Cosmological effects in waveform modeling

Giulia Cusin

12th-16th April 2021

UNIVERSITÉ
DE GENÈVE

Outline

- The new era of gravitational wave physics: theory and observations

- Solution of GW equation in vacuum: counting dofs and TT gauge blackboard

- Derivation of the quadrupole formula: binary system in quasi-circular orbit

- Waveform at cosmological distance → blackboard

- Effects of redshift variation over observation time

- Effect of lensing on the waveform: amplification factor blackboard

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References

Textbooks:

M. Maggiore. Gravitational waves, Vol. 1

C. M. Will. Theory and Experiment in Gravitational Physics

P. Schneider J. Ehlers E. E. Falco. Gravitational Lenses

Articles:

Bonvin et al. 2016 T*he effect of matter structure on the gravitational waveform*

Nakamura, Deguchi 1999 *Wave optics in gravitational lensing*

Cusin, Lagos 2019 *Gravitational wave propagation beyond geometric optics*

Dalang, Cusin, Lagos 2021 *Polarisation distortion beyond geometric optics*

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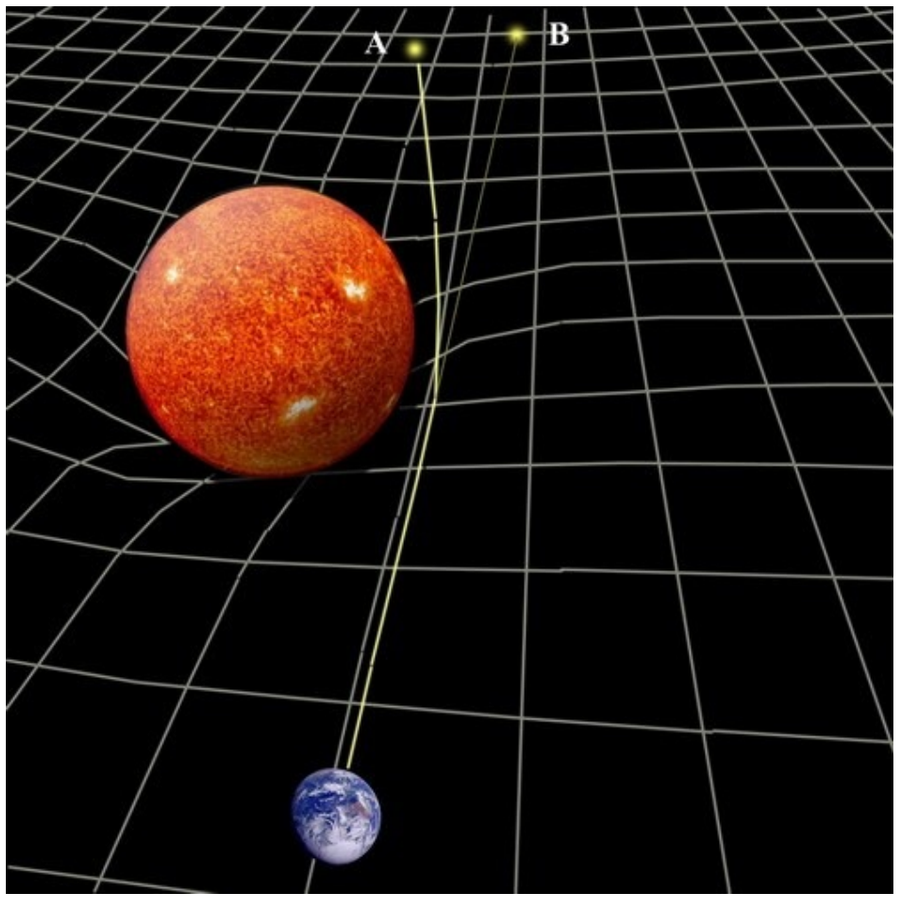
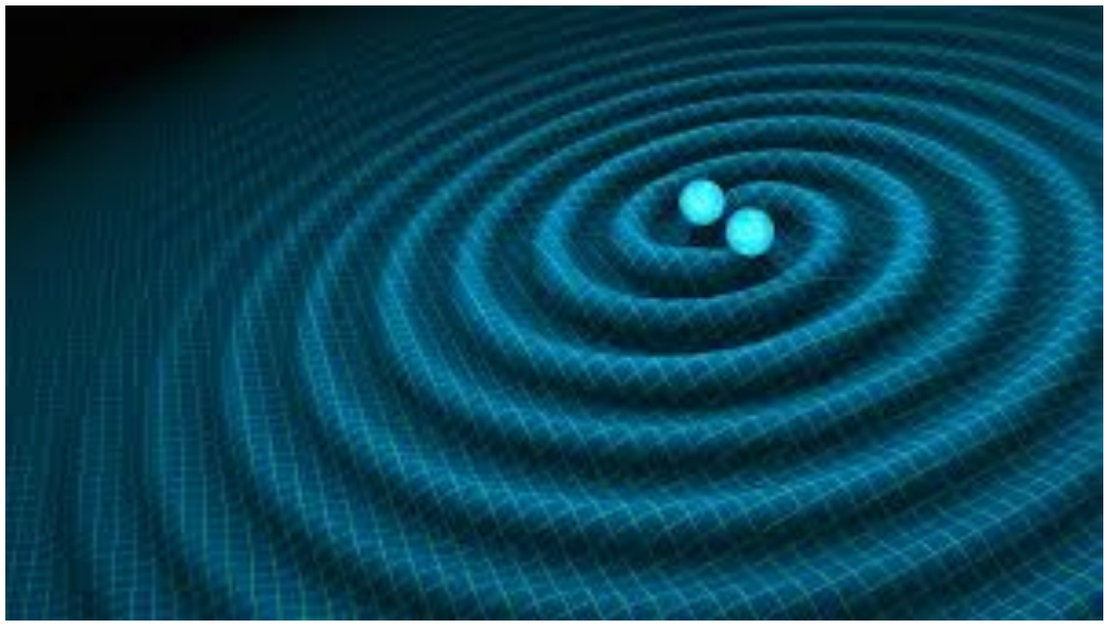
Gravitational waves vs electromagnetic radiation



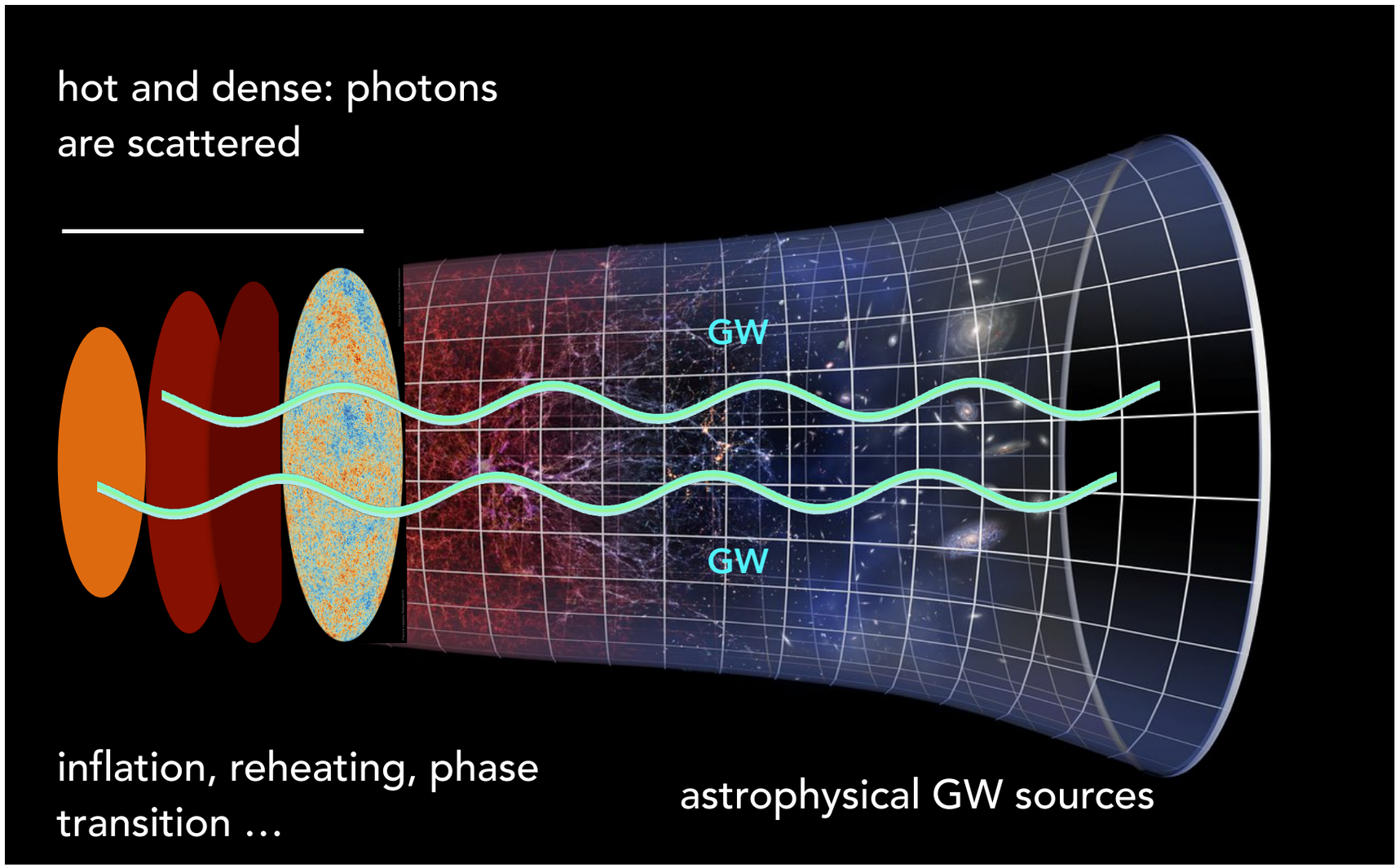


electromagnetic radiation:
signals propagate in a curved spacetime

gravitational waves:
deformation of the space time itself

Gravitational wave sources: a new window on our Universe

Gravitational waves from early Universe bring us information of “very first” instants of time

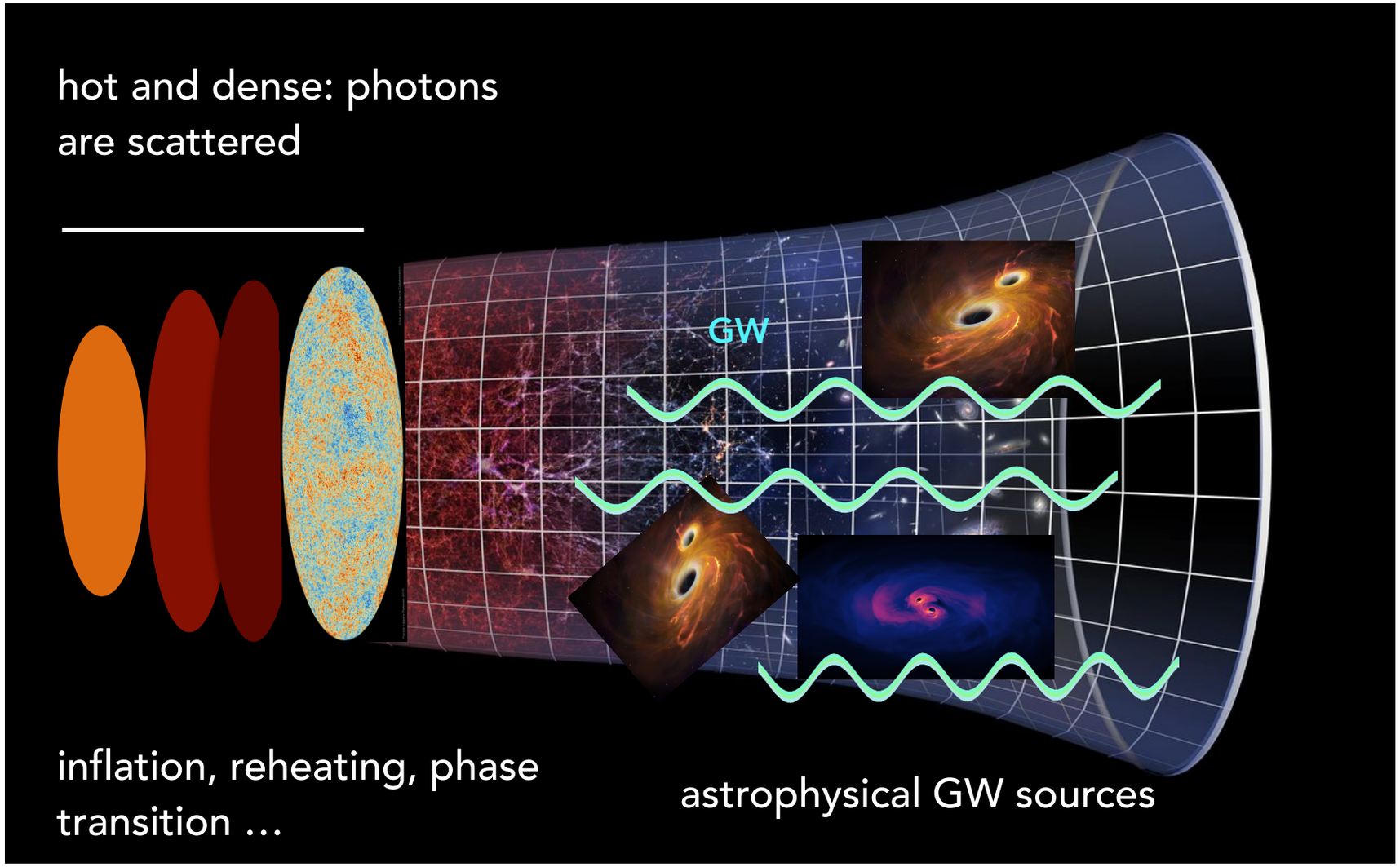


Big Bang → observer

time

Gravitational wave sources: a new window on our Universe

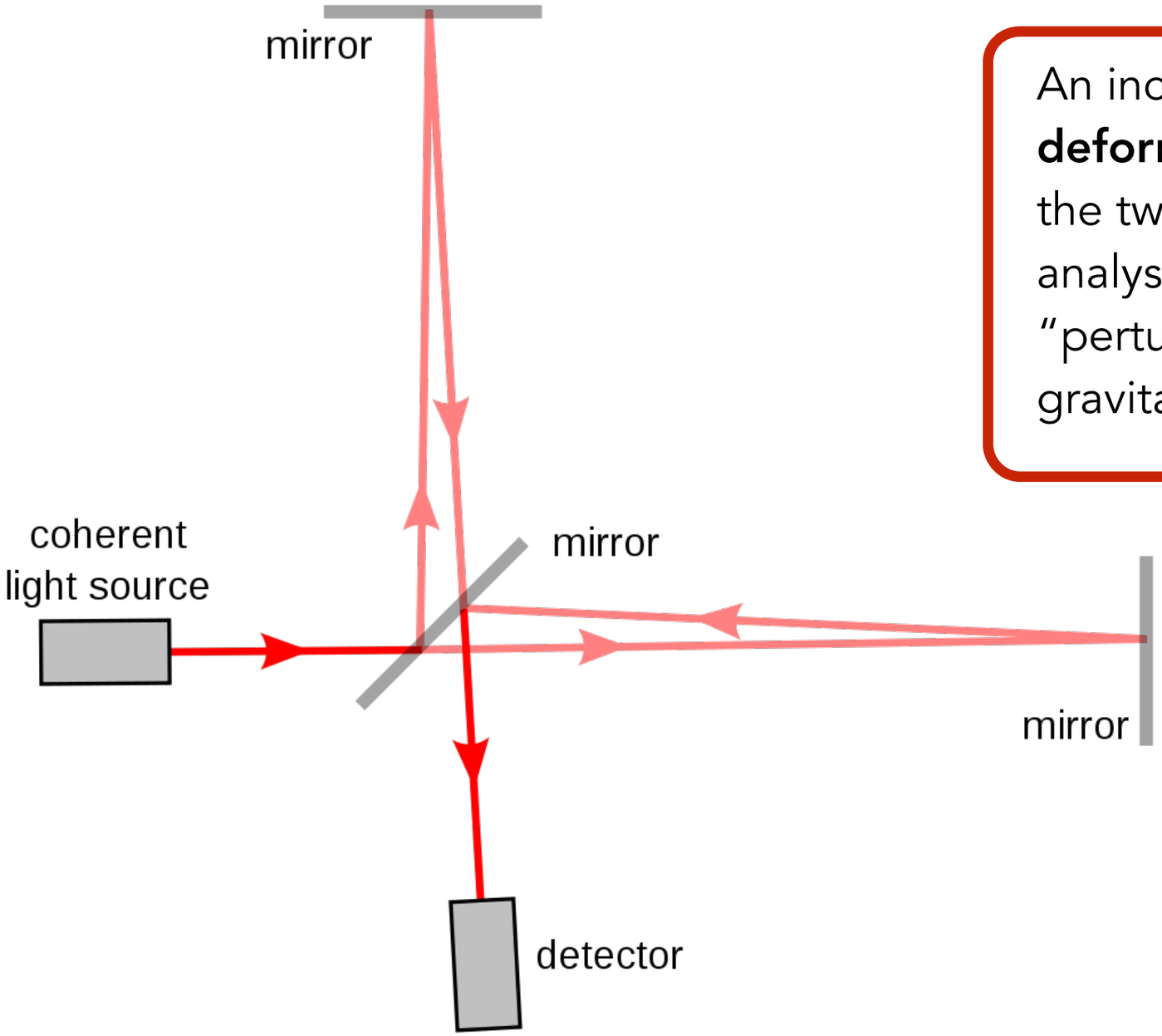
Any time-varying mass distribution is source of gravitational radiation



