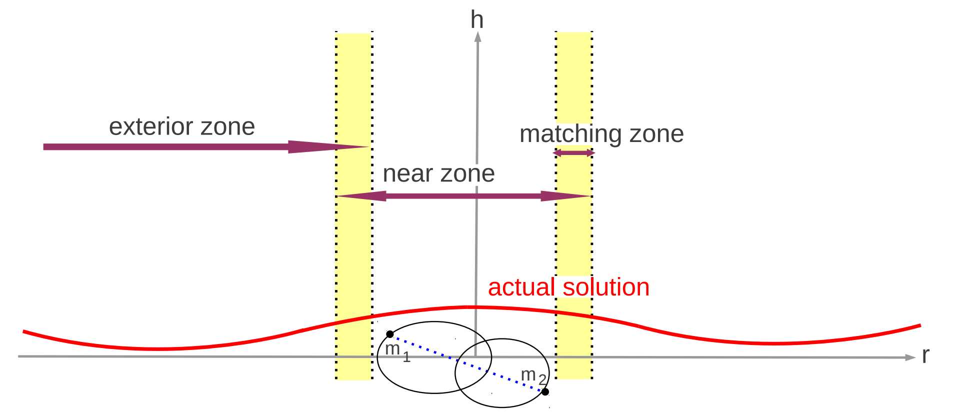

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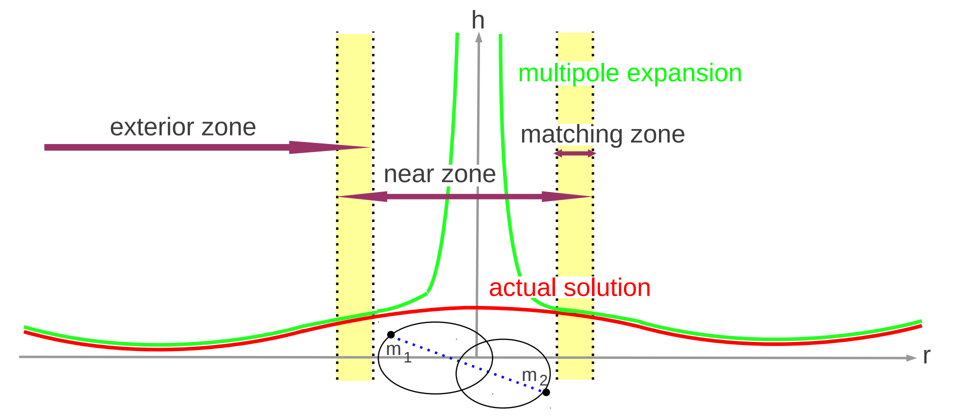

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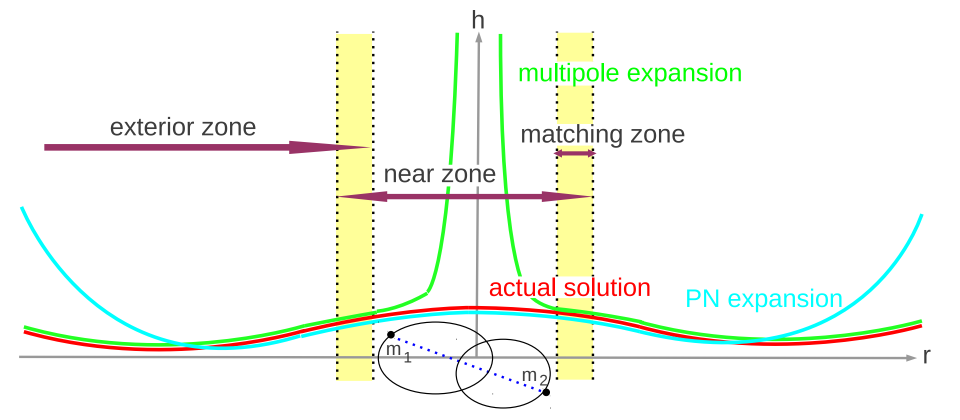

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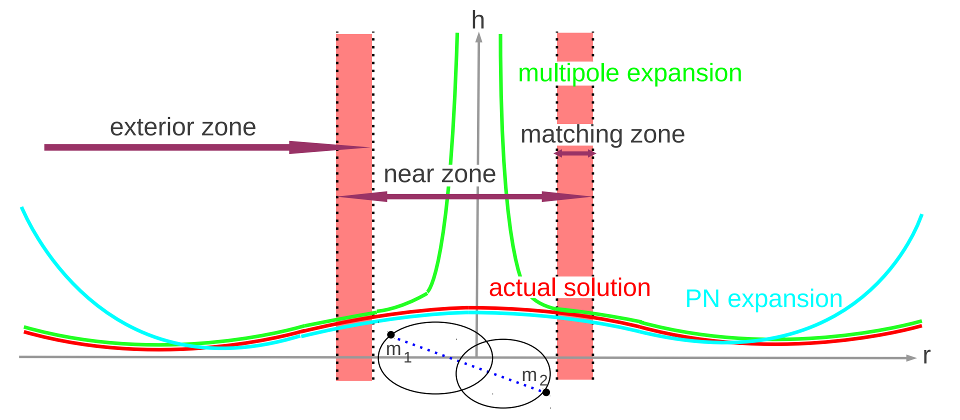


matching equation $\Longrightarrow$ $\boxed{\overline{\mathcal{M}(h)} = \mathcal{M}(\bar{h})}$

Near-zone expansion of the multipole expansion

Lemma 1

$$\overline{\underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\text{ret}}\left[\left(\frac{r}{r_0}\right)^B\mathcal{M}(\Lambda)\right]} = \underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\text{sym}}\left[\left(\frac{r}{r_0}\right)^B\overline{\mathcal{M}(\Lambda)}\right]$$
$$\underbrace{-\frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{\mathcal{R}_L(t-r/c)-\mathcal{R}_L(t+r/c)}{2r}\right\}}_{\text{antisymmetric type homogeneous solution}}$$

where the radiation reaction multipole moments are

$$\boxed{\mathcal{R}_L(u) = \underset{B=0}{\mathrm{FP}}\int \mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B \hat{x}_L\int_1^{+\infty}\mathrm{d}z\,\gamma_\ell(z)\;\underbrace{\mathcal{M}(\tau)(\mathbf{x},t-zr/c)}_{\text{multipole expansion of the pseudo-tensor}}}$$

The finite part at $B=0$ plays the role of an UV regularization $(r\to 0)$

Far-zone expansion of the PN expansion

Lemma 2

$$\mathcal{M}\left(\mathop{\mathrm{FP}}_{B=0}\Box^{-1}_{\text{sym}}\left[\left(\frac{r}{r_0}\right)^B\bar{\tau}\right]\right) = \mathop{\mathrm{FP}}_{B=0}\Box^{-1}_{\text{sym}}\left[\left(\frac{r}{r_0}\right)^B\mathcal{M}(\bar{\tau})\right] \underbrace{- \frac{1}{4\pi}\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{\mathcal{F}_L(t-r/c)+\mathcal{F}_L(t+r/c)}{2r}\right\}}_{\text{symmetric type homogeneous solution}}$$

$$\boxed{\mathcal{F}_L(u) = \mathop{\mathrm{FP}}_{B=0}\int \mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B\hat{x}_L\int_{-1}^{1}\mathrm{d}z\,\delta_\ell(z)\,\underbrace{\bar{\tau}(\mathbf{x},t-zr/c)}_{\text{PN expansion of the pseudo-tensor}}}$$