Big Bang → observer

time

How we detect: interferometry in one slide

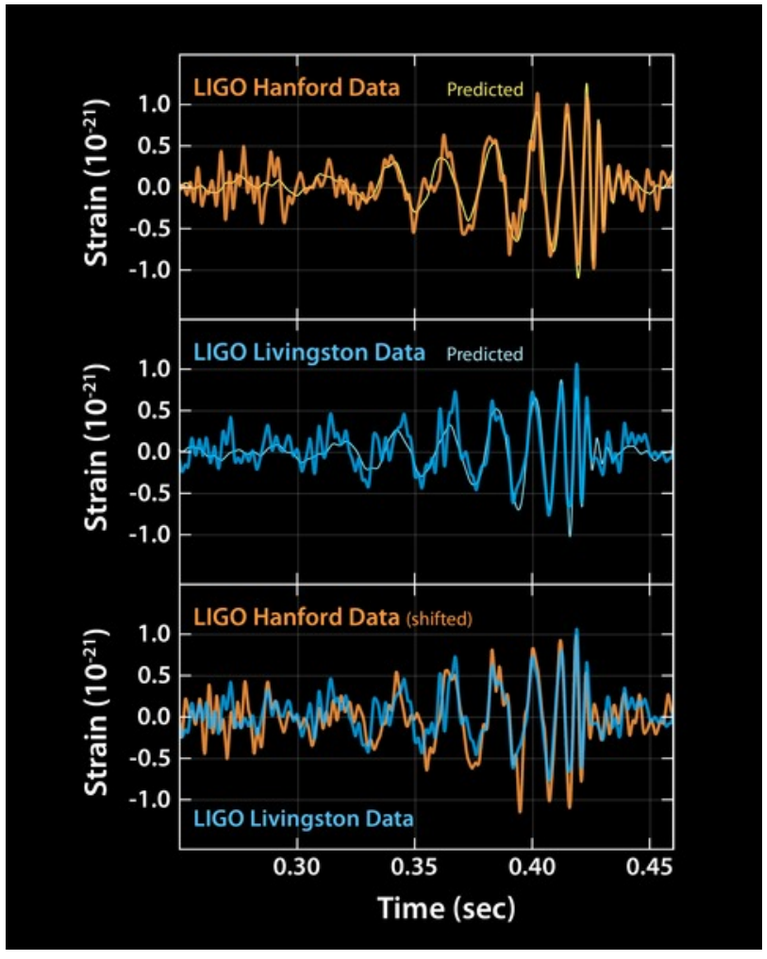


Simplified idea

An incoming wave gives **anisotropic deformation** of the two arms: combining the two reflected light beams, we can analyse whether the instrument has been "perturbed" or not and reconstruct the gravitational wave signal

Resolvable signal vs stochastic background

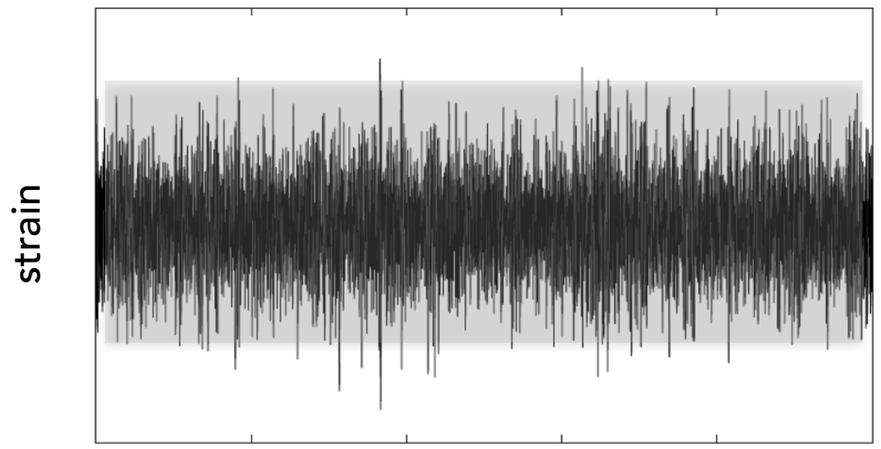
GW from resolvable event
(catalogue approach)



LVC

quantity directly observable is the strain

stochastic background: incoherent superposition of GW signals



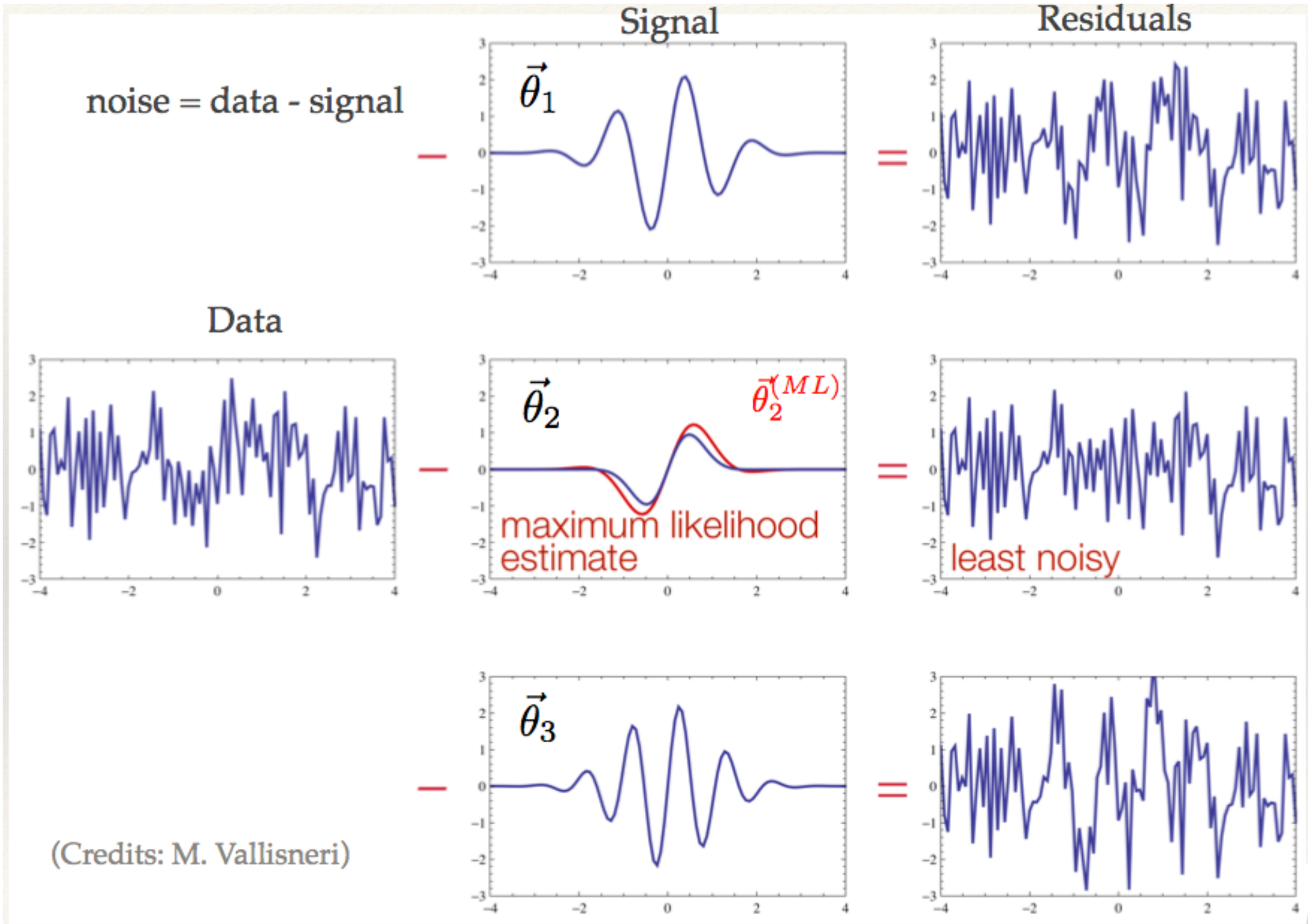
"waveform" similar to noise

typically within instrumental noise threshold

↓

strain is not directly observable:
we only observe quantities **quadratic in the strain**
(we need to match outputs of two detectors to extract background signal)

Match filtering and parameter estimation



Which is the arm length needed

$$10^{-13}\mathrm{pc} \simeq 1\mathrm{km}$$

Frequency (Hz)	Wavelength (pc)	
$10^{0} - 10^{3}$	$10^{-8} - 10^{-11}$	merger of stellar mass compact objects
$10^{-4} - 10^{-1}$	$10^{-4} - 10^{-7}$	merger of massive compact objects
$10^{-9} - 10^{-6}$	$10^{-1} - 10^{1}$	inspiraling of stellar mass compact objects
		massive compact objects inspiraling

As a rule of thumb: $M \propto 1/f$

Which is the arm length needed

$$10^{-13}\mathrm{pc} \simeq 1\mathrm{km}$$

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massive compact objects inspiraling

The instrument “does not see” the wave for

$\lambda \gg (\text{arm length})$

The longer the wavelength, the longer the arm length has to be

Which is the arm length needed

$10^{-13}\mathrm{pc} \simeq 1\mathrm{km}$

Detector	Frequency (Hz)	Wavelength (pc)	
LIGO/Virgo	$10^{0} - 10^{3}$	$10^{-8} - 10^{-11}$	merger of stellar mass compact objects
LISA	$10^{-4} - 10^{-1}$	$10^{-4} - 10^{-7}$	merger of massive compact objects
IPTA	$10^{-9} - 10^{-6}$	$10^{-1} - 10^{1}$	inspiraling of stellar mass compact objects

on Earth

in space

using array of pulsars

massive compact objects inspiraling

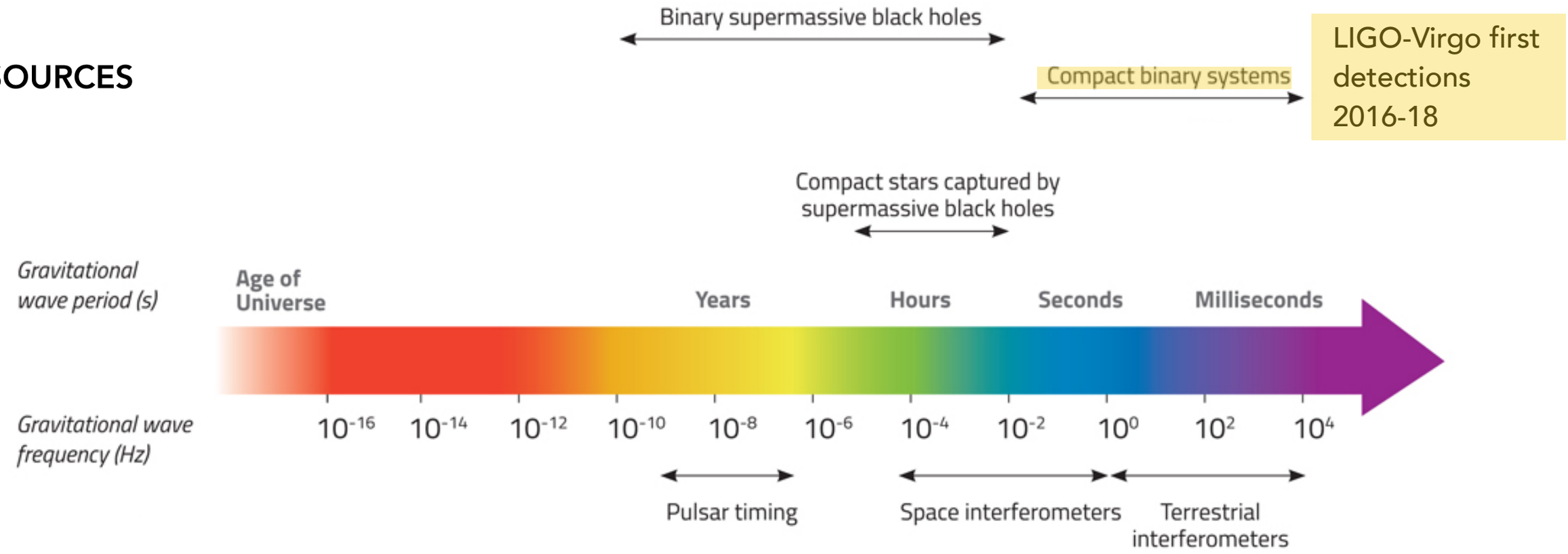
The instrument “does not see” the wave for

$\lambda \gg (\text{arm length})$

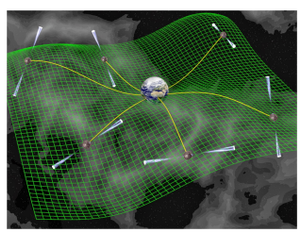
The longer the wavelength, the longer the arm length has to be

The new era of gravitational wave astronomy

SOURCES



DETECTORS

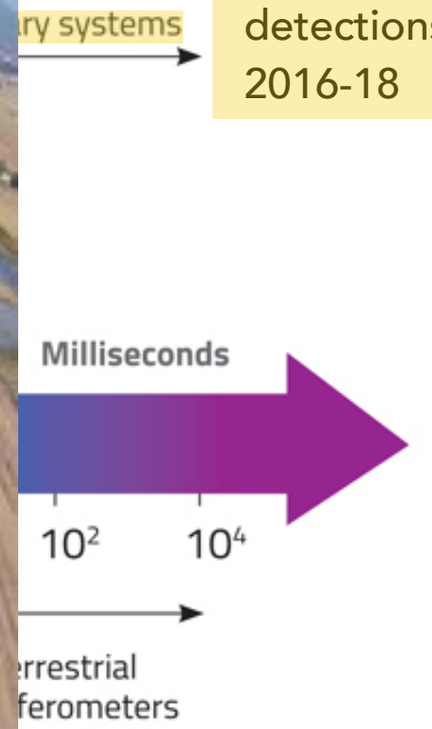


Pulsar-Timing Arrays

LISA (>2030)

LIGO, Virgo, LIGO-India, Kagra, **Cosmic Explorer, Einstein Telescope** (>2030)

The Hz band



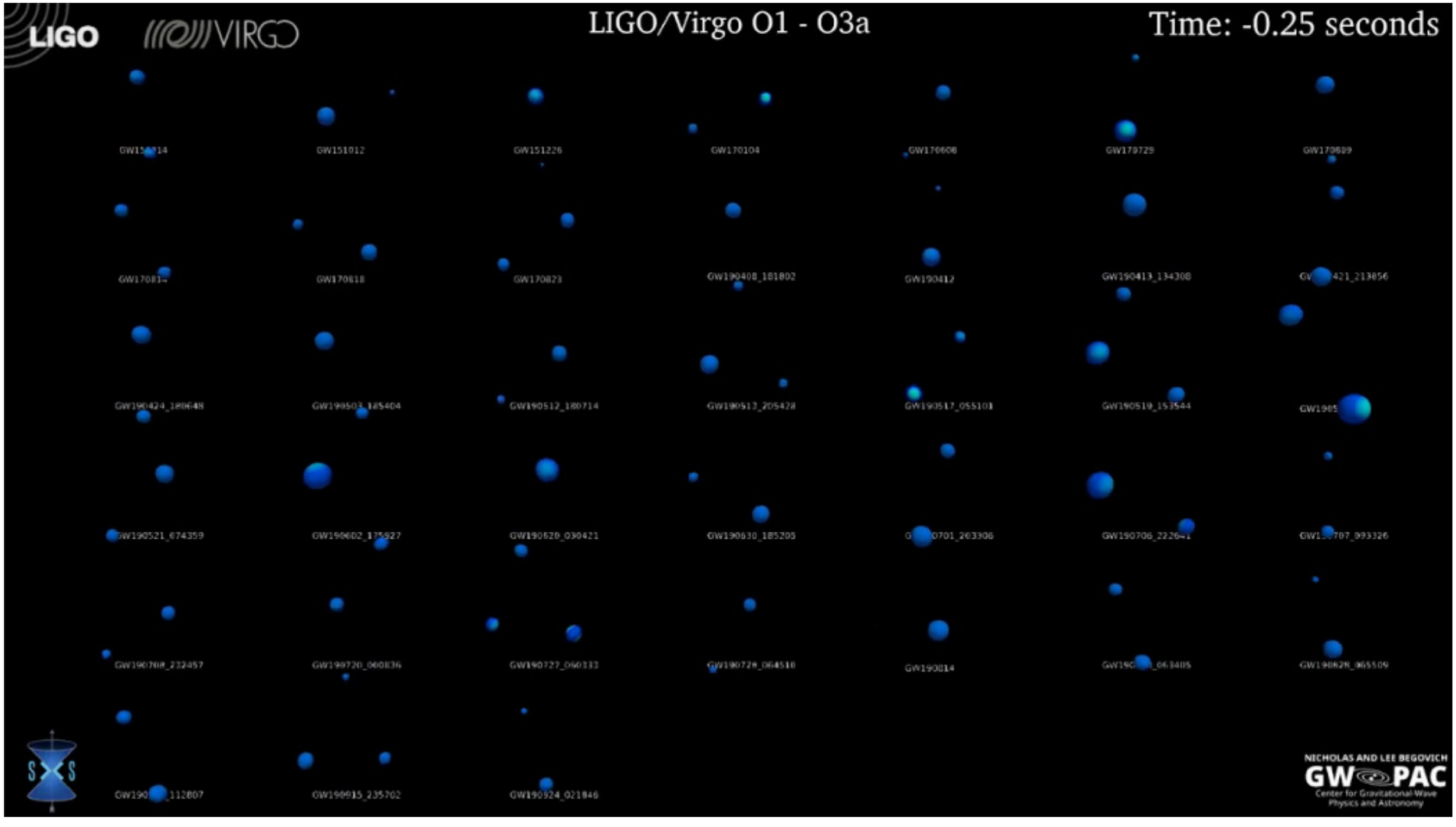
LIGO-Virgo first detections 2016-18

ground based detectors: arm length few km

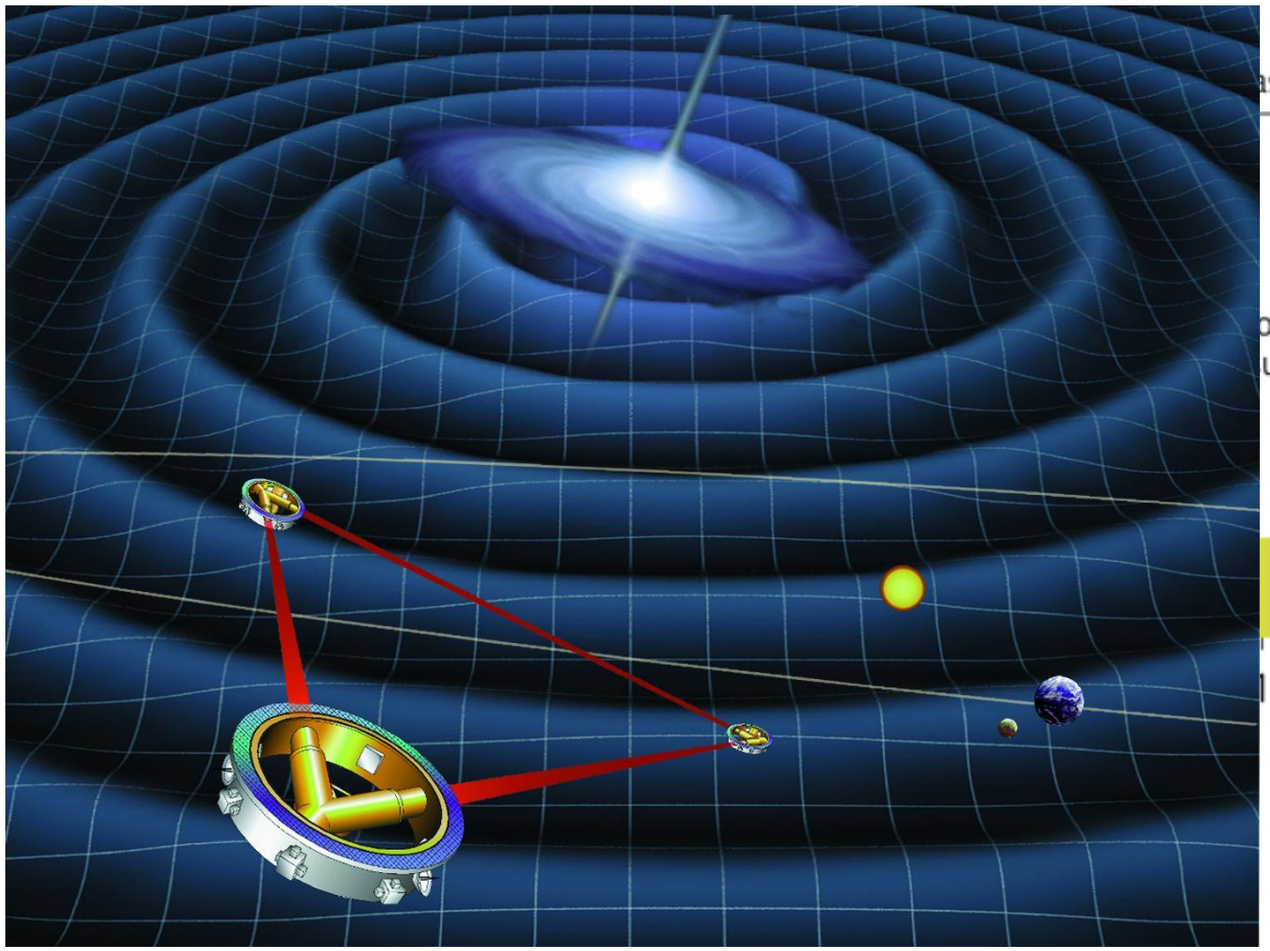
LIGO, Virgo, LIGO-India, Kagra,
Einstein Telescope (>2035)
Cosmic Explorer

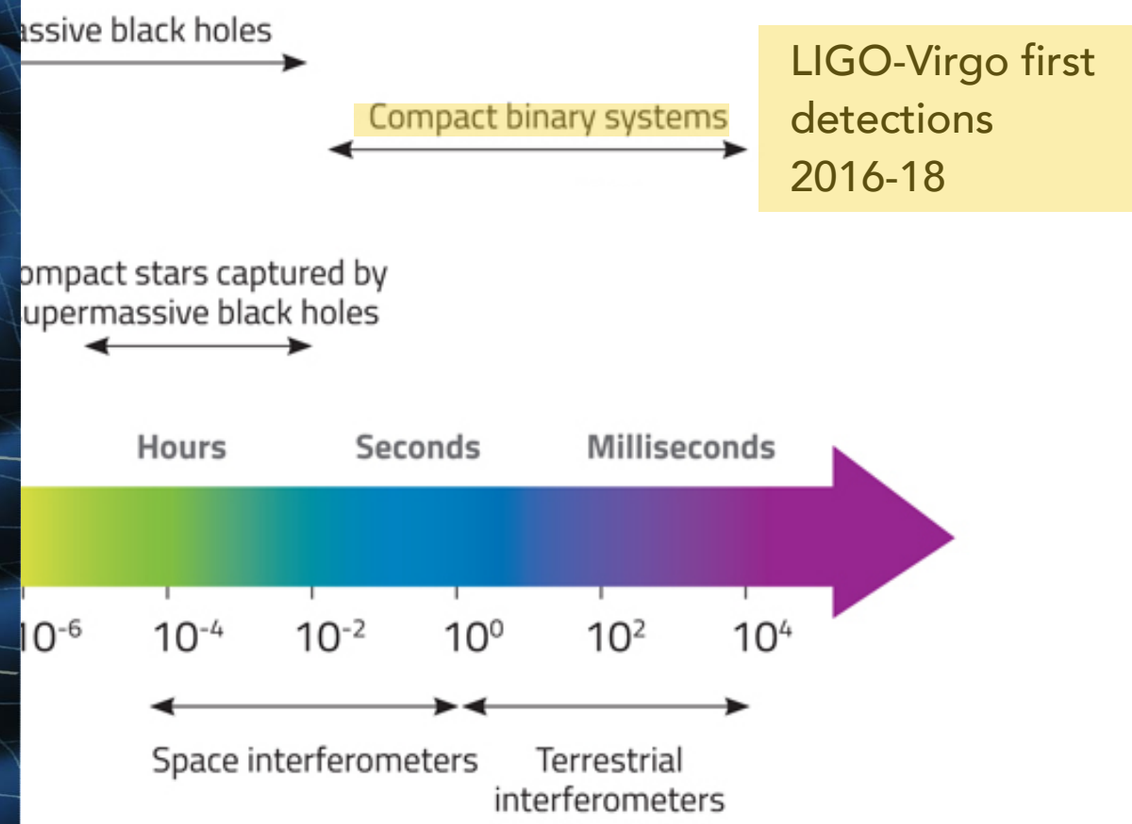
Catalogue of detected binary black holes

O1-O3 runs



The mHz band

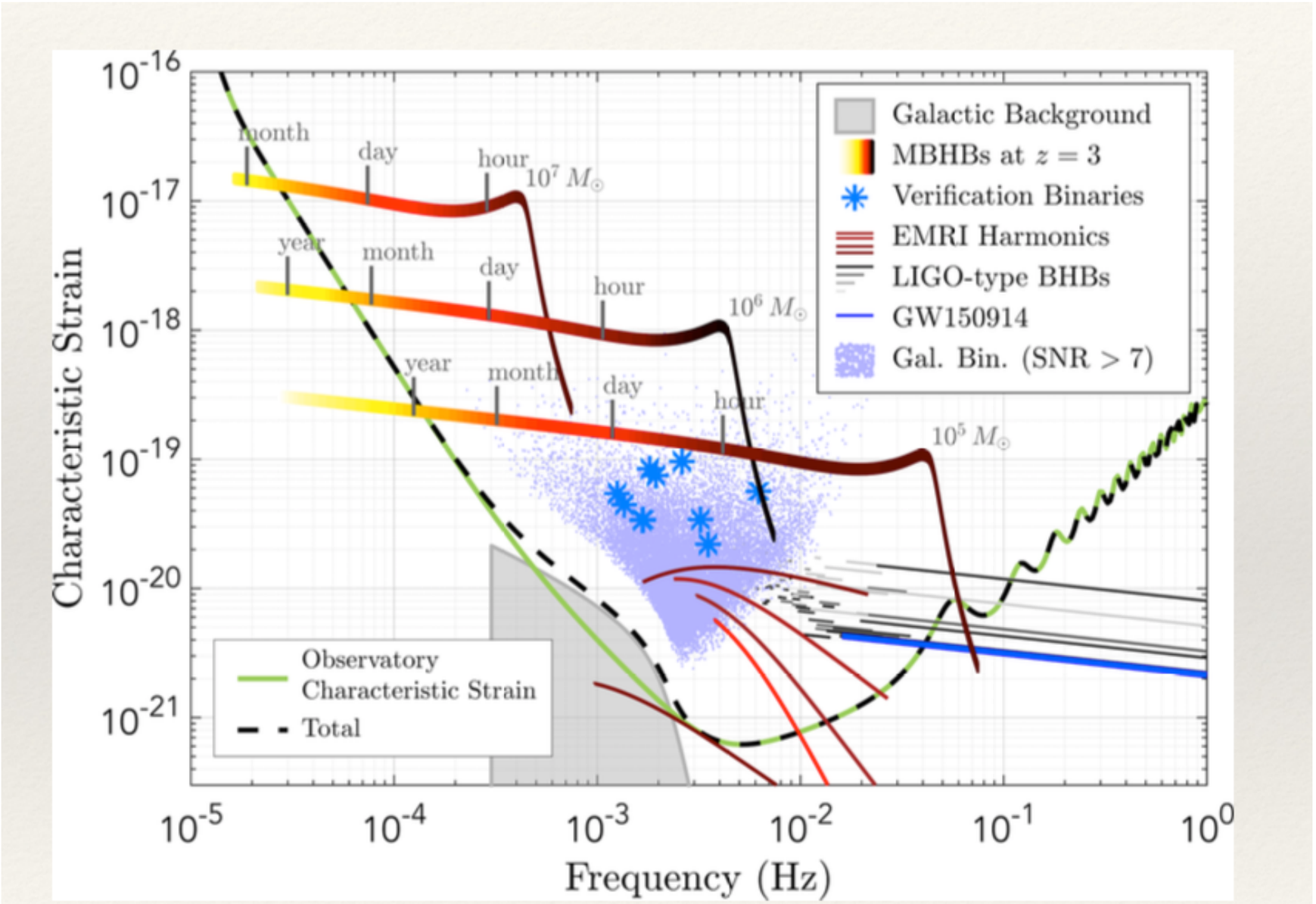




LISA: three satellites in triangular configuration at 2.5 Gm

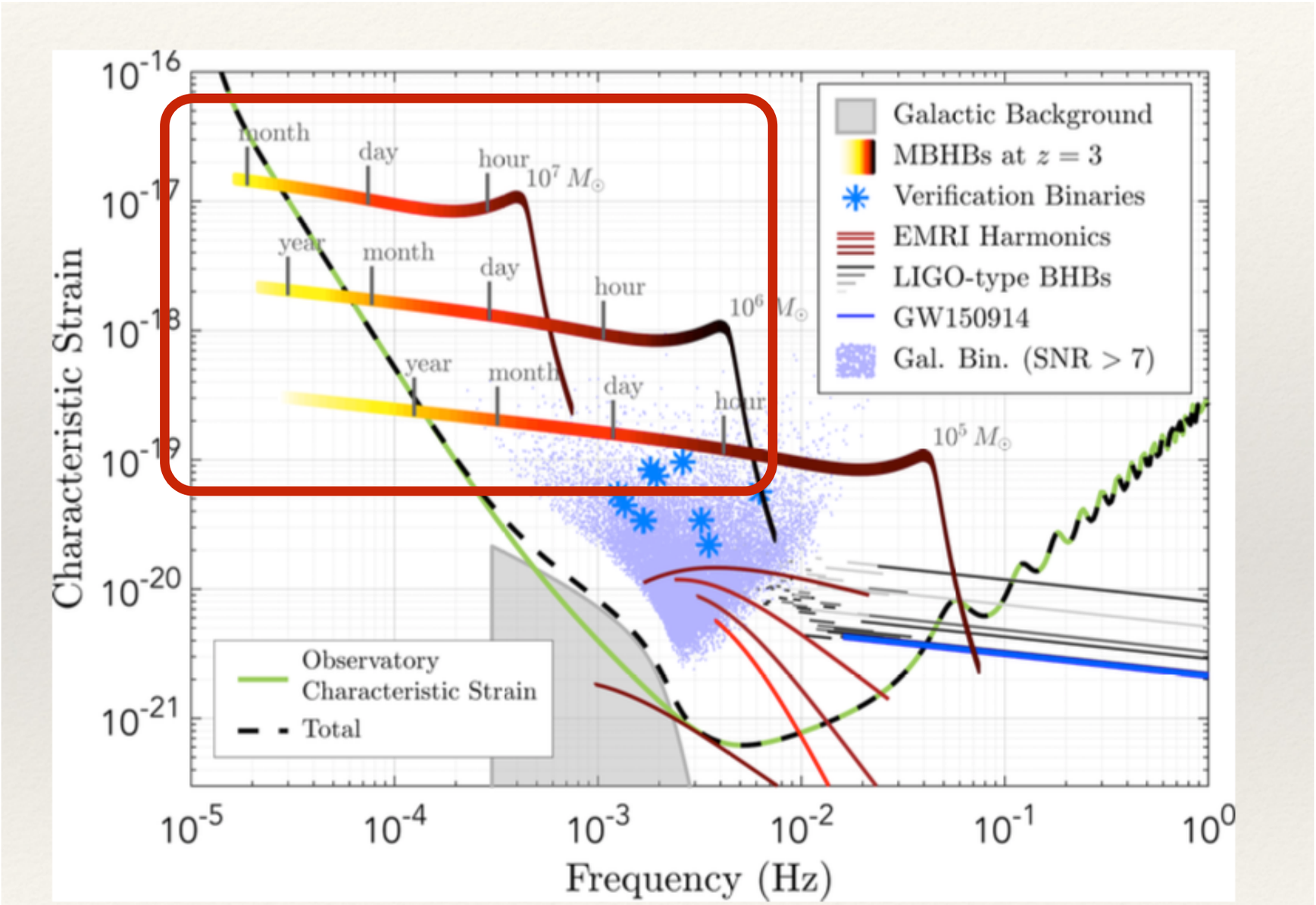
We detect frequencies in the mHz band

Expected sources from LISA



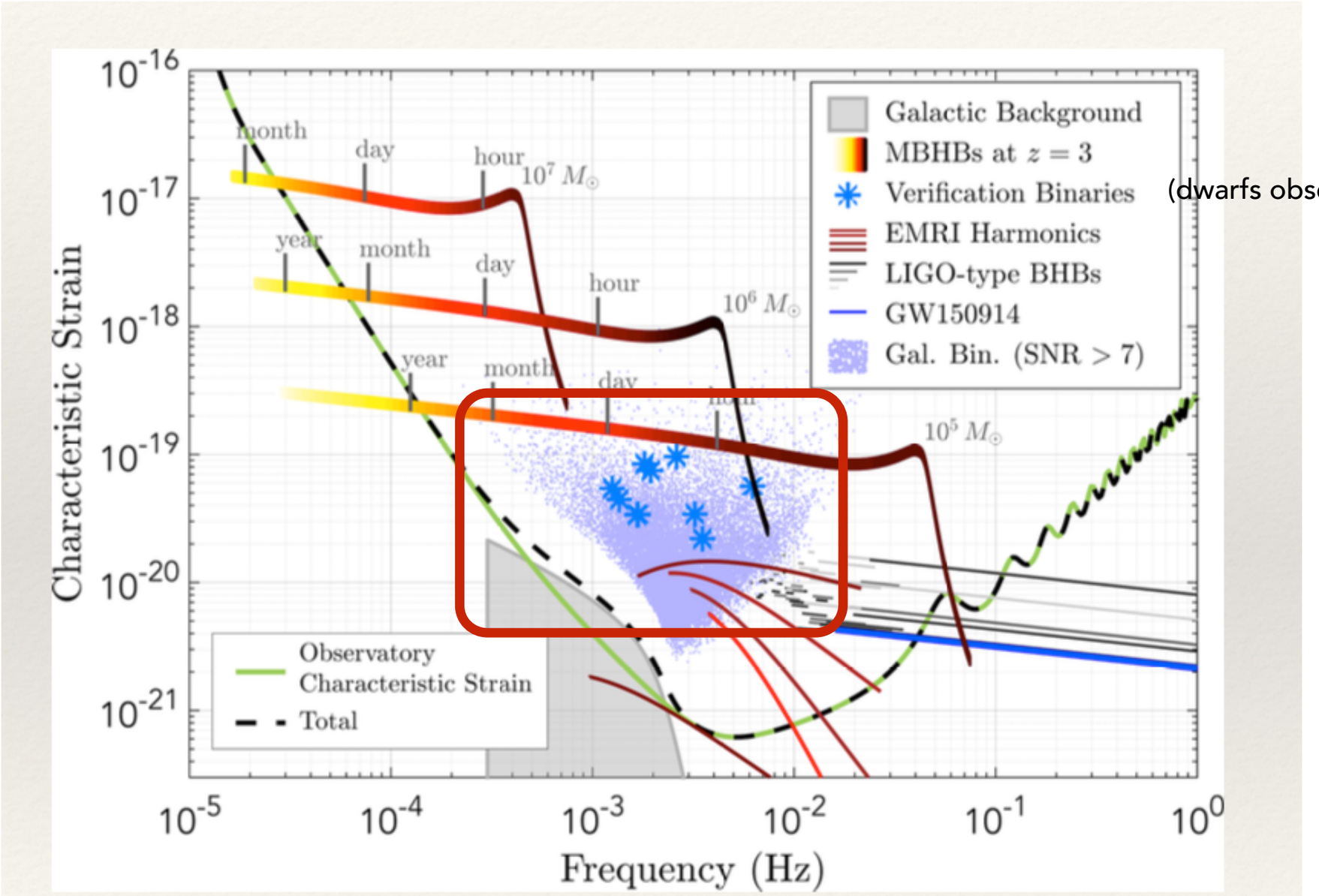
Expected sources from LISA

The strongest sources: (i) MBHBs (massive black hole binaries)



Expected sources from LISA

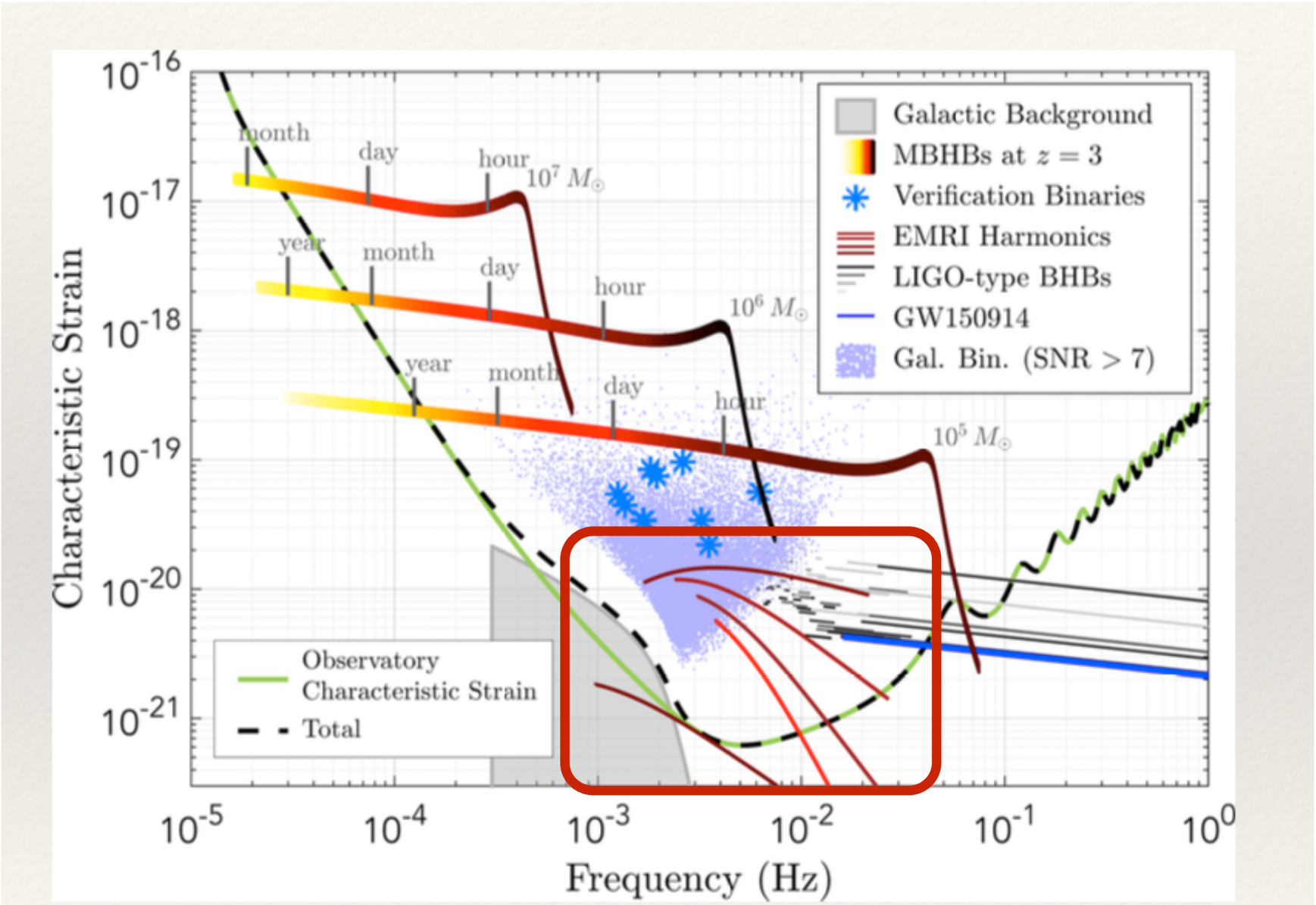
The strongest sources: (ii) Galactic white dwarf binaries