The finite part at $B=0$ plays the role of an IR regularization ($r\to+\infty$)

General solution of the matching equation

1. In the far zone

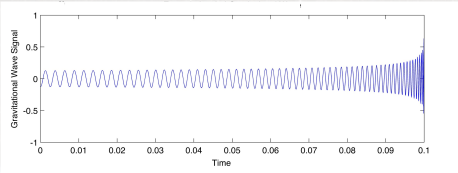
$$\boxed{\mathcal{M}(h) = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\tfrac{r}{r_0}\right)^B \mathcal{M}(\Lambda) \right] - \underbrace{\frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{F}_L(t-r/c)}{r} \right\}}_{\text{source's multipole moments}}}$$

2. In the near zone [Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]

$$\boxed{\bar{h} = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\tfrac{r}{r_0}\right)^B \bar{\tau} \right] - \underbrace{\frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{R}_L(t-r/c) - \mathcal{R}_L(t+r/c)}{r} \right\}}_{\text{non-local tail term (4PN order)}}}$$

POST-NEWTONIAN PARAMETERS

PN parameters in the orbital phase evolution



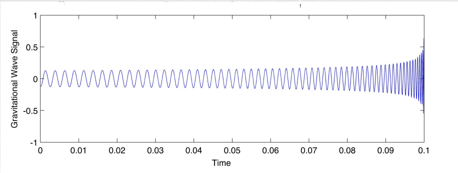
- The PN parameters come from a mixture of conservative and dissipative effects through the energy balance equation

$$\frac{\mathrm{d}\overbrace{E}^{\text{conservative energy}}}{\mathrm{d}t} = - \underbrace{\mathcal{F}^{\text{GW}}}_{\text{dissipative energy flux}}$$

- The orbital phase $\phi = \int \omega \, \mathrm{d}t$ is obtained as a function of $x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$ and the mass ratio $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$

$$\boxed{\phi(x) = \phi_0 - \frac{x^{-5/2}}{32\nu} \sum_p \left(\varphi_{p\text{PN}}(\nu) + \varphi^{(l)}_{p\text{PN}}(\nu) \log x\right) x^p + \mathcal{O}\left[(\log x)^2\right]}$$

PN parameters in the orbital phase evolution



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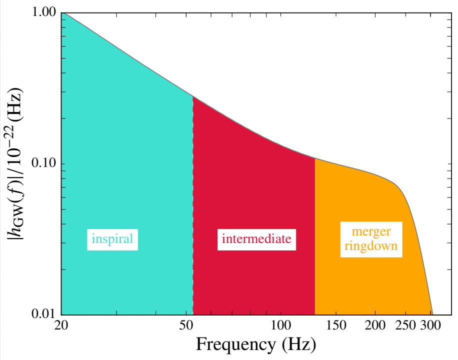
$$\boxed{\phi(x) = \phi_0 - \frac{x^{-5/2}}{32\nu} \sum_p \Big(\varphi_{p\mathsf{PN}}(\nu) + \varphi^{(l)}_{p\mathsf{PN}}(\nu)\,\log x\Big)\, x^p + \mathcal{O}\left[(\log x)^2\right]}$$

The inspiral-merger-ringdown (IMR) model



Effective methods that interpolate between the different phases play a crucial role

- The effective-one-body (EOB) approach [Buonanno & Damour 1999]
- The inspiral-merger-ringdown (IMR) [Ajith *et al.* 2008]

$$\Big\{\underbrace{\text{PN parameters;}}_{\text{inspiral}}\ \underbrace{\beta_2,\beta_3}_{\text{intermediate}}\ ;\ \underbrace{\alpha_2,\alpha_3,\alpha_4}_{\text{merger-ringdown}}\Big\}$$

The known 3.5PN parameters

They were computed from the MPM-PN approach [Blanchet 2014 for a review]

$$\varphi_{\mathrm{0PN}} = 1 \qquad \Longleftarrow \quad \text{Einstein quadrupole formula}$$

$$\varphi_{\mathrm{1PN}} = \frac{3715}{1008} + \frac{55}{12}\nu$$

$$\varphi_{\mathrm{1.5PN}} = -10\pi$$

$$\varphi_{\mathrm{2PN}} = \frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2$$

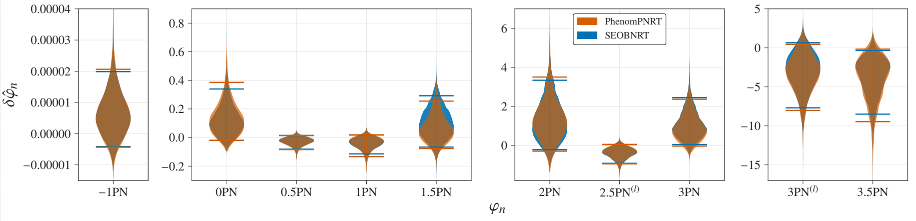
$$\varphi^{(l)}_{\mathrm{2.5PN}} = \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi$$

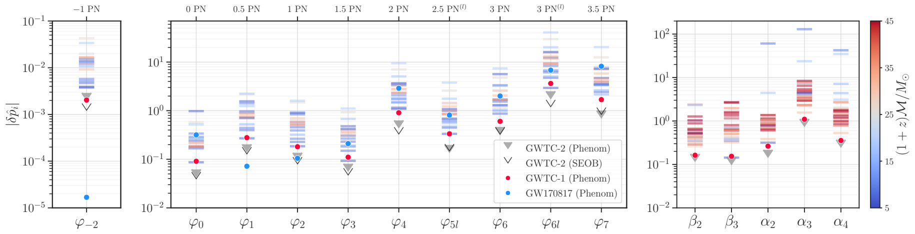
$$\begin{aligned}\varphi_{\mathrm{3PN}} &= \frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_{\mathrm{E}} - \frac{3424}{21}\ln 2 \\ &\quad + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3\end{aligned}$$

$$\varphi^{(l)}_{\mathrm{3PN}} = -\frac{856}{21}$$

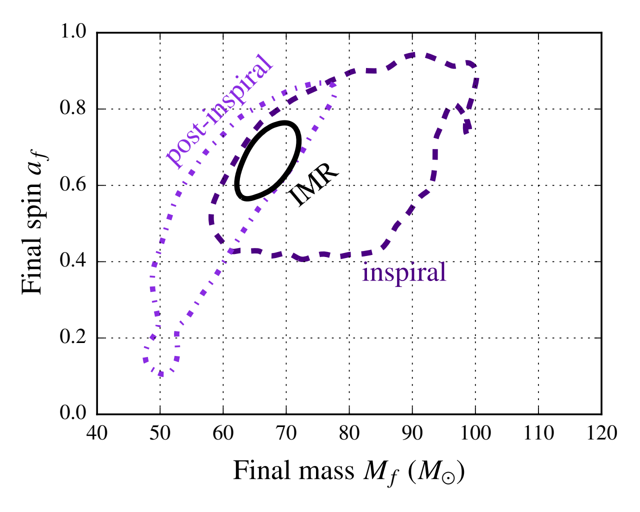
$$\varphi_{\mathrm{3.5PN}} = \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi$$

Measurement of PN parameters [LIGO/Virgo 2017, 2020]

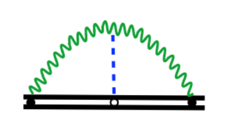


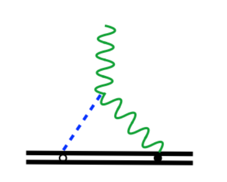


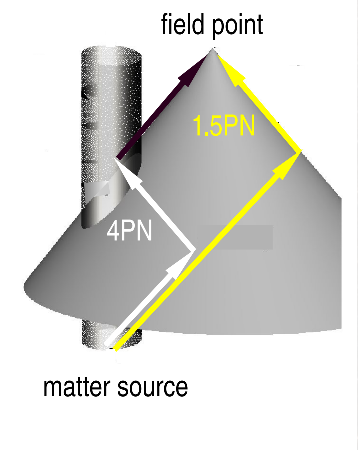
Inspiral-Merger-Ringdown consistency test [LIGO/Virgo 2016]



The gravitational wave tail effect [Blanchet & Damour 1988, 1992]







- In the near zone (4PN effect)

$$\boxed{S^{\text{tail}} = \frac{G^2 M}{5c^8} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} I^{(3)}_{ij}(t)\, I^{(3)}_{ij}(t')}$$

- In the far zone (1.5PN effect)

$$\boxed{h^{\text{tail}}_{ij} = \frac{4G}{c^4 r}\frac{GM}{c^3}\int_{-\infty}^{u} \mathrm{d}u' I^{(4)}_{ij}(u') \ln\left(\frac{u-u'}{P}\right)}$$

Tail effects in PN parameters

tail terms

$$\varphi_{0\text{PN}} = 1$$

$$\varphi_{1\text{PN}} = \frac{3715}{1008} + \frac{55}{12}\nu$$

$$\varphi_{1.5\text{PN}} = \boxed{-10\pi}$$

$$\varphi_{2\text{PN}} = \frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2$$

$$\varphi^{(l)}_{2.5\text{PN}} = \boxed{\left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi}$$

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Tail effects in PN parameters

tail terms

tail-of-tail terms

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The 4.5PN radiative quadrupole moment

$$U_{ij}(t) = I^{(2)}_{ij}(t) + \underbrace{\frac{GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau\, I^{(4)}_{ij}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail integral}}$$

$$+\frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty}\mathrm{d}\tau\, I^{(3)}_{a<i}I^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\}$$

$$+\underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty}\mathrm{d}\tau\, I^{(5)}_{ij}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail integral}}$$

$$+\underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty}\mathrm{d}\tau\, I^{(6)}_{ij}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail integral}}$$

$$+\mathcal{O}\left(\frac{1}{c^{10}}\right)$$

Toward 4.5PN parameters

- The 4.5PN term is also known and due to the 4.5PN tail-of-tail-of-tail integral for circular orbits [Marchand, Blanchet & Faye 2017; Messina & Nagar 2017]

$$\varphi_{4.5\mathrm{PN}} = \Bigg(-\frac{93098188434443}{150214901760} + \frac{80}{3}\pi^2 + \frac{1712}{21}\gamma_\mathrm{E} + \frac{3424}{21}\ln 2 + \left[\frac{1492917260735}{1072963584} - \frac{2255}{48}\pi^2\right]\nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3\Bigg)\pi$$

$$\varphi^{(l)}_{4.5\mathrm{PN}} = \frac{856}{21}\pi$$

tail-of-tail-of-tail terms

- However the 4PN term is only known from perturbative BH theory in the test-mass limit $\nu \to 0$ [Tagoshi & Sasaki 1994; Tanaka, Tagoshi & Sasaki 1996]

$$\varphi_{4\mathrm{PN}} = \frac{2550713843998885153}{2214468081745920} - \frac{45245}{756}\pi^2 - \frac{9203}{126}\gamma_\mathrm{E} - \frac{252755}{2646}\ln 2 - \frac{78975}{1568}\ln 3 + \mathcal{O}(\nu)$$

$$\varphi^{(l)}_{4\mathrm{PN}} = -\frac{9203}{252} + \mathcal{O}(\nu)$$

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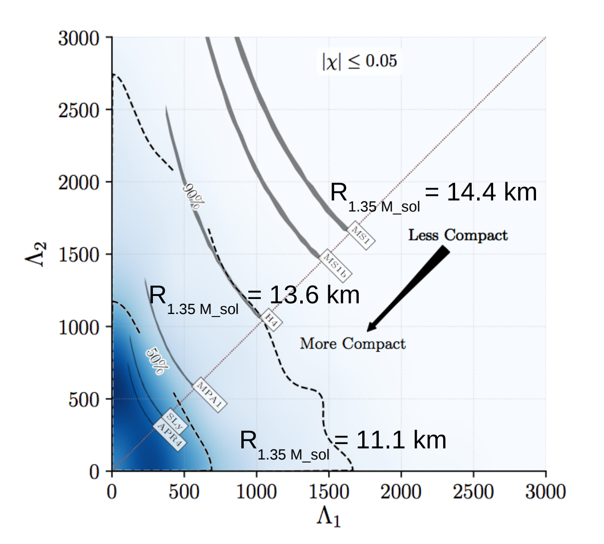
FINITE-SIZE EFFECTS IN COMPACT BINARIES

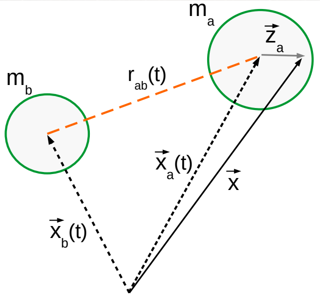
Constraining the neutron star equation of state

[LIGO/Virgo 2017]



$$\Lambda_a = \frac{2}{3} \underbrace{k_a}_{\text{Love number}} \left(\frac{c^2 R_a}{G m_a}\right)^5$$