(dwarfs observable in EM)

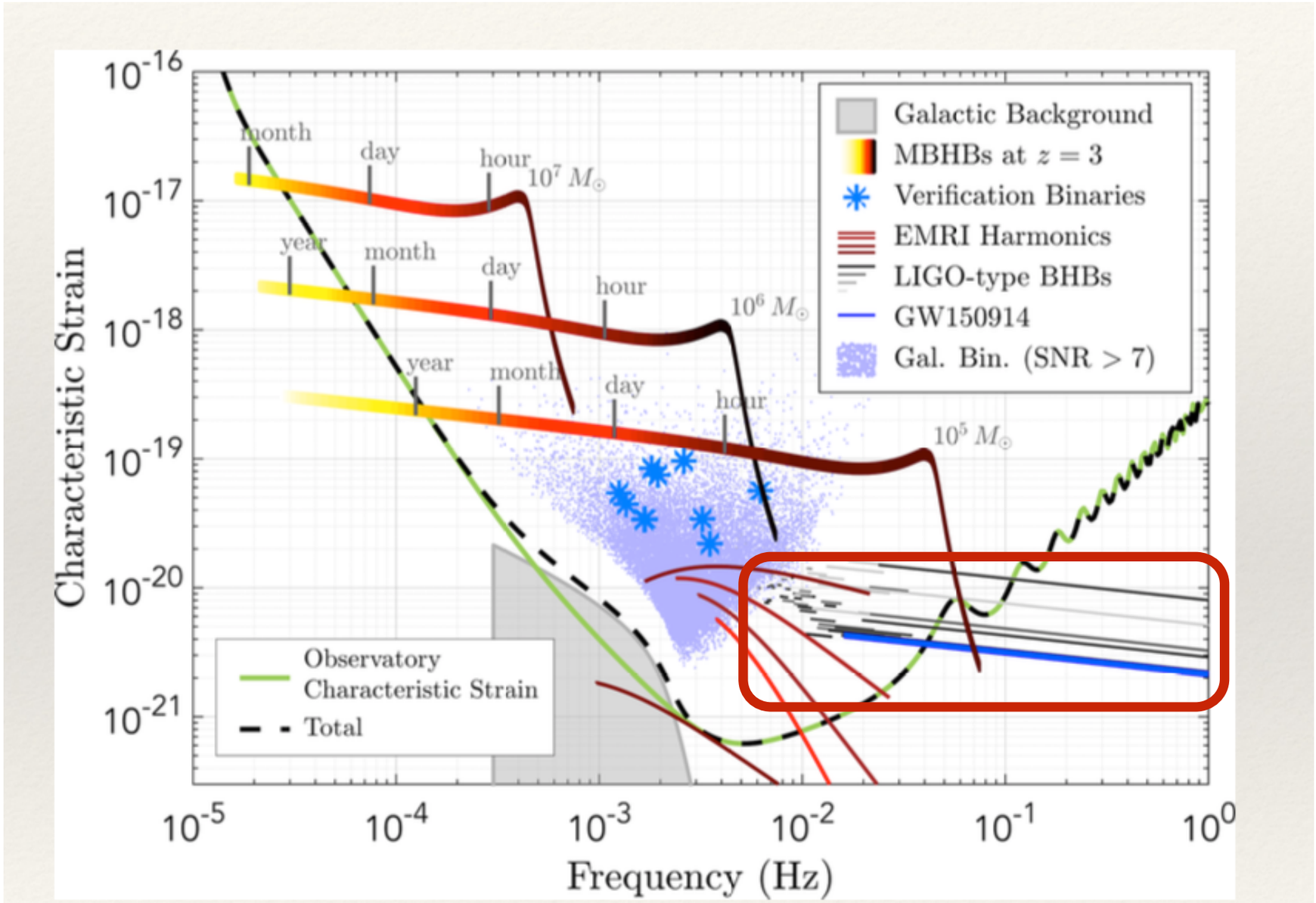
Expected sources from LISA

Weak and hard to detect sources: (i) extreme mass ratio inspirals

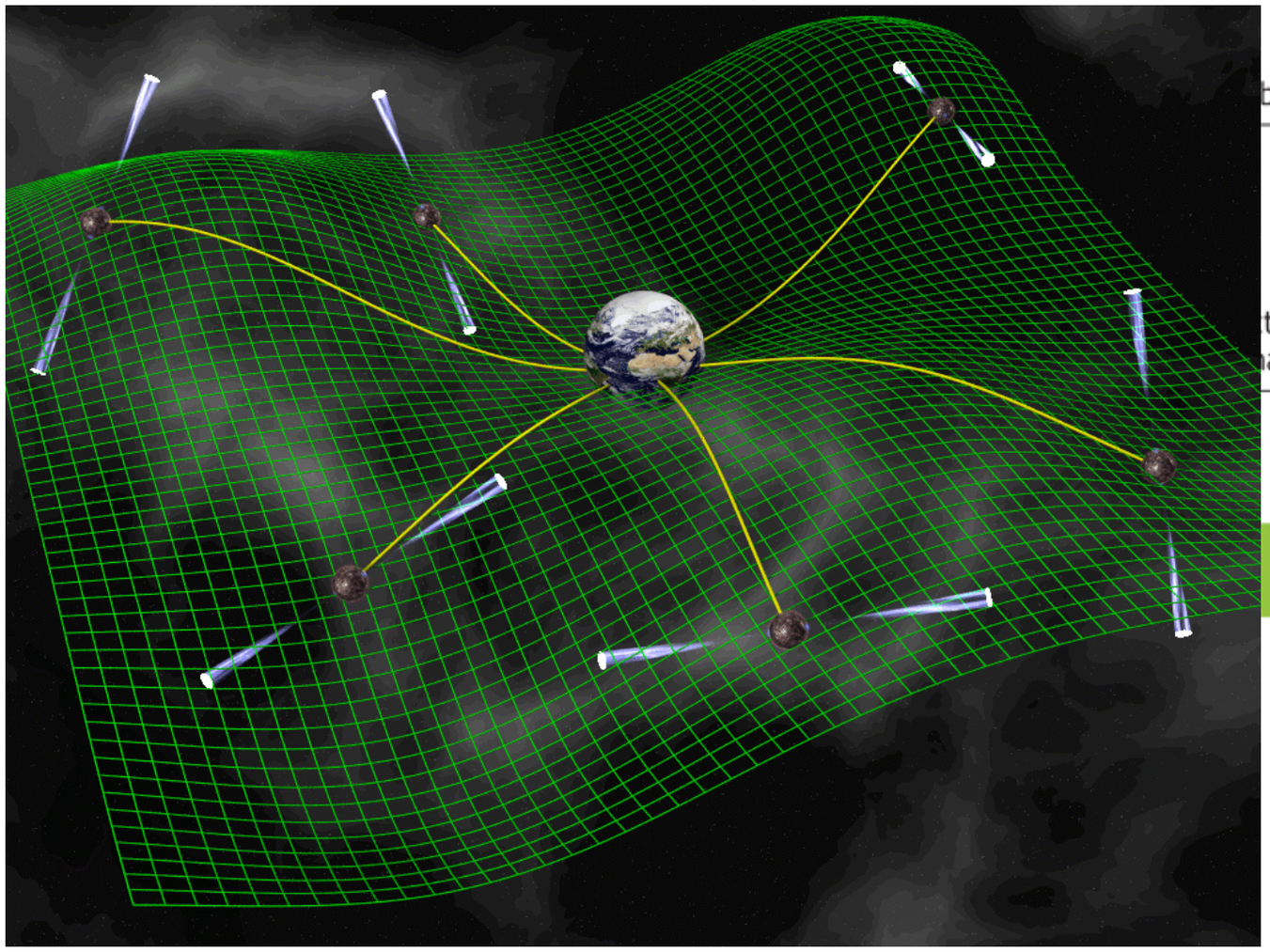


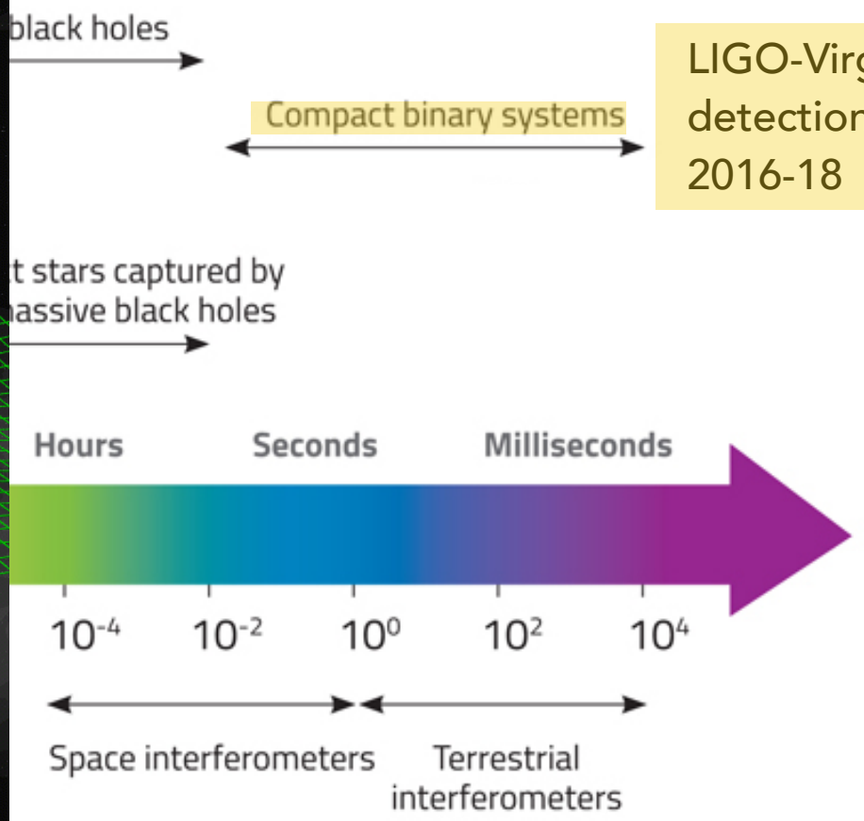
Expected sources from LISA

Weak and hard to detect sources: (ii) stellar mass binaries (iii) stochastic GW signal



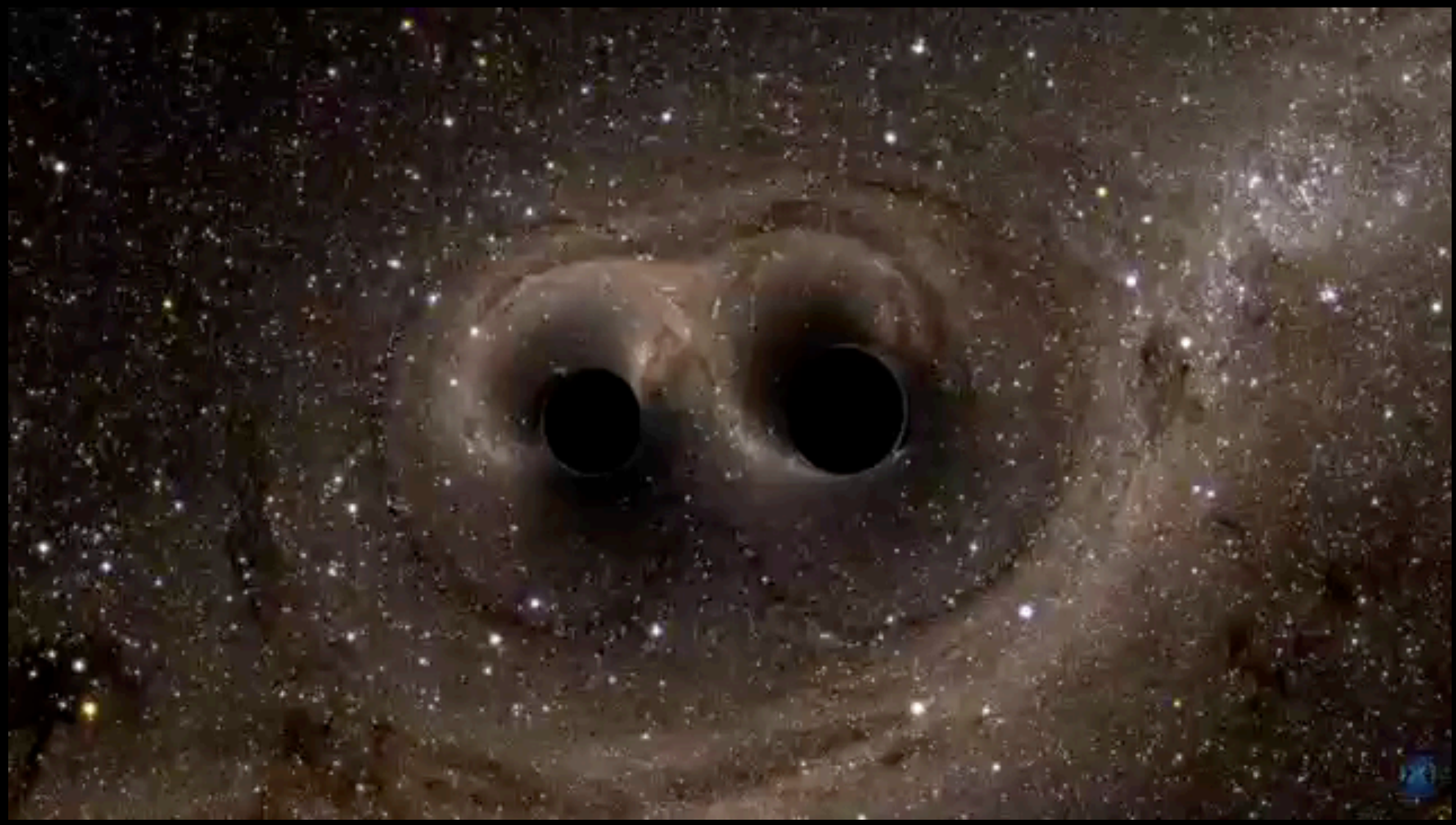
nano Hz band





Pulsar timing arrays: we monitor the period of an array of pulsars.
Perturbations in the period is indication that spacetime Earth-pulsar has been deformed

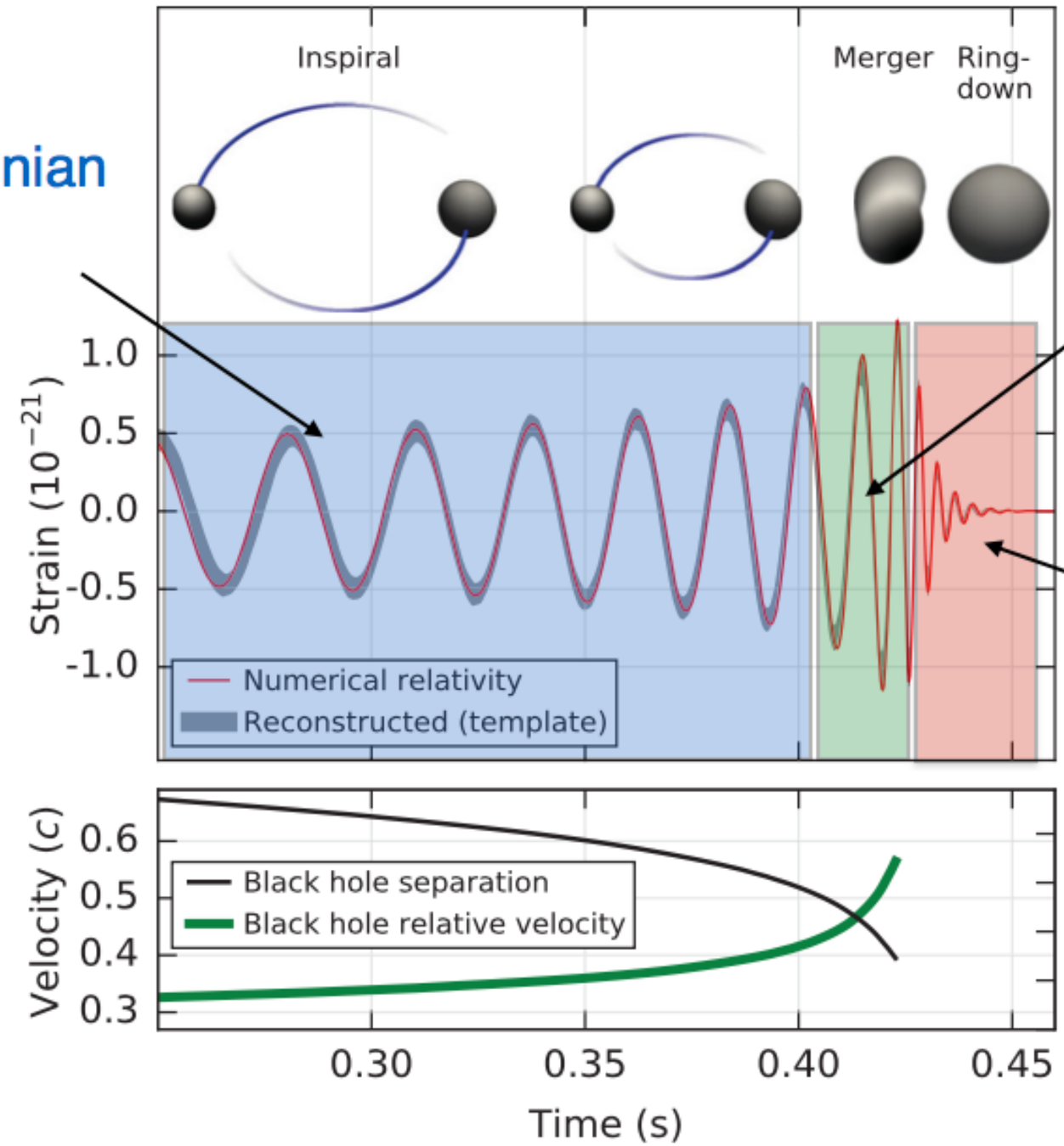
Mergers of compact objects



Title: BBH gravitational lensing of gw150914
Author: Simulating eXtreme Spacetimes Lensing (SXS)