Equations of motion of N extended bodies



$$m_a = \int_{\mathcal{V}_a} \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x},t) \qquad \boldsymbol{x}_a(t) = \frac{1}{m_a}\int_{\mathcal{V}_a} \mathrm{d}^3\mathbf{x}\,\mathbf{x}\,\rho(\mathbf{x},t)$$

$$\mathbf{x} = \boldsymbol{x}_a(t) + \boldsymbol{z}_a(\mathbf{x},t) \qquad Q_a^{ij} = \int_{\mathcal{V}_a} \mathrm{d}^3\boldsymbol{z}_a\,\rho_a(\boldsymbol{z}_a,t)\Big(z_a^i z_a^j - \frac{1}{3}\delta^{ij}\boldsymbol{z}_a^2\Big)$$

$$\boxed{\alpha \sim \frac{|\boldsymbol{z}_a|}{r_{ab}} \ll 1}$$

Equations of motion of N extended bodies

1. The Newtonian equations of motion of extended (spinless) bodies are

$$m_a \frac{\mathrm{d}v_a^i}{\mathrm{d}t} = G \sum_{b\neq a} \left[m_a m_b \frac{\partial}{\partial x_a^i}\left(\frac{1}{r_{ab}}\right) + \overbrace{\frac{1}{2}\left(m_a\, Q_b^{jk} + m_b\, Q_a^{jk}\right) \frac{\partial^3}{\partial x_a^i \partial x_a^j \partial x_a^k}\left(\frac{1}{r_{ab}}\right)}^{\text{effect of the quadrupole moments}} \right]$$

2. The conserved energy of the N-body system is the sum of the internal energies E_a and of the orbital contributions

$$E = \sum_a \left\{ E_a + \frac{1}{2} m_a \boldsymbol{v}_a^2 - \frac{G}{2} \sum_{b\neq a} \frac{m_a m_b}{r_{ab}} - \frac{1}{2}\, Q_a^{ij}\, \mathcal{G}_a^{ij} \right\}$$

3. The tidal quadrupole moment felt by the body a is

$$\boxed{\mathcal{G}_a^{ij} = \frac{\partial^2 U_a}{\partial x_a^i \partial x_a^j} \quad \text{where} \quad U_a = \sum_{b\neq a} \frac{G m_b}{r_{ab}}}$$

Equations of motion of N extended bodies

1. The coupling of the quadrupole moments with the external tidal field $\mathcal{G}_a^{ij}$ implies a variation of the internal energy given by

$$\frac{\mathrm{d}E_a}{\mathrm{d}t} = \frac{1}{2}\dot{Q}_a^{ij}\,\mathcal{G}_a^{ij}$$

2. Neglecting tidal dissipation we assume that the quadrupole moment is aligned with the tidal field

$$\boxed{Q_a^{ij} = \mu_a\,\mathcal{G}_a^{ij}}$$

   where μ_a is a deformability or polarizability coefficient

3. The conserved energy of the system simplifies in this case

$$E = \sum_a \left\{ \frac{1}{2} m_a \boldsymbol{v}_a^2 - \frac{G}{2}\sum_{b\neq a}\frac{m_a m_b}{r_{ab}} - \frac{\mu_a}{4}\,\mathcal{G}_a^{ij}\mathcal{G}_a^{ij} \right\}$$

4. Very importantly the dynamics admits a Lagrangian formulation

$$\boxed{L = \sum_a \left\{ \frac{1}{2} m_a \boldsymbol{v}_a^2 + \frac{G}{2}\sum_{b\neq a}\frac{m_a m_b}{r_{ab}} + \frac{\mu_a}{4}\,\mathcal{G}_a^{ij}\mathcal{G}_a^{ij} \right\}}$$

GW flux of extended two-body systems

1. We compute the GW flux using the quadrupole formula, where the total quadrupole moment of the system is ($x^i = x_1^i - x_2^i$)

$$Q^{ij} = \overbrace{m\nu\left(x^i x^j - \frac{1}{3}\delta^{ij} r^2\right)}^{\text{orbital quadrupole moment}} + Q_1^{ij} + Q_2^{ij}$$

2. For two bodies moving on a circular orbit this yields

$$\boxed{\mathcal{F}^{\mathrm{GW}} = \frac{32G}{5c^5} r^4 \omega^6 m^2 \nu^2 \left[1 + 6\left(m_1^4 \Lambda_1 + m_2^4 \Lambda_2\right) \frac{G^5 m}{r^5 c^{10}}\right]}$$

3. The internal structure is characterized by the dimensionless parameter

$$\Lambda_a = \frac{c^{10} \mu_a}{G^4 m_a^5} = \frac{2}{3} k_a \left(\frac{c^2 R_a}{G m_a}\right)^5$$

Influence of the internal structure on the phase

1. Applying the energy balance equation $\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}^{\mathsf{GW}}$ we obtain the modification of the phase due to the internal structure as

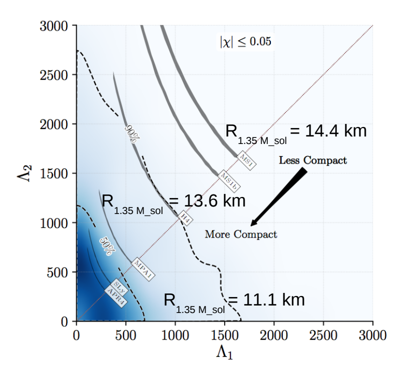
$$\boxed{\phi = \phi_0 - \frac{x^{-5/2}}{32\nu}\left[1 + \overbrace{\frac{39}{8}\tilde{\Lambda}\, x^5}^{\text{5PN effect}}\right]} \qquad x = \left(\frac{Gm\omega}{c^3}\right)^{2/3}$$

2. The tidal interaction on two bodies moving on a circular orbit depends on [Flanagan & Hinderer 2008]

$$\tilde{\Lambda} = \frac{16}{13}\left[\frac{(m_1 + 11m_2)m_1^3}{m^4}\Lambda_1 + \frac{(m_2 + 11m_1)m_2^3}{m^4}\Lambda_2\right]$$

3. The effect of the internal structure is formally a very small effect for compact objects comparable to an orbital correction of the order 5PN $\sim 1/c^{10}$

Dominant quadrupole tidal effect in BNS



- Tidal contribution to the GW chirp

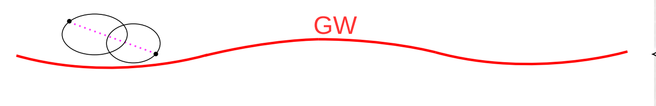
$$\boxed{\begin{aligned} x(t) &= \frac{1}{4}\theta^{-1/4}\left[1 + \frac{39}{8192}\tilde{\Lambda}\,\theta^{-5/4}\right] \\ \phi(t) &= \phi_0 - \frac{x^{-5/2}}{32\nu}\left[1 + \underbrace{\frac{39}{8}\tilde{\Lambda}\,x^5}_{\text{5PN effect}}\right] \end{aligned}}$$

with $x = (\frac{Gm\omega}{c^3})^{2/3}$ and $\theta = \frac{\nu c^3}{5Gm}(t_c - t)$

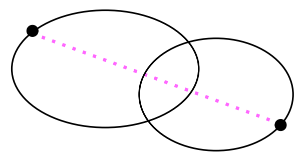
- The polarizability $\tilde{\Lambda}$ depends on the source mass of the NS (for a given EoS) while the point-particle part of the signal depends on the redshifted mass

Effective action for compact binary systems

- Hierarchy of length scales in a compact binary system



$$\Longleftarrow \quad \boxed{a \ll \lambda_{\text{GW}} \sim \frac{a}{v/c}}$$



$$\Longleftarrow \quad \boxed{R_1\,,\; R_2 \ll a}$$

- The Newtonian result can be reformulated as an effective matter action

$$\boxed{S_{\text{eff}} = \sum_a \int \mathrm{d}t \left[\overbrace{\frac{1}{2} m_a \boldsymbol{v}_a^2 + \frac{1}{2} \sum_{b \neq a} \frac{G m_b}{r_{ab}}}^{\text{point-particle action}} + \overbrace{\underbrace{\frac{\mu_a}{4} \mathcal{G}_a^{ij} \mathcal{G}_a^{ij}}_{\text{5PN}}}^{\text{internal structure effect}} \right]}$$

Effective field theory for extended compact objects

[Goldberger & Rothstein 2006; Damour & Nagar 2009]

- Matter action with non-minimal world-line couplings

$$S_{\text{eff}} = \sum_a \int \mathrm{d}\tau_a \Bigg\{ -m_a + \sum_{\ell=2}^{+\infty} \frac{1}{2\ell!} \Bigg[\underbrace{\mu_a^{(\ell)}}_{\substack{\text{mass type}\\ \text{polarizability}}} \big(\mathcal{G}^a_{\hat{L}}\big)^2 + \frac{\ell}{\ell+1} \underbrace{\sigma_a^{(\ell)}}_{\substack{\text{current type}\\ \text{polarizability}}} \big(\mathcal{H}^a_{\hat{L}}\big)^2 \Bigg] + \cdots \Bigg\}$$

- Tidal multipole moments [Thorne & Hartle 1985; Zhang 1986]

$$\mathcal{G}^a_{\hat{L}} = -\Big[\nabla_{\langle \hat{i}_1} \cdots \nabla_{\hat{i}_{\ell-2}} C_{\hat{i}_{\ell-1}\underline{\hat{0}}\hat{i}_\ell\rangle\hat{0}}\Big]_a$$
$$\mathcal{H}^a_{\hat{L}} = 2\Big[\nabla_{\langle \hat{i}_1} \cdots \nabla_{\hat{i}_{\ell-2}} C^*_{\hat{i}_{\ell-1}\underline{\hat{0}}\hat{i}_\ell\rangle\hat{0}}\Big]_a$$

where $C_{\hat{i}0\hat{j}0}$ are the components of the Weyl tensor $C_{\mu\nu\rho\sigma}$ projected on a local tetrad and evaluated at the location of the particle using a self-field regularization

High-order PN tidal effects

A recent result [Henry, Faye & Blanchet 2020abc] is the orbital SPA phase at the next-to-next-to-leading order for equal NS binaries on circular orbit

$$\boxed{\begin{aligned}\psi_{\text{tidal}} = -\frac{117}{2}v^5\Bigg\{&\widetilde{\mu}^{(2)} + \overbrace{\left(\frac{3115}{1248}\widetilde{\mu}^{(2)} + \frac{370}{117}\widetilde{\sigma}^{(2)}\right)v^2}^{\text{NLO}}\\ &-\pi\widetilde{\mu}^{(2)}v^3 + \underbrace{\left(\frac{379931975}{44579808}\widetilde{\mu}^{(2)} + \frac{935380}{66339}\widetilde{\sigma}^{(2)} + \frac{500}{351}\widetilde{\mu}^{(3)}\right)v^4}_{\text{NNLO}}\\ &-\pi\left(\frac{2137}{546}\widetilde{\mu}^{(2)} + \frac{592}{117}\widetilde{\sigma}^{(2)}\right)v^5\Bigg\}\end{aligned}}$$

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SYNERGY WITH THE EFFECTIVE FIELD THEORY

Fokker action versus effective action

$$S_{\mathrm{g}}[\boldsymbol{x}, h] = \frac{c^3}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} \Big[\overbrace{R}^{\text{Einstein-Hilbert Lagrangian}} \underbrace{-\frac{1}{2}\Gamma^\mu \Gamma_\mu}_{\text{gauge-fixing term}} \Big] - \sum_a \underbrace{m_a \int \mathrm{d}\tau_a}_{\text{point particles}}$$

- **Traditional PN approach:** compute the Fokker action by inserting an explicit iterated PN solution of the Einstein field equations

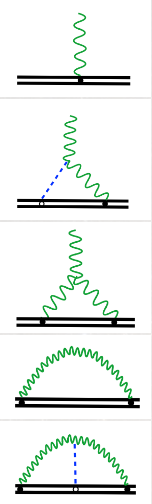
$$\boxed{\begin{aligned} &h^{\mu\nu}(\mathbf{x}, t) \longrightarrow \overline{h}^{\mu\nu}(\mathbf{x}; \boldsymbol{x}_a(t), \boldsymbol{v}_a(t), \cdots) \\ &S_{\mathsf{Fokker}}[\boldsymbol{x}] = S_{\mathrm{g}}[\boldsymbol{x}, \overline{h}(\boldsymbol{x})] \end{aligned}}$$

- **Effective field theory:** compute the effective action by integrating over the gravitational degrees of freedom

$$\boxed{\mathrm{e}^{\mathrm{i}S_{\mathsf{eff}}[\boldsymbol{x}]} = \int \mathcal{D}[h]\, \mathrm{e}^{\mathrm{i}S_{\mathrm{g}}[\boldsymbol{x}, h]}}$$

Diagrammatic expansion in EFT

Effective Field Theory



$\Longleftrightarrow$

Post-Newtonian

- emission from a quadrupole source
- tail effect in radiation field (1.5PN)
- non-linear memory effect (2.5PN)
- radiation reaction (2.5PN)
- tail in radiation reaction (4PN)

The EFT is equivalent to the traditional PN at the level of tree diagrams

Action for simple non-local tails derived by EFT

[Foffa & Sturani 2019]

- Using the relation between the tail self-energy diagram and the imaginary part of the tail radiation diagram

$$\boxed{S^{\text{tail}} = \sum_{\ell=2}^{+\infty} \frac{G^2 M}{c^{2\ell+4}} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} \left[a_\ell \, I_L^{(\ell+1)}(t) I_L^{(\ell+1)}(t') + \frac{b_\ell}{c^2} J_L^{(\ell+1)}(t) J_L^{(\ell+1)}(t') \right]}$$

- The coefficients are those which appear in the multipole expansion of the gravitational wave energy flux [Thorne 1980]
- The proof of this action by PN methods is tedious and limited to 1PN
- However the multipole moments I_L and J_L are computed up to high PN order by traditional PN methods [Blanchet & Iyer 2004]

High-order logarithmic tails in the circular energy

[Blanchet, Foffa, Larrouturou & Sturani 2020]