GW model for mergers of compact objects

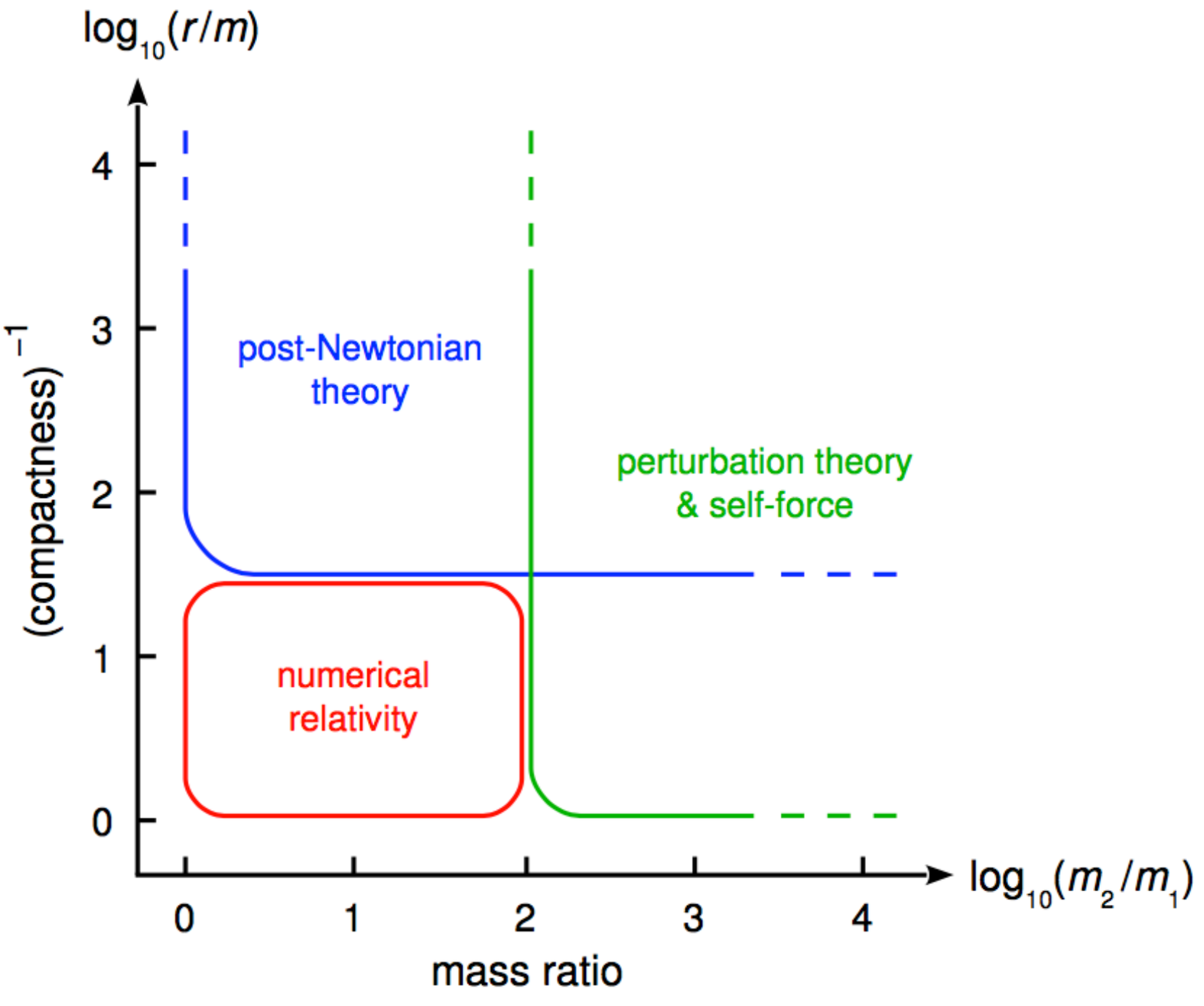
Post-Newtonian theory



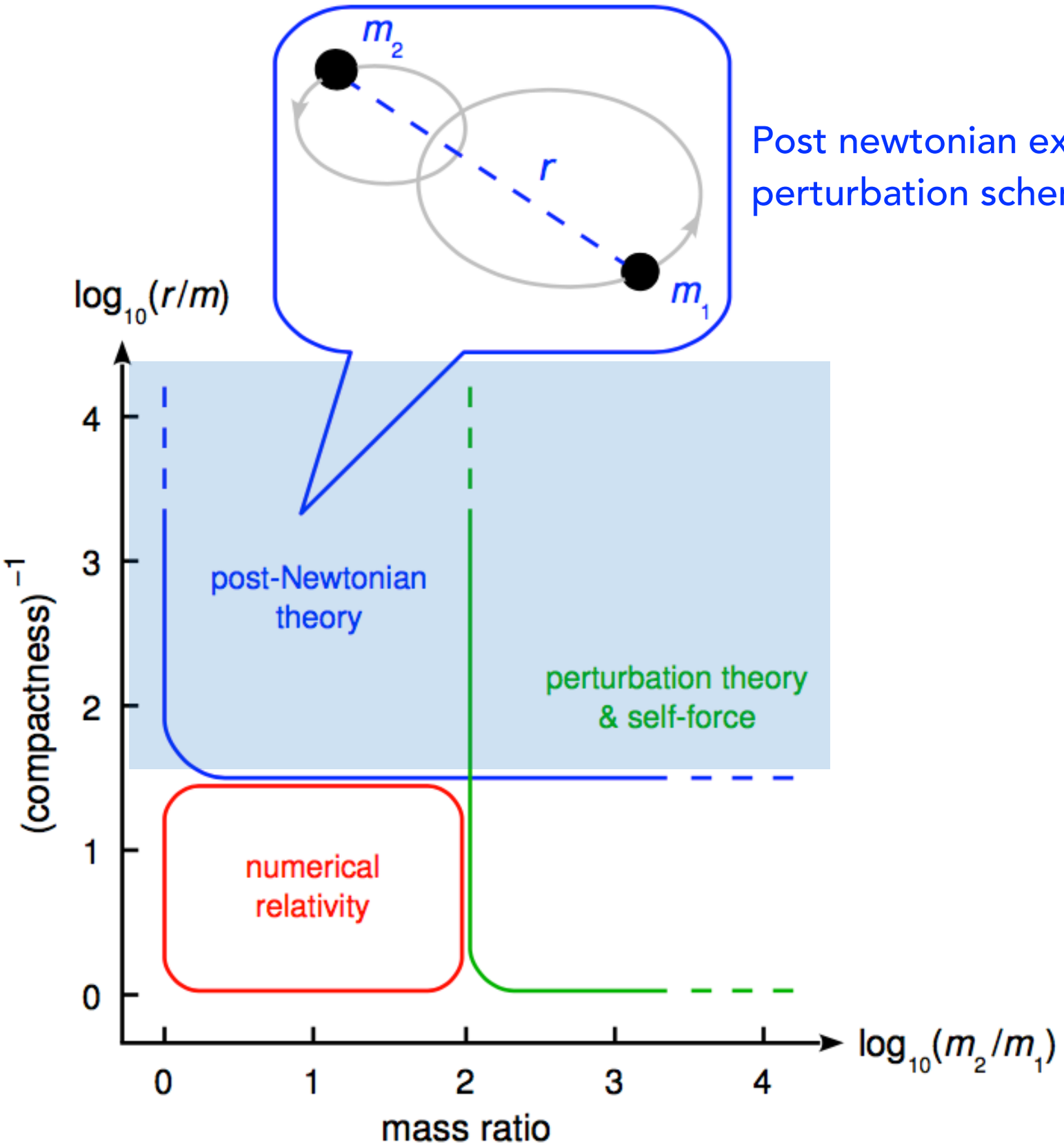
Numerical relativity

BH perturbations
QNM

PRL 116, 061102 (2016)



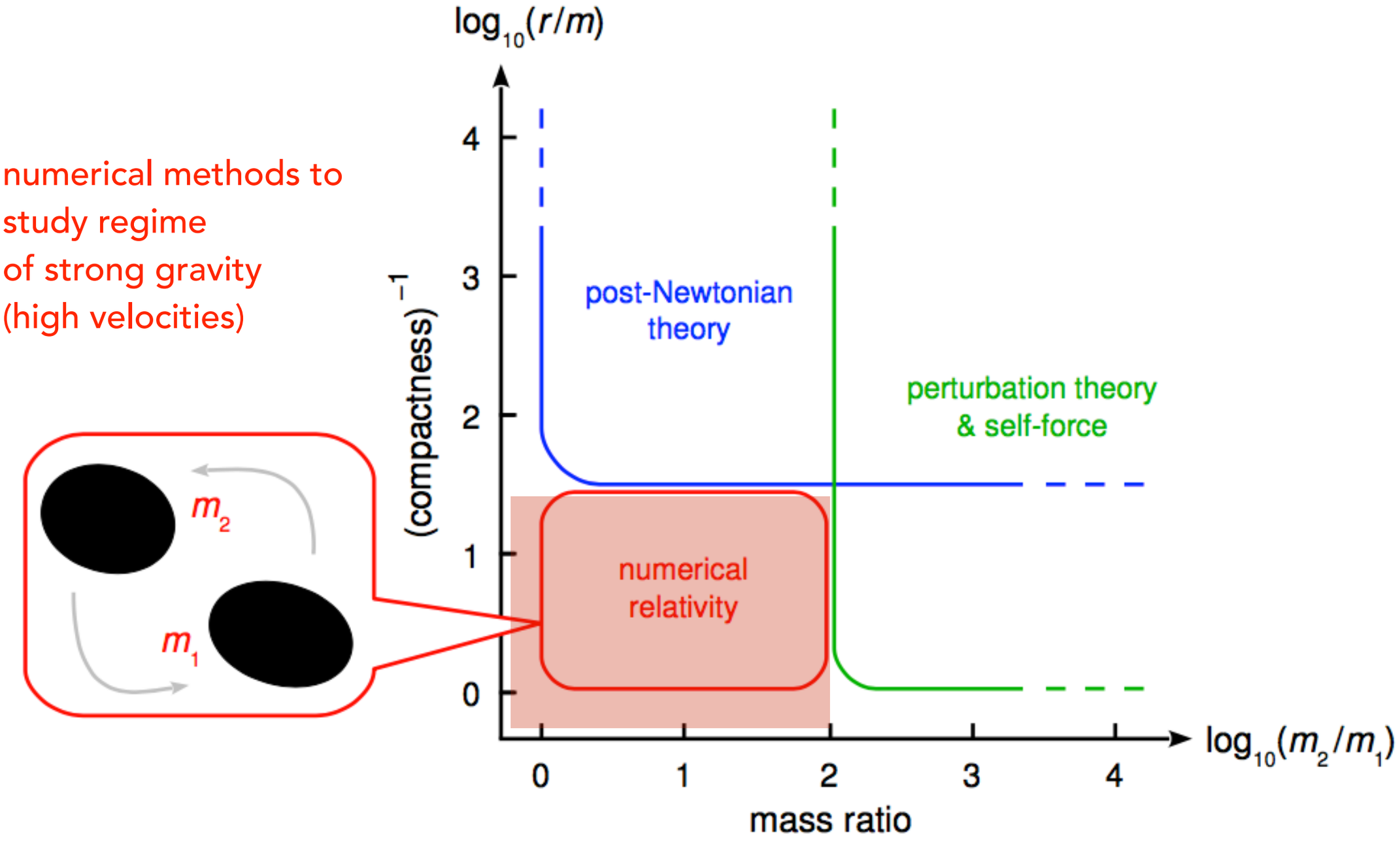
credit: A. Le Tiec



Post newtonian expansion
perturbation scheme in (v/c)

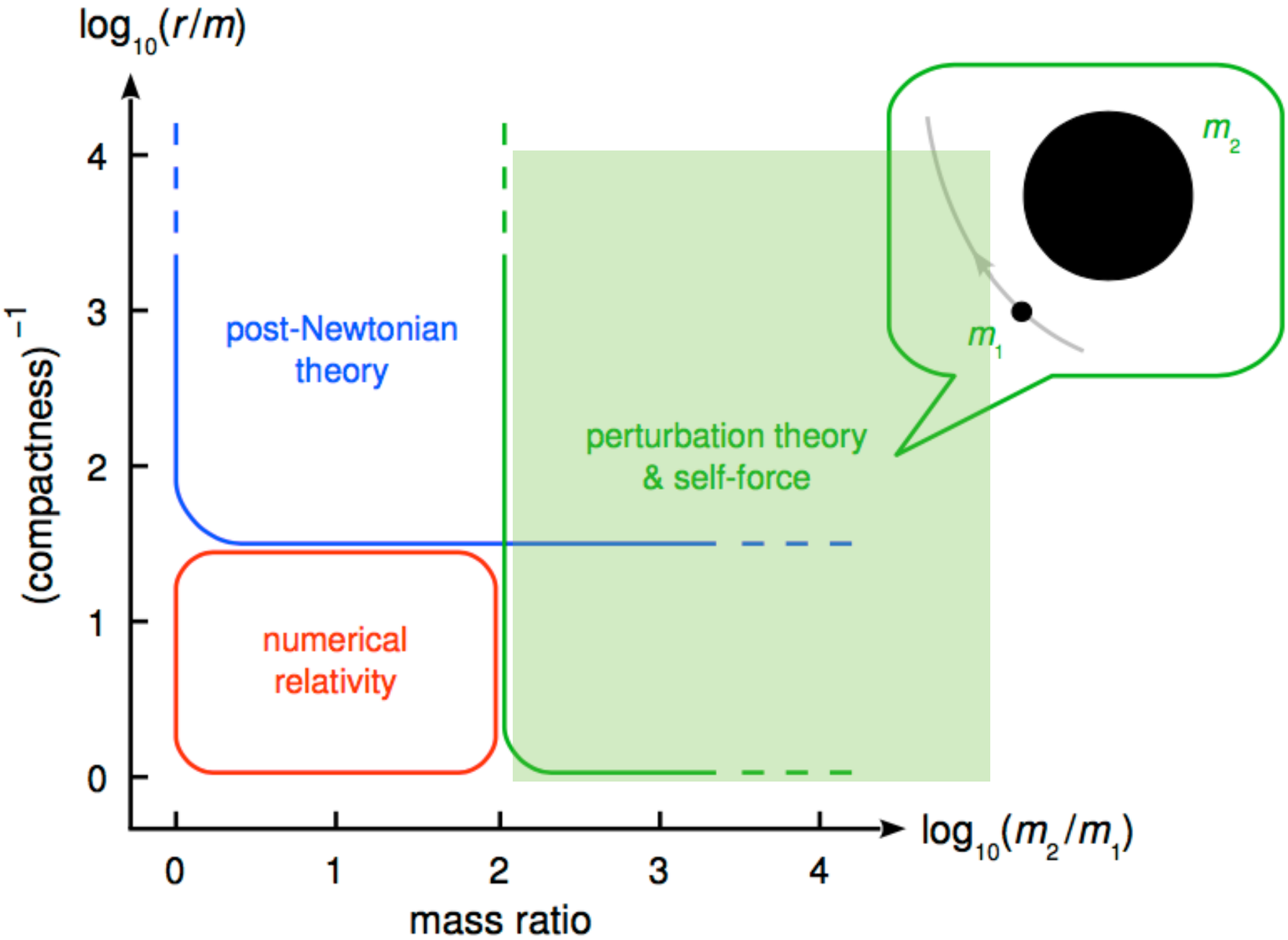
credit: A. Le Tiec

numerical methods to
study regime
of strong gravity
(high velocities)



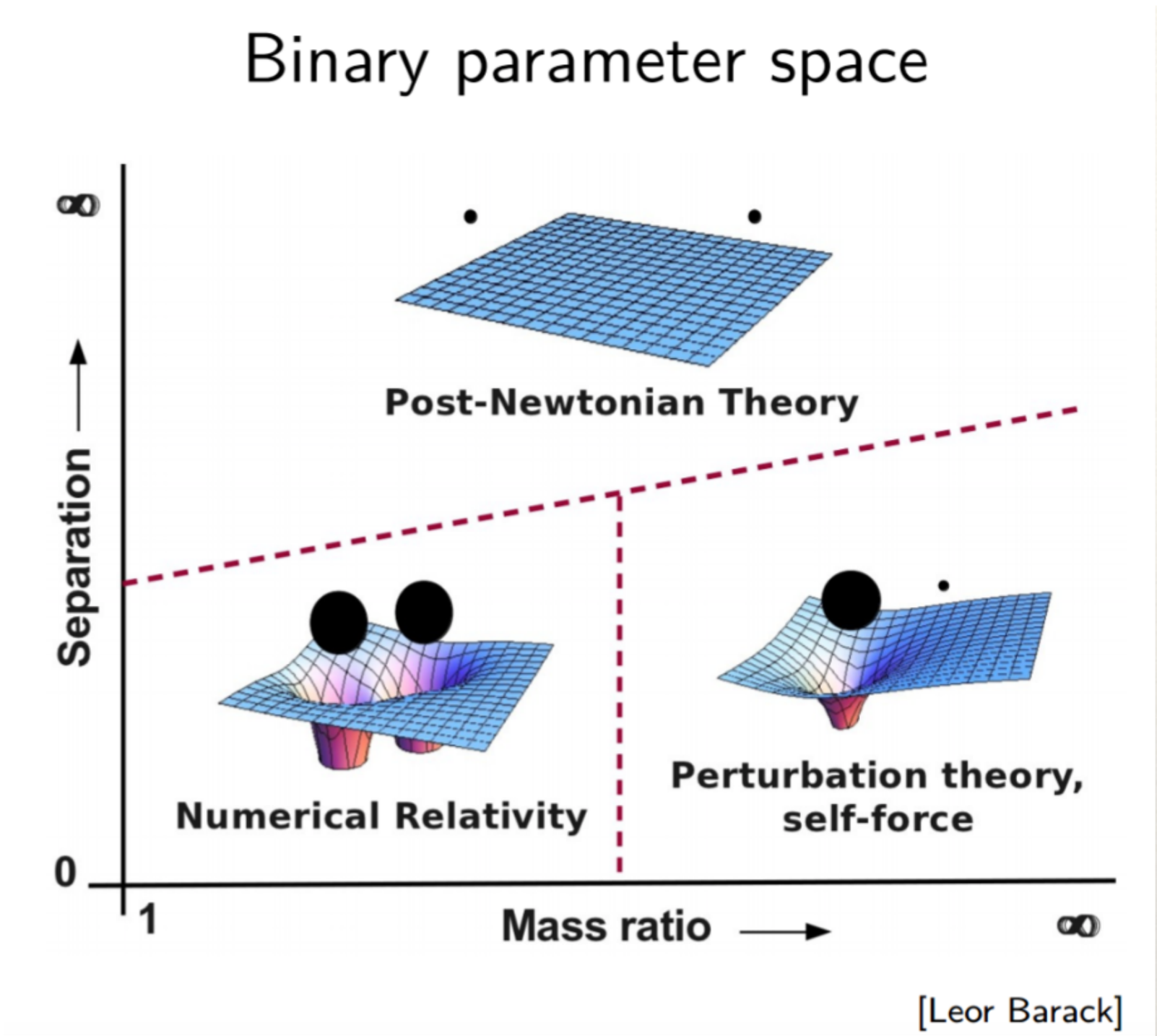
credit: A. Le Tiec

Expansion in inverse mass ratio. To zeroth order, the path of the lighter object is a geodesic in Kerr spacetime. Corrections due to the finite mass of the lighter object are included as an effective force on the object **(self-force)**

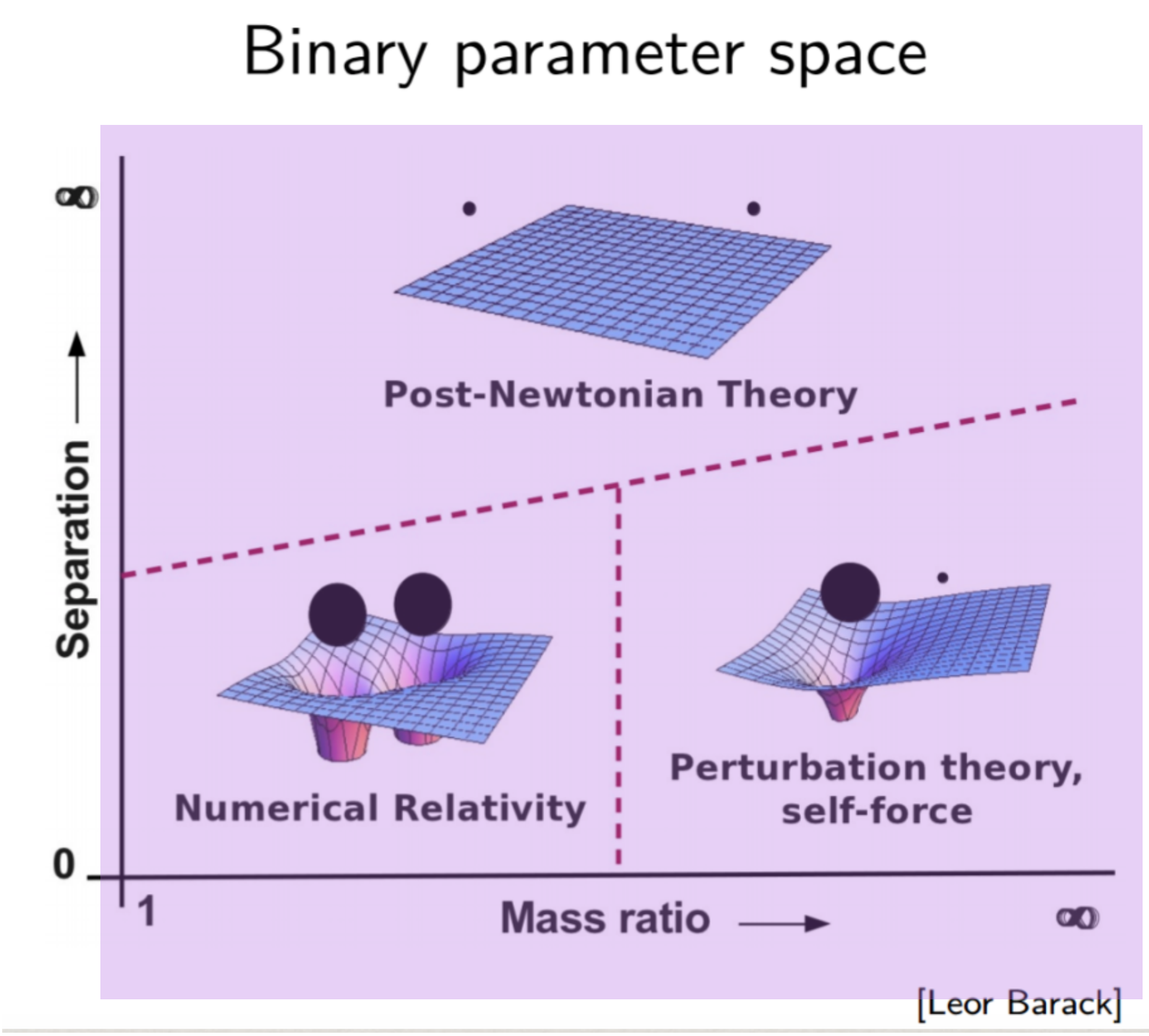


credit: A. Le Tiec

GW model(s)


Binary parameter space
Post-Newtonian Theory
Numerical Relativity
Perturbation theory,
self-force
Separation
0
1
Mass ratio
[Leor Barack]

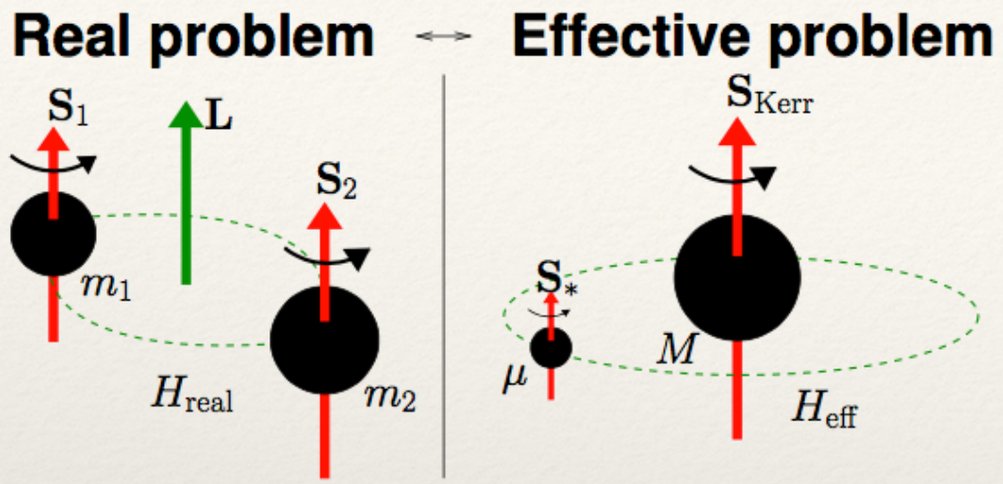
GW model(s)



effective one body and phenomenological models describe entire parameter space

GW model(s)

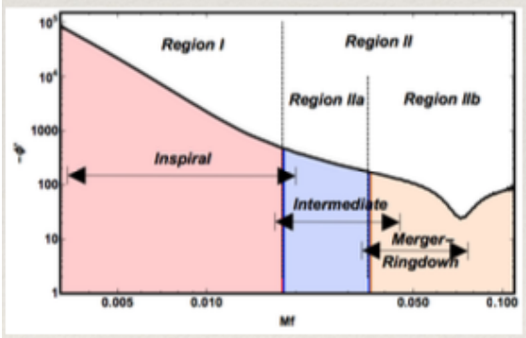
Approaches that potentially describe the entire dynamics

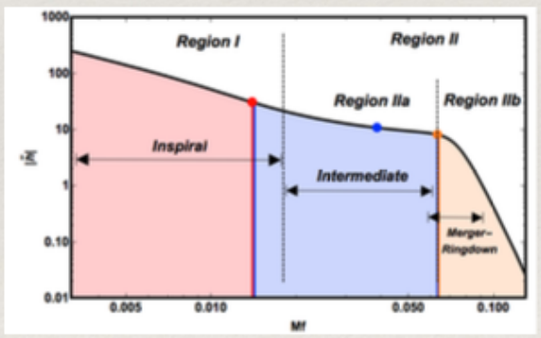


EOB

[Buonanno & Damour 2000]

Effective-One-Body approach:
mapping real two body dynamics onto
motion of a test mass in the s/t of a deformed BH





IMRPhenom

Phenomenological model:
uses PN at low frequency + analytic fit for merger
Generated in frequency, very fast

[Khan et al. 2018-2019; García-Quíros et al. 2020, Pratten et al. 2020]

End of overview

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Solution of GW equation in vacuum

blackboard

Mass quadrupole radiation

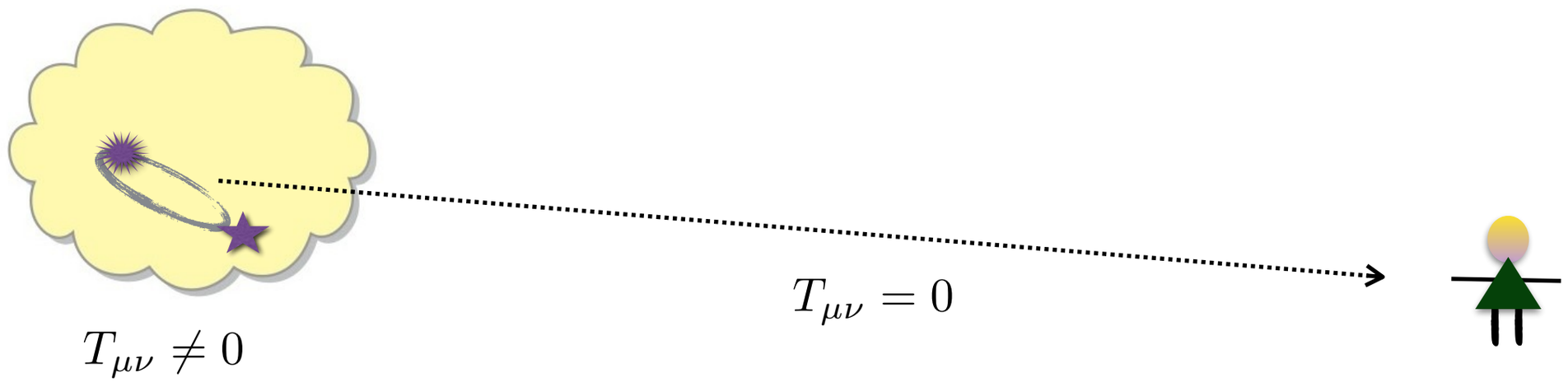
Weak-field sources with arbitrary velocities

If the gravitational field generated is sufficiently weak: expansion around flat space-time. In Lorentz gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

$h_{\mu\nu} - h/2\eta_{\mu\nu}$

energy momentum tensor of matter



General solution of this equation

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \int d^4x \, G(x - x') T_{\mu\nu}(x') \qquad \Box_x G(x - y) = \delta^4(x - y)$$

General solution of this equation

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \int d^4x\, G(x - x') T_{\mu\nu}(x') \qquad \Box_x G(x - y) = \delta^4(x - y)$$

The solution depends on boundary conditions. Just like in EM case, the appropriate solution for a radiation problem is the retarded one (we only integrate along past light cone observer)

$$G(x - x') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \delta(ct_{\text{ret}} - ct') \qquad t_{\text{ret}} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}$$

We can plug this in the equation above and fix the gauge to TT gauge with usual projector

$$h_{ij}^{TT} = \Lambda_{ijpq} h_{pq} = \Lambda_{ijpq} \bar{h}_{pq}$$

Weak-field sources with arbitrary velocities

Fixing completely the gauge freedom, the solution can be written as

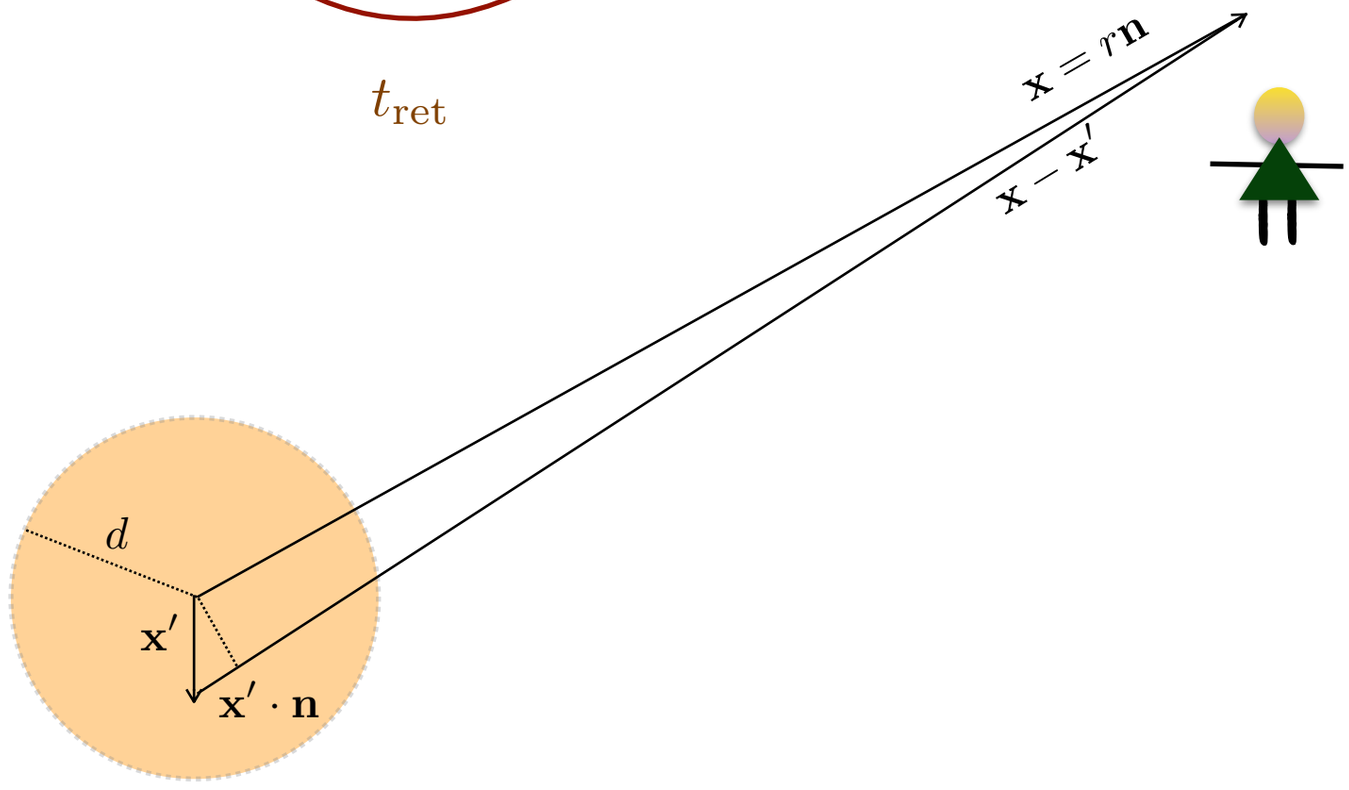
$$h_{ij}^{TT}(t,\mathbf{x}) = \frac{4G}{c^4}\Lambda_{ij,kl}(\mathbf{n})\int d^3x' \, \frac{1}{|\mathbf{x}-\mathbf{x}'|} T_{kl}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\right)$$

projector to TT gauge

$t_{\rm ret}$

At large distance from the source

$$|\mathbf{x}-\mathbf{x}'| = r - \mathbf{x}'\cdot\mathbf{n} + \mathcal{O}\left(\frac{d^2}{r}\right)$$



Low-velocity expansion

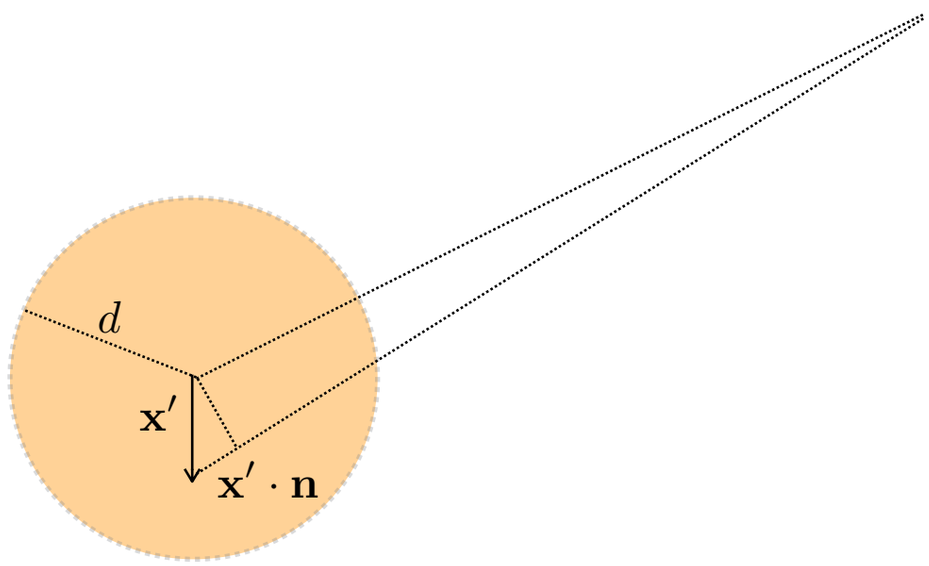
$$T_{kl}\left(t-\frac{r}{c}+\frac{\mathbf{x}'\cdot\mathbf{n}}{c},\mathbf{x}'\right)=\int\frac{d^4k}{(2\pi)^4}\,\tilde{T}_{kl}(\omega,\mathbf{k})e^{-i\omega(t-r/c+\mathbf{x}'\cdot\mathbf{n}/c)+i\mathbf{k}\cdot\mathbf{x}'}$$

Fourier transform

$\omega\sim\omega_s\sim v/d$

frequency emitted

frequency of motion



Low-velocity expansion

$$T_{kl}\left(t-\frac{r}{c}+\frac{\mathbf{x}'\cdot\mathbf{n}}{c},\mathbf{x}'\right)=\int\frac{d^4k}{(2\pi)^4}\,\tilde{T}_{kl}(\omega,\mathbf{k})e^{-i\omega(t-r/c+\mathbf{x}'\cdot\mathbf{n}/c)+i\mathbf{k}\cdot\mathbf{x}'}$$