Current knowledge for the mass quadrupole moment is limited to 3PN order so we can compute the logarithmic tail terms to NNNL/7PN order

$$\boxed{\begin{aligned} E^{\text{tail}} = -\frac{m\nu^2}{2}x^5\log x\Big\{ & \tfrac{448}{15} + \left(-\tfrac{4988}{35} - \tfrac{656}{5}\nu\right)x \\ & + \left(-\tfrac{1967284}{8505} + \tfrac{914782}{945}\nu + \tfrac{32384}{135}\nu^2\right)x^2 \\ & + \Big[\tfrac{85229654387}{16372125} + \left(\tfrac{2132}{45}\pi^2 - \tfrac{41161601}{51030}\right)\nu - \tfrac{13476541}{5670}\nu^2 - \tfrac{289666}{1215}\nu^3 \\ & \qquad - \tfrac{1424384}{1575}\left(\gamma_\mathsf{E} + \log 4\right) - \tfrac{356096}{1575}\log x\Big]x^3 \\ & + \tfrac{64}{15}\textstyle\sum_{n=3}^{+\infty} \frac{(6n+1)(4\beta_I)^{n-1}}{n!}\, x^{3(n-1)}(\log x)^{n-1} + \cdots \Big\} \end{aligned}}$$

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4PN-7PN tails

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4PN-7PN tails

7PN tail-of-tail-of-tails

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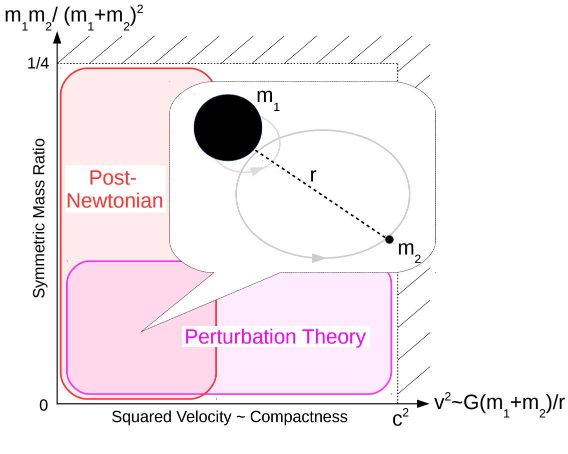
4PN-7PN tails | 7PN tail-of-tail-of-tails | leading $(3n+1)$PN $(\log x)^n$ terms

Consistency with gravitational self-force calculations

The 7PN tail-of-tail-of-tail terms are computed by combining information from high-order GSF calculations of the redshift invariant [Kavanagh, Ottewill & Wardell 2015]



Leading powers of logarithms from RG theory

- The renormalization group equations for mass and angular momentum are (with μ the renormalization scale) [Goldberger, Ross & Rothstein 2014]

$$\frac{\mathrm{d}\log M(\mu)}{\mathrm{d}\log\mu} = -\frac{2G^2}{5}\left[2I^{(1)}_{ij}I^{(5)}_{ij} - 2I^{(2)}_{ij}I^{(4)}_{ij} + I^{(3)}_{ij}I^{(3)}_{ij}\right]$$

$$\frac{\mathrm{d}J^i(\mu)}{\mathrm{d}\log\mu} = -\frac{8G^2 M}{5}\varepsilon^{ijk}\left[I_{jl}I^{(5)}_{kl} - I^{(1)}_{jl}I^{(4)}_{kl} + I^{(2)}_{jl}I^{(3)}_{kl}\right]$$

- The quadrupole moment itself undergoes a logarithmic renormalization under the RG flow (in the Fourier domain) [Blanchet 1998; Goldberger & Ross 2010]

$$\boxed{\tilde{I}_{ij}(\omega,\mu) = \bar{\mu}^{\beta_I (GM\omega)^2}\,\tilde{I}_{ij}(\omega,\mu_0)}$$

with $\bar{\mu} \equiv \mu/\mu_0$ and $\beta_I = -\frac{214}{105}$ is the beta function coefficient

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Leading powers of logarithms from RG theory

1. Integrating and averaging over one orbital scale, then specializing to quasi-circular orbits

$$E = \frac{1}{2} m\nu\, r^2\omega^2 - \frac{G m^2 \nu}{r} - 8 m\nu^2 \frac{\gamma^2}{\beta_I} \sum_{n=1}^{+\infty} \frac{1}{n!} \left(8\beta_I \gamma^3 \log v\right)^n$$

$$J = m\nu\, r^2\omega - \frac{48}{5} G^2 m^3 \nu^2 \frac{\omega}{\beta_I \gamma} \sum_{n=1}^{+\infty} \frac{1}{n!} \left(8\beta_I \gamma^3 \log v\right)^n$$

2. For circular orbits the two invariants $E(\omega)$ and $J(\omega)$ are linked by the "thermodynamic" relation or first law of binary mechanics

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Leading powers of logarithms from RG theory

This gives three relations for the three unknowns $E(\omega)$ and $J(\omega)$ and $r(\omega)$

$$\boxed{\begin{aligned} E^{\text{leading }(\log)^n} &= -\frac{m\nu\, x}{2}\left[1+\frac{64\nu}{15}\sum_{n=1}^{+\infty}\frac{6n+1}{n!}(4\beta_I)^{n-1}\, x^{3n+1}(\log x)^n\right] \\ J^{\text{leading }(\log)^n} &= \frac{m^2\nu}{\sqrt{x}}\left[1-\frac{64\nu}{15}\sum_{n=1}^{+\infty}\frac{3n+2}{n!}(4\beta_I)^{n-1}\, x^{3n+1}(\log x)^n\right] \end{aligned}}$$

in agreement with high-order GSF calculations up to 22PN order !

RADIATION REACTION AND BALANCE EQUATIONS

Radiation reaction and balance equations

1. Conserved Newtonian energy in the source

$$E = \int \mathrm{d}^3\mathbf{x}\,\rho\left[\frac{\mathbf{v}^2}{2} + \Pi - \frac{U}{2}\right]$$

2. Eulerian equations of motion in the source

$$\rho\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\partial_i P + \rho\partial_i U - \overbrace{\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5}}^{\boldsymbol{F}^{\text{reac}}}$$

3. Energy loss is due to the work of the radiation reaction force

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = \int \mathrm{d}^3\mathbf{x}\,\boldsymbol{v}\cdot\boldsymbol{F}^{\text{reac}} = -\frac{G}{5c^5}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \text{total time derivative}}$$

4. Obtain the balance equation after averaging over one period

$$\langle\frac{\mathrm{d}E}{\mathrm{d}t}\rangle = -\langle\mathcal{F}^{\text{GW}}\rangle \quad\Longrightarrow\quad \phi = \int \omega\,\mathrm{d}t = \int \frac{\omega}{\dot{\omega}}\,\mathrm{d}\omega$$

Radiation reaction to 4PN order [Blanchet 1993, 1997]