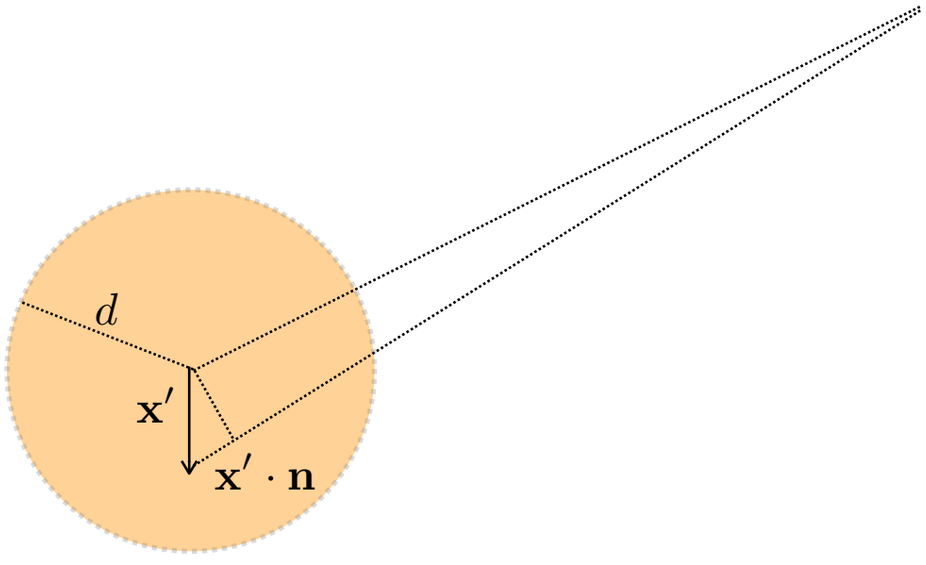
$\omega_s d \ll c$ for non-relativistic sources

$$\frac{\omega}{c}\mathbf{x}'\cdot\mathbf{n}\leq\frac{\omega_s d}{c}\ll 1$$

$$e^{-i\omega(t-r/c)}\times\left[1-i\frac{\omega}{c}x'^i n^i+\frac{1}{2}\left(-i\frac{\omega}{c}\right)^2 x'^i x'^j n^i n^j+\ldots\right]$$

$\omega\sim\omega_s\sim v/d$

frequency emitted — frequency of motion



Mass quadrupole radiation

Plugging this expansion in the expression for the GW field, at lowest order

$$h_{ij}^{TT}(t,\mathbf{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\mathbf{n})\left[S^{kl} + \frac{1}{c}n_m \dot{S}^{kl,m} + \dots\right]_{\rm ret}$$

$$\begin{cases} S^{ij}(t) = \int d^3x\, T^{ij}(t,\mathbf{x}) \\ S^{ij,k}(t) = \int d^3x\, T^{ij}(t,\mathbf{x}) x^k \end{cases}$$

using conservation of energy momentum tensor one can eliminate momenta of T^{ij} in favor of momenta of energy density T^{00} *(check it!)*

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$$\begin{cases} M = \dfrac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x}) \\ M^i = \dfrac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x}) x^i \\ M^{ij} = \dfrac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x}) x^i x^j \end{cases}$$

using conservation of energy momentum tensor one can eliminate momenta of T^{ij} in favor of momenta of energy density T^{00} *(check it!)*

$$\left[h_{ij}^{TT}(t,\mathbf{x})\right]_{\rm quad} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\mathbf{n})\ddot{M}^{kl}(t-r/c)$$

second moment of energy density distribution

Why a quadrupole?

There is no monopole M and no dipole P^i radiation for GWs. Why?

group theory argument

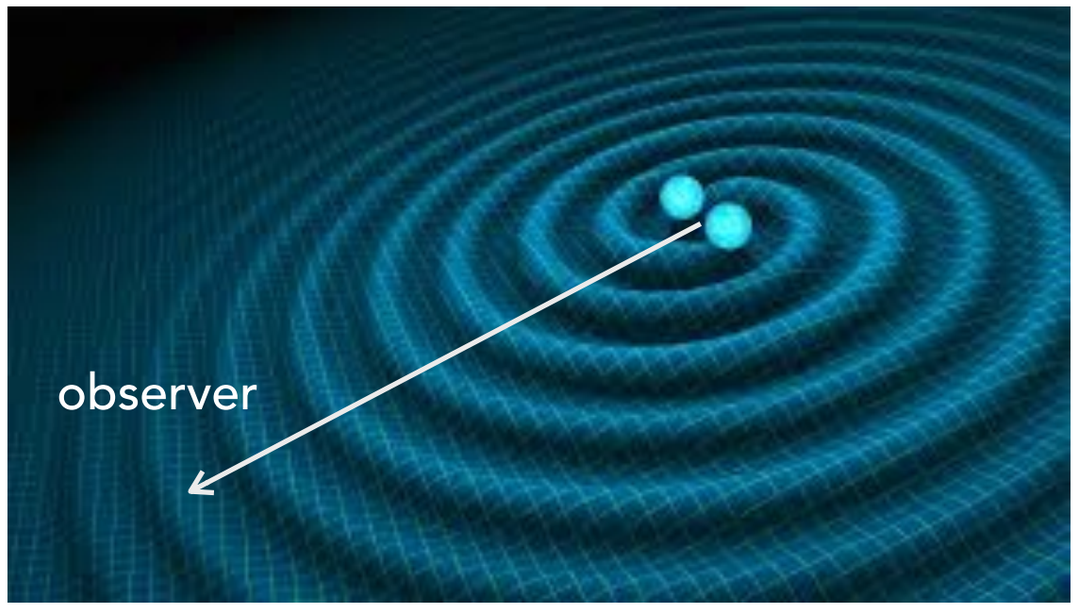
It is impossible to put a graviton in a state with total angular momentum $j = 0, 1$ (graviton is massless particle with helicities ± 2). Monopole and dipole radiation would correspond to a total angular momentum of 0 and 1 respectively

(see analogy for photons, no monopole)

linear theory

h_{ij}^{TT} depends on time derivatives of the multipole moments (static source does not radiate and and M and P^i are conserved quantities)

Weak-field and velocity expansions



- Generation of gravity waves in the context of the linear theory: weak field

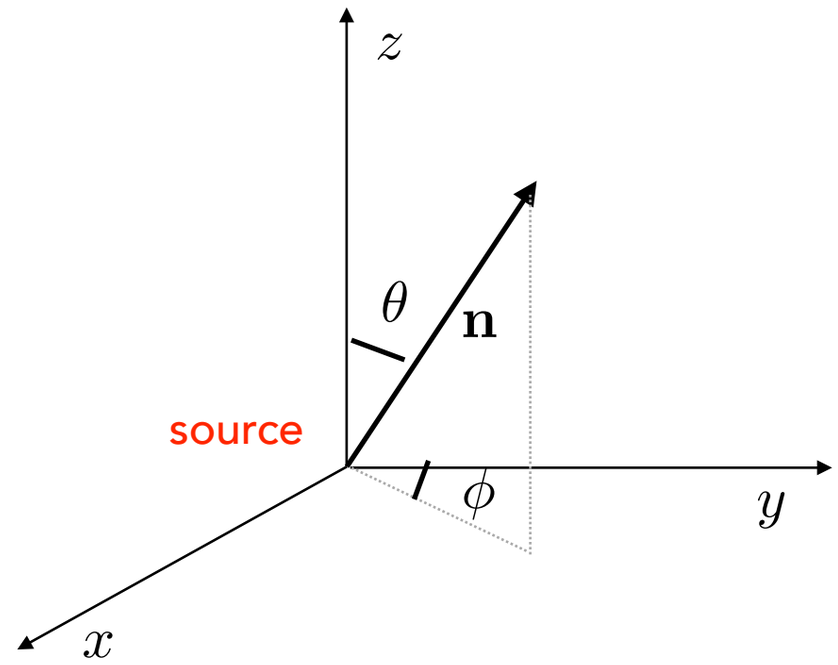
- For self gravitating systems being $E_{\rm kin} = -(1/2)U \quad \rightarrow \quad v^2/c^2 = R_s/(2r)$

weak field $\leftrightarrow$ low velocities

Mass quadrupole radiation: polarizations

$$h_{ij}^{TT}(t,\mathbf{x}) = \sum_{A=+,\times} h_A(t,\mathbf{x}) e_{ij}^A(\mathbf{n})$$

Exercise



Writing explicit expression for TT projector in spherical coordinates, one obtains

$$h_+(t,\theta,\phi) = \frac{1}{r}\frac{G}{c^4}\left[\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2\theta) + \ddot{M}_{22}\ldots\right]$$

$$h_\times(t,\theta,\phi) = \frac{1}{r}\frac{G}{c^4}\left[(\ddot{M}_{11} - \ddot{M}_{22})\sin 2\phi\cos\theta + 2\ddot{M}_{12}\cos 2\phi\cos\theta -\right]$$

(see M. Maggiore's book, chapter 3.3, Vol. 1)

Compact binary systems

Mass quadrupole for binary system

Binary system with masses m_1 and m_2. We consider the reference frame of center of mass. The relative coordinate is performing circular motion in (x, y) plane

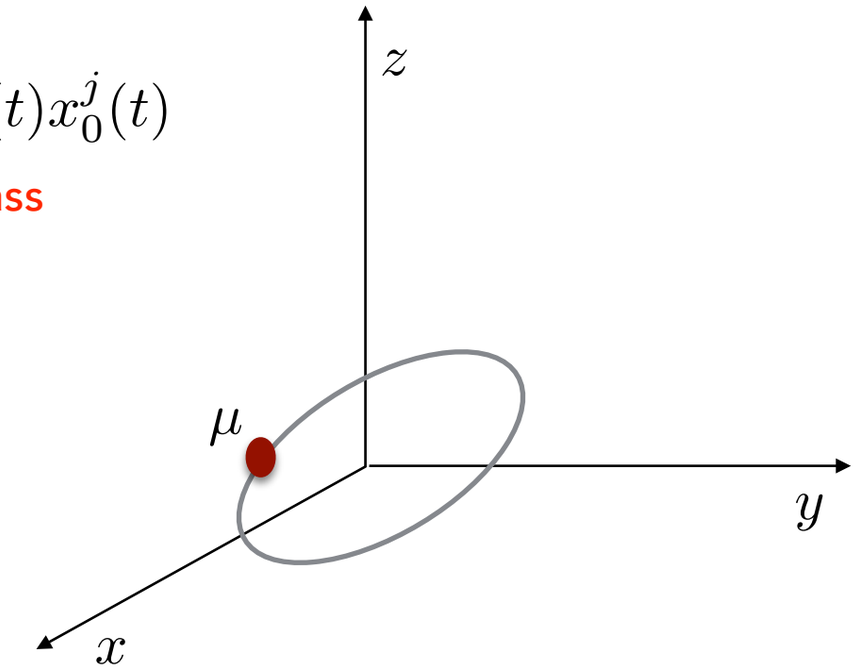
$$x_0(t) = R\cos\left(\omega_S t + \frac{\pi}{2}\right)$$

$$y_0(t) = R\sin\left(\omega_S t + \frac{\pi}{2}\right)$$

$$z_0(t) = 0$$

$$M^{ij} = \mu x_0^i(t) x_0^j(t)$$

reduced mass



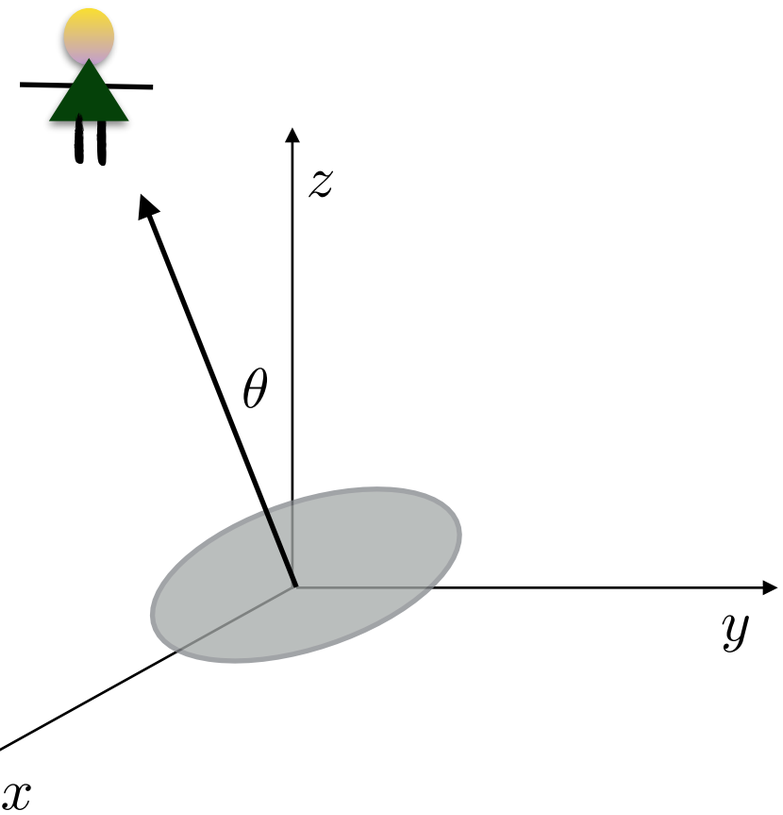
$$\mu = m_1 m_2/(m_1 + m_2)$$

we use this quadrupole moment in the expression for the two polarisations with angular dependence

Mass quadrupole for binary system

$$h_\times(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos\theta\sin(2\omega_s t_{\rm ret} + 2\phi)$$

$$h_+(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega_s t_{\rm ret} + 2\phi)$$



the system is not invariant under rotations around the z axis

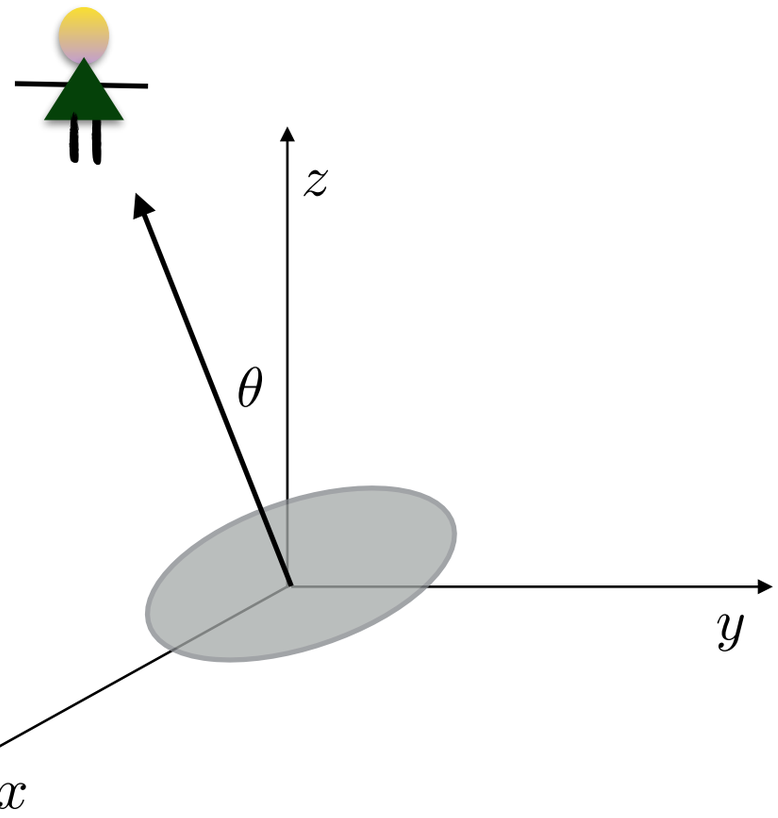
since the orbit is circular, a rotation of the source around z is equivalent to a time translation $\Delta\phi = \omega_s \Delta t$ ⟶ dependence on ϕ can be reabsorbed in time redefinition

Mass quadrupole for binary system

radiation observed depends on the angle
between line of sight and normal to the orbit

$$h_\times(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos\theta\sin(2\omega_s t_{\rm ret} + 2\phi)$$

$$h_+(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega_s t_{\rm ret} + 2\phi)$$



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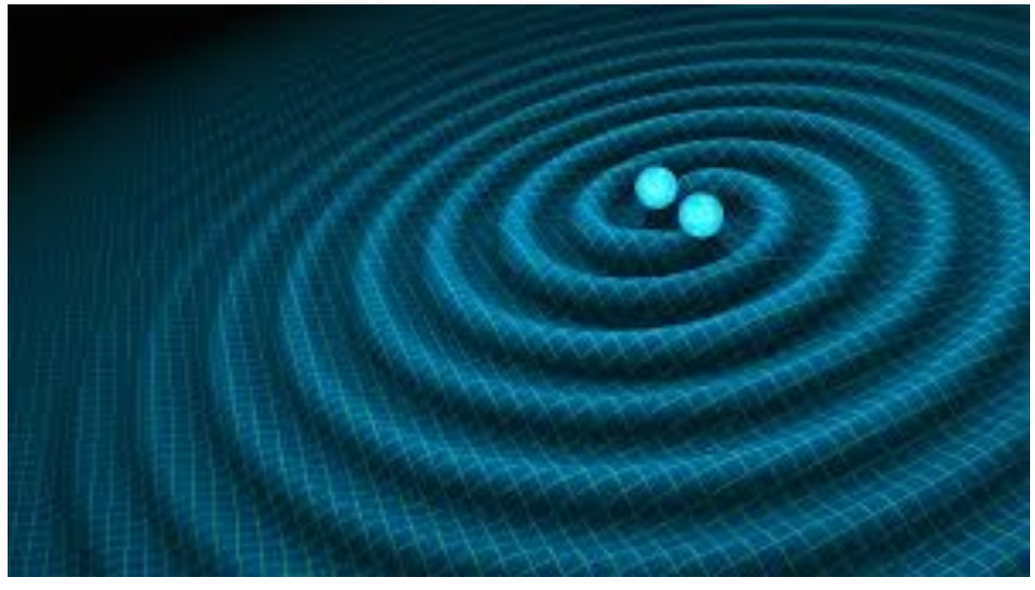
Quasi-circular orbit



Emission of GWs coasts energy

$$E_{\rm orbit} = E_{\rm kin} + E_{\rm pot} = -\frac{Gm_1m_2}{2R}$$

to compensate the loss of energy by GW emission, the radius R has to decrease

For circular orbits

$$\omega_s^2 = \frac{G(m_1+m_2)}{R^3}$$

if the radius decreases, the frequency increases

Quasi-circular orbit

We can find the evolution of frequency and radius by writing

$$[P]_{\rm quadr} = -\frac{dE_{\rm orbit}}{dt}$$

We find (…)

$$\dot{f}_{\rm gw} = \frac{96}{5}\pi^{8/3}\left(\frac{GM_c}{c^3}\right)^{5/3} f_{\rm gw}^{11/3}$$

$$f_{\rm gw} = \omega_{\rm gw}/(2\pi)$$

$$M_c = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

chirp mass

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$$M_c = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