1. At 2.5PN order for general matter systems the radiation reaction force in a specific gauge is purely scalar [Burke & Thorne 1970]

$$F_i^{\text{reac}} = \rho\,\partial_i V^{\text{reac}}$$

2. At the 3.5PN order the radiation reaction derives from scalar and vector radiation reaction potentials

$$F_i^{\text{reac}} = \rho \left[\partial_i V^{\text{reac}} - \frac{4}{c^2} v^j \left(\partial_i V_j^{\text{reac}} - \partial_j V_i^{\text{reac}} \right) - \frac{4}{c^3} \varepsilon_{ijk} v^j \frac{\mathrm{d}V_k^{\text{reac}}}{\mathrm{d}t} \right]$$

3. At 4PN order the radiation reaction contains a tail term (again scalar)

Radiation reaction to 4PN order [Blanchet 1993, 1997]

$$V^{\text{reac}} = -\overbrace{\frac{G}{5c^5} x^{ij} I^{(5)}_{ij}}^{\text{2.5PN radiation reaction}} + \overbrace{\frac{G}{c^7}\left[\frac{1}{189} x^{ijk} I^{(7)}_{ijk} - \frac{1}{70} r^2 x^{ij} I^{(7)}_{ij}\right]}^{\text{3.5PN scalar correction}}$$

$$\underbrace{-\frac{4G^2 M}{5c^8} x^{ij} \int_0^{+\infty} \mathrm{d}\tau\, I^{(7)}_{ij}(t-\tau)\left[\ln\left(\frac{\tau}{2\tau_0}\right) + \frac{11}{12}\right]}_{\text{4PN radiation reaction tail}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

$$V^{\text{reac}}_i = \underbrace{\frac{G}{c^5}\left[\frac{1}{21}\hat{x}^{ijk} I^{(6)}_{jk} - \frac{4}{45}\varepsilon_{ijk}\, x^{jl} J^{(5)}_{kl}\right]}_{\text{3.5PN vector correction}} + \mathcal{O}\left(\frac{1}{c^7}\right)$$

This result permits to prove the balance equations for general isolated systems up to the 4PN order or 1.5PN relative order beyond the quadrupolar radiation

Radiation reaction derivation revisited [Blanchet & Faye 2018]

1. Metric accurate to 1PN order for conservative effects and to 3.5PN order for dissipative radiation reaction effects

$$
\begin{aligned}
g_{00} &= -1 + \frac{2\mathcal{V}}{c^2} - \frac{2\mathcal{V}^2}{c^4} + \mathcal{O}^{\text{cons}}\left(\frac{1}{c^6}\right)\\
g_{0i} &= -\frac{4\mathcal{V}_i}{c^3} + \mathcal{O}^{\text{cons}}\left(\frac{1}{c^5}\right)\\
g_{ij} &= \delta_{ij}\left(1 + \frac{2\mathcal{V}}{c^2}\right) + \frac{4}{c^4}\Big(W_{ij} - \delta_{ij} W_{kk}\Big) + \mathcal{O}^{\text{cons}}\left(\frac{1}{c^6}\right)
\end{aligned}
$$

2. Potentials are composed of a conservative part and a dissipative one

$$
\mathcal{V}_\mu = V_\mu^{\text{cons}} + \boxed{V_\mu^{\text{reac}}}
$$

3. Integrate the matter equations of motion $\nabla_\nu T^{\mu\nu} = 0$ over the source

$$
\partial_\nu\left(\sqrt{-g}\,T^\nu_\mu\right) = \frac{1}{2}\sqrt{-g}\,\partial_\mu g_{\rho\sigma} T^{\rho\sigma}
$$

Radiation reaction derivation revisited [Blanchet & Faye 2018]

- Recover well known results for the fluxes of energy and angular momentum [Epstein & Wagoner 1975; Thorne 1980; Blanchet & Damour 1989]

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{G}{c^5}\left(\frac{1}{5}I^{(3)}_{ij}I^{(3)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}I^{(4)}_{ijk}I^{(4)}_{ijk} + \frac{16}{45}J^{(3)}_{ij}J^{(3)}_{ij}\right]\right) + \mathcal{O}\left(\frac{1}{c^8}\right)$$

$$\frac{\mathrm{d}J_i}{\mathrm{d}t} = -\frac{G}{c^5}\varepsilon_{ijk}\left(\frac{2}{5}I^{(2)}_{jl}I^{(3)}_{kl} + \frac{1}{c^2}\left[\frac{1}{63}I^{(3)}_{jlm}I^{(4)}_{klm} + \frac{32}{45}J^{(2)}_{jl}J^{(3)}_{kl}\right]\right) + \mathcal{O}\left(\frac{1}{c^8}\right)$$

- And also for the linear momentum which is a subdominant 3.5PN effect [Papapetrou 1971; Bekenstein 1973]

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = -\frac{G}{c^7}\left[\frac{2}{63}I^{(4)}_{ijk}I^{(3)}_{jk} + \frac{16}{45}\varepsilon_{ijk}I^{(3)}_{jl}J^{(3)}_{kl}\right] + \mathcal{O}\left(\frac{1}{c^9}\right)$$

What about the position of the center of mass?

- For an isolated conservative system the conserved integrals are E, J_i, P_i and also the initial position of the center of mass

$$Z_i = G_i - P_i t$$

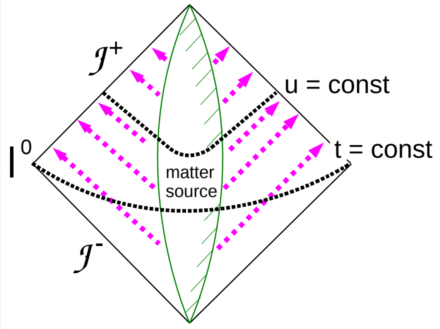
where G_i is the position of the center of mass multiplied by the mass

- The conservation of Z_i is associated with the invariance under Lorentz boosts
- We also find a balance equation for the center-of-mass position

$$\boxed{\frac{\mathrm{d}G_i}{\mathrm{d}t} = P_i - \frac{2G}{21c^7}\, I^{(3)}_{ijk} I^{(3)}_{jk} + \mathcal{O}\left(\frac{1}{c^9}\right)}$$

- This formula has never appeared in standard texbooks on GR or gravitational waves, nor on specialized reviews, it appeared only recently in the GW litterature [Kozameh, Nieva & Quirega 2018; Nichols 2018; Blanchet & Faye 2018]

Direct calculation of the GW fluxes at infinity



1. Introduce a retarded null coordinate u satisfying

$$\boxed{g^{\mu\nu}\partial_\mu u \partial_\nu u = 0}$$

2. For instance choose $u = t - r_*/c$ with the tortoise coordinate

$$r_* = r + \frac{2GM}{c^2}\ln\left(\frac{r}{r_0}\right) + \mathcal{O}\left(\frac{1}{r}\right)$$

Direct calculation of the GW fluxes at infinity

1. Perform a coordinate change $(t, \mathbf{x}) \to (u, \mathbf{x})$ in the conservation law of the pseudo-tensor $\partial_\nu \tau^{\mu\nu} = 0$ to get

$$\frac{\partial}{c\partial u}\Big[\tau^{\mu 0}(\mathbf{x}, u + r_*/c) - n_*^i \tau^{\mu i}(\mathbf{x}, u + r_*/c)\Big] + \partial_i \Big[\tau^{\mu i}(\mathbf{x}, u + r_*/c)\Big] = 0$$

2. Integrating over a volume $\mathcal{V}$ tending to infinity with $u =$const

$$\boxed{\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}u} &= -c \int_{\partial\mathcal{V}} \mathrm{d}S_i \, \tau^{0i}_{\mathsf{GW}}(\mathbf{x}, u + r_*/c) \\
\frac{\mathrm{d}J_i}{\mathrm{d}u} &= -\varepsilon_{ijk} \int_{\partial\mathcal{V}} \mathrm{d}S_l \, x^j \, \tau^{kl}_{\mathsf{GW}}(\mathbf{x}, u + r_*/c) \\
\frac{\mathrm{d}P^i}{\mathrm{d}u} &= - \int_{\partial\mathcal{V}} \mathrm{d}S_j \, \tau^{ij}_{\mathsf{GW}}(\mathbf{x}, u + r_*/c) \\
\frac{\mathrm{d}G_i}{\mathrm{d}u} &= P_i - \frac{1}{c} \int_{\partial\mathcal{V}} \mathrm{d}S_j \left(x^i \, \tau^{0j}_{\mathsf{GW}} - r_* \, \tau^{ij}_{\mathsf{GW}} \right) (\mathbf{x}, u + r_*/c)
\end{aligned}}$$

Direct calculation of the GW fluxes at infinity

A long calculation to control the leading $1/r^2$ and subleading $1/r^3$ terms in the GW pseudo-tensor when $r \to +\infty$ gives the fluxes as full multipole series parametrized by the multipole moments I_L and J_L up to order $\mathcal{O}(G^2)$

$$\boxed{\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}u} &= -\sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left\{ \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\ell!(2\ell+1)!!} \overset{(\ell+1)}{I}_L \overset{(\ell+1)}{I}_L \right. \\
&\qquad\qquad \left. + \frac{4\ell(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} \overset{(\ell+1)}{J}_L \overset{(\ell+1)}{J}_L \right\} \\
\frac{\mathrm{d}J_i}{\mathrm{d}u} &= -\varepsilon_{ijk} \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left\{ \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!(2\ell+1)!!} \overset{(\ell)}{I}_{jL-1} \overset{(\ell+1)}{I}_{kL-1} \right. \\
&\qquad\qquad \left. + \frac{4\ell^2(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} \overset{(\ell)}{J}_{jL-1} \overset{(\ell+1)}{J}_{kL-1} \right\}
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\frac{\mathrm{d}P_i}{\mathrm{d}u} = -\sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+3}} \Bigg\{ & \frac{2(\ell+2)(\ell+3)}{\ell(\ell+1)!(2\ell+3)!!} \overset{(\ell+2)}{I}_{iL} \overset{(\ell+1)}{I}_{L} \\
& + \frac{8(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!} \varepsilon_{ijk} \overset{(\ell+1)}{I}_{jL-1} \overset{(\ell+1)}{J}_{kL-1} \\
& + \frac{8(\ell+3)}{c^2(\ell+1)!(2\ell+3)!!} \overset{(\ell+2)}{J}_{iL} \overset{(\ell+1)}{J}_{L} \Bigg\}
\end{aligned}
$$

$$
\frac{\mathrm{d}G_i}{\mathrm{d}u} = P_i \underbrace{- \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+3}} \left\{ \frac{2(\ell+2)(\ell+3)}{\ell\,\ell!(2\ell+3)!!} \overset{(\ell+1)}{I}_{iL} \overset{(\ell+1)}{I}_{L} + \frac{8(\ell+3)}{c^2 \ell!(2\ell+3)!!} \overset{(\ell+1)}{J}_{iL} \overset{(\ell+1)}{J}_{L} \right\}}_{\text{[Blanchet \& Faye 2018]}}
$$

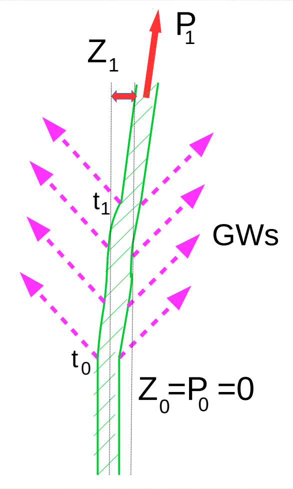
Any implication for the total recoil of a source?

1. We have obtained the balance equations

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = -\boldsymbol{F}_P\,,$$
$$\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}t} = \boldsymbol{P} - \boldsymbol{F}_G\,,$$

2. Integrating these equations for a burst of GWs with finite duration we obtain

$$\boldsymbol{P}_1 = -\int_{t_0}^{t_1} \mathrm{d}t'\,\boldsymbol{F}_P(t')\,,$$
$$\boldsymbol{Z}_1 = \int_{t_0}^{t_1} \mathrm{d}t'\Big[t'\,\boldsymbol{F}_P(t') - \boldsymbol{F}_G(t')\Big]\,.$$



3. The total recoil depends only on the linear momentum flux (as in usual calculations)

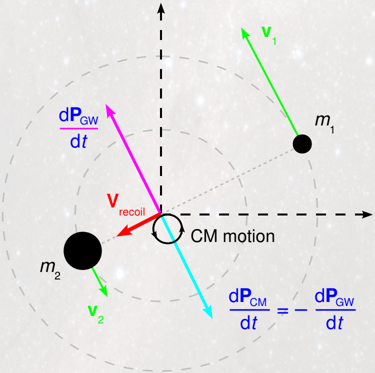
The instantaneous CM position of a circular binary

1. The linear momentum is evaluated for a Newtonian circular binary as usual [Fitchett 1983]

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \frac{464}{105}\frac{G^4 m^5 \omega}{c^7 r^4}\sqrt{1-4\nu}\,\nu^2\,\boldsymbol{\lambda}$$
$$\boldsymbol{P} = \frac{464}{105}\frac{G^4 m^5}{c^7 r^4}\sqrt{1-4\nu}\,\nu^2\,\boldsymbol{n}$$

2. However in order to obtain the instantaneous CM position we must also use the CM flux

$$\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}t} = \boldsymbol{P} + \frac{544}{105}\frac{G^4 m^5}{c^7 r^4}\sqrt{1-4\nu}\,\nu^2\,\boldsymbol{n}$$
$$\boldsymbol{G} = -\frac{48}{5}\frac{G^4 m^5}{c^7 r^4 \omega}\sqrt{1-4\nu}\,\nu^2\,\boldsymbol{\lambda}$$



3. It would be interesting to compare this prediction to very accurate NR computations of the CM position [Gerosa, Hébert & Stein 2018; Woodford, Boyle & Pfeiffer 2019]