$f_{\rm gw}$ formally diverges at a finite time that we call $t_{\rm coal}$

$\tau = t - t_{\rm coal}$ time to coalescence

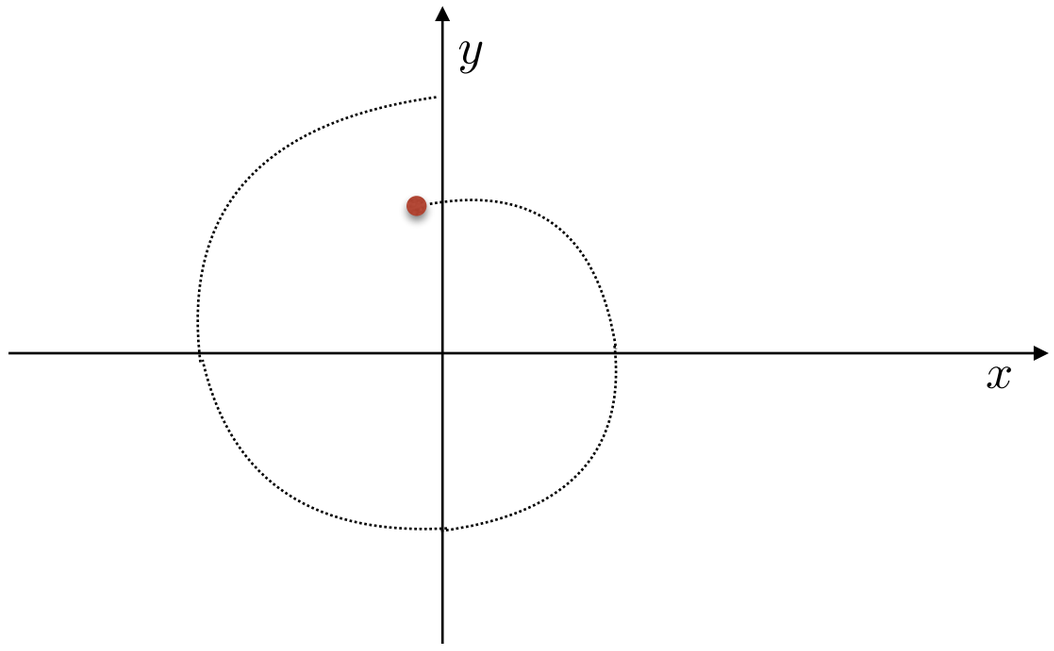
Changes in the waveform

We examine how the corresponding waveform of the GW changes. Let us go back to motion of relative coordinate

$$x(t) = R(t)\cos(\Phi(t)/2)$$

$$y(t) = R(t)\sin(\Phi(t)/2)$$

$$\Phi(t) = \int_{t_0}^{t} dt' \omega_{\rm gw}(t')$$



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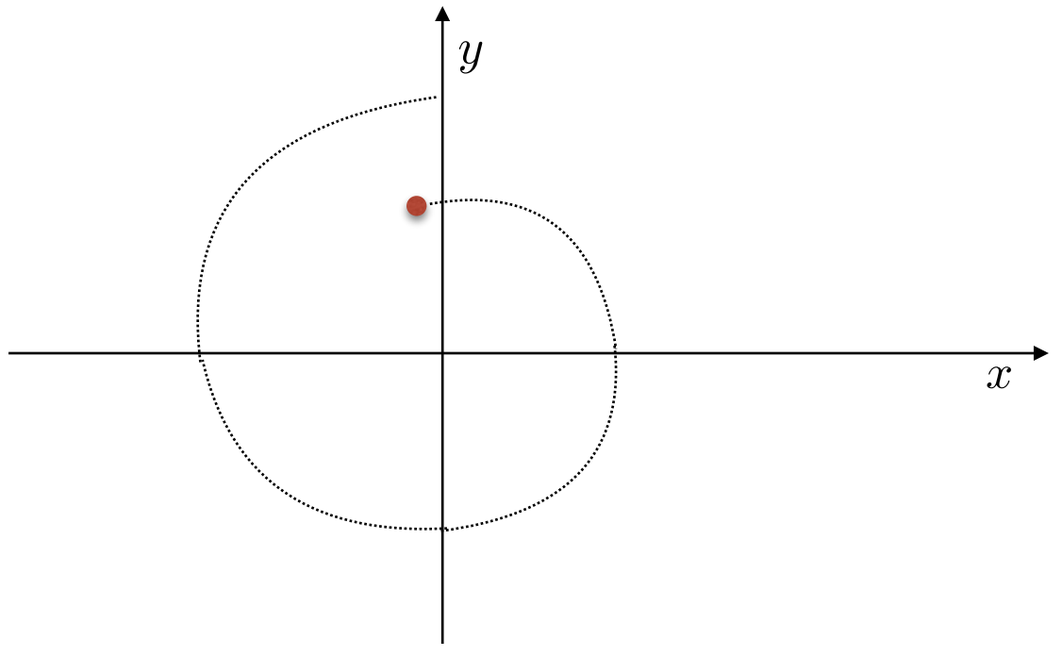

When computing the waveform (for quasi-circular motion) → contribution of time derivatives is negligible

- in the argument of the trigonometric functions $\omega_{\rm gw} t \to \Phi(t)$
- in the amplitude $\omega_{\rm gw} \to \omega_{\rm gw}(t)$

Final results for quasi-circular motion

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t)}{c}\right)^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos[\Phi(t)]$$

$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}(t)}{c}\right)^{2/3}\cos\theta\sin[\Phi(t)]$$

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$$\Phi(t) = 2\pi\int_{t_0}^{t} dt' f_{\rm gw}(t')$$

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$$\dot{f}_{\rm gw} = \frac{96}{5}\pi^{8/3}\left(\frac{GM_c}{c^3}\right)^{5/3} f_{\rm gw}^{11/3} \quad \rightarrow \quad f_{\rm gw}(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}$$

$$\Phi(t) = 2\pi\int_{t_0}^{t} dt' f_{\rm gw}(t') \quad \longrightarrow \quad \Phi(\tau) = -2\left(\frac{5GM_c}{c^3}\right)^{-5/8}\tau^{5/8} + \Phi_0$$

Final results for quasi-circular motion

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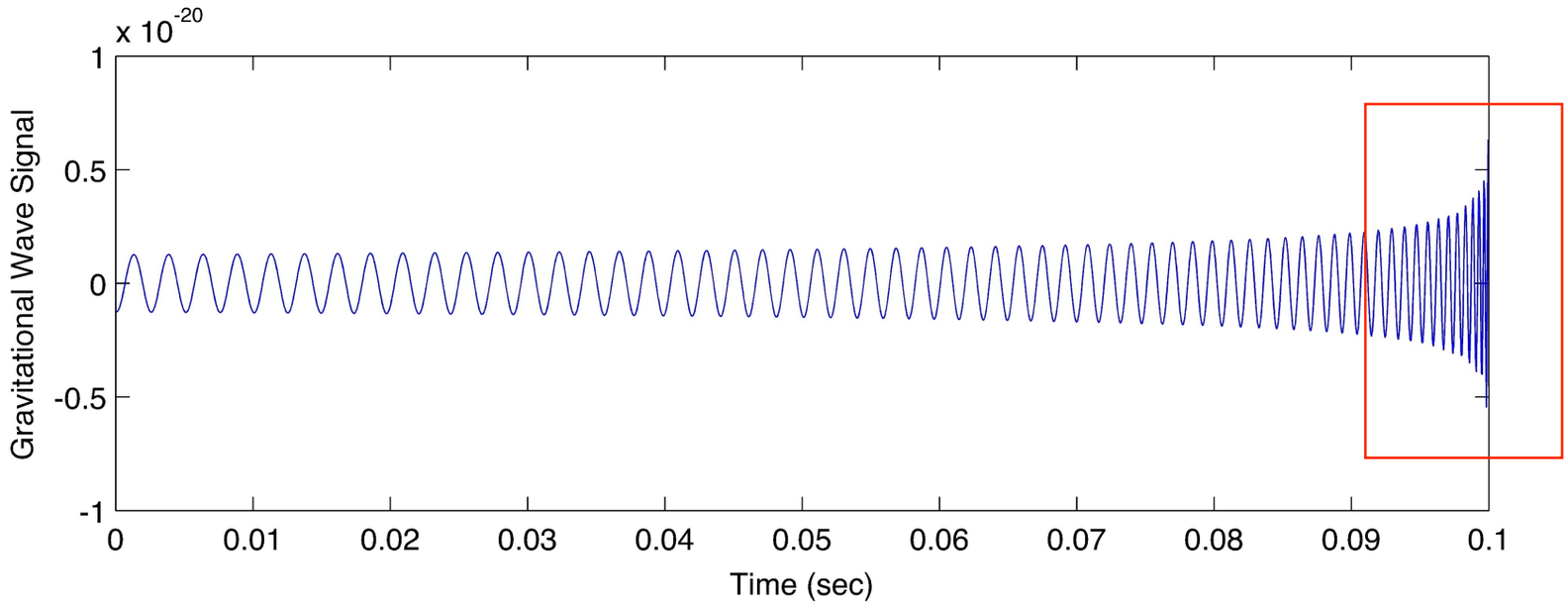
Both the frequency and the amplitude increase as the coalescence is approached



chirping

Binary systems as standard sirens

Gravitational standard sirens

In (not so good) analogy with type Ia Supernovae (SNIa)

Supernovae: standard candles

they emit the same energy when exploding: we know their luminosity

by measuring the flux, we infer $d_L = \sqrt{L/(4\pi F)}$

we have a separate measurement of redshift

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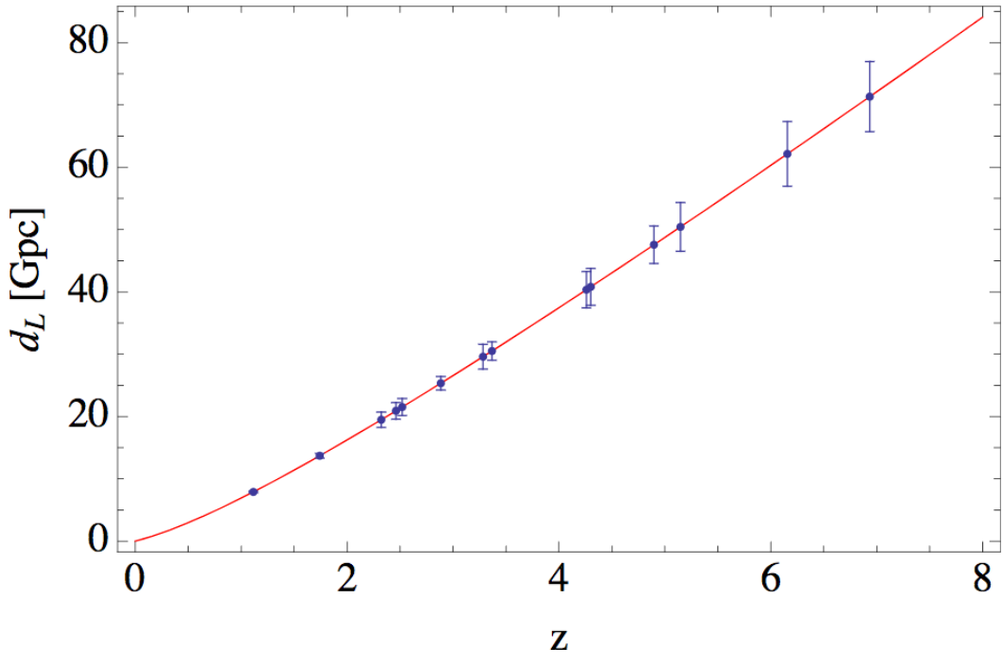
Hubble diagram



Fitting points in this plot we test luminosity distance-redshift relation: we set constraints on cosmological models and cosmological parameters

$$d_L(z|\Omega_m, H_0, \Omega_\Lambda, ...)$$

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Binary systems: standard candles?

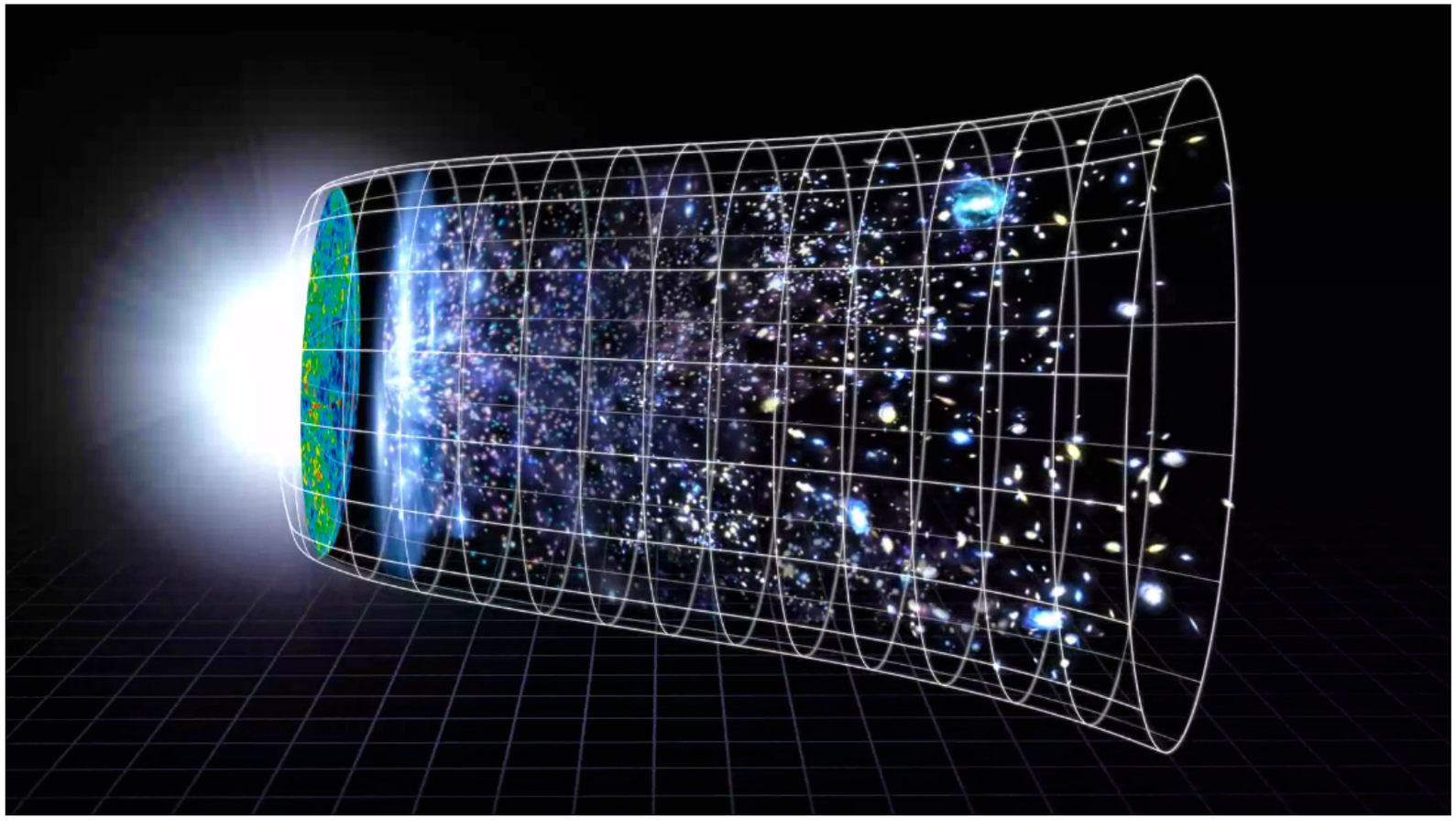
Question: can we use binary system of compact objects to test the luminosity distance-redshift relation?

We need to write the waveform at the observer and test whether we can have independent reconstruction of **redshift** and **luminosity distance**

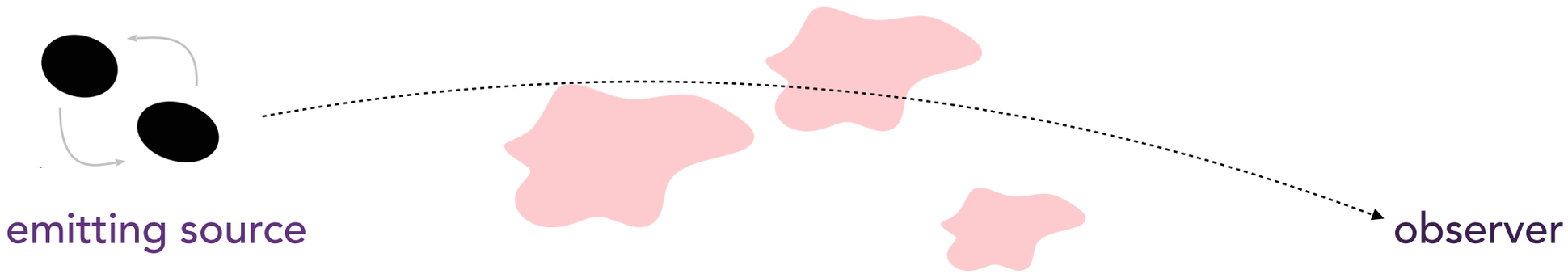
Waveform in a cosmological context

Up to now we considered: flat universe with no expansion and no perturbations

Which is the effect of the expansion of the Universe and of cosmological perturbations on the waveform?



Waveform modeling in a cosmological context



(1) accurate description of the **emitted signal**

+

(2) description of the evolution of the signal during **propagation**

Only recently studied

= **observed signal**

Different effects, e.g.

1) effects due to peculiar velocities

2) evolution of redshift during the observation time

3) lensing by massive structures

4) wave effects beyond ray optics

Waveform in a cosmological setting

Preliminary: waveform close to the source

We consider a FLRW universe with **scalar perturbations**

$$ds^2 = -(1+2\psi)dt^2 + a^2(1-2\phi)\delta_{ij}dx^i dx^j$$

The solution found expanding around a flat background is used to describe the GW **close to the source**

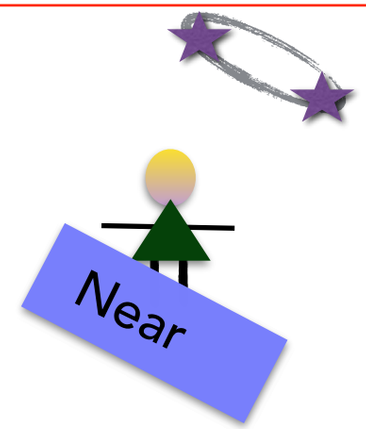
$r \to r_{\rm phys} = a_s r$

Preliminary: waveform close to the source

Close to the source we have: ($r \to r_{\rm phys} = a_s r$)

$$h_+(\tau_s) = \frac{4}{a_s r}(GM_c)^{5/3}(\pi f_s(\tau_s))^{2/3}\frac{1+\cos^2\theta}{2}\cos(\phi_s(\tau_s))$$

$$h_\times(\tau_s) = \frac{4}{a_s r}(GM_c)^{5/3}(\pi f_s(\tau_s))^{2/3}\cos\theta\sin(\phi_s(\tau_s))$$



how can we write this waveform at the observer: blackboard presentation

From the source to the observer

I want to rewrite the solution in terms of quantities defined at the observer (far from the source)

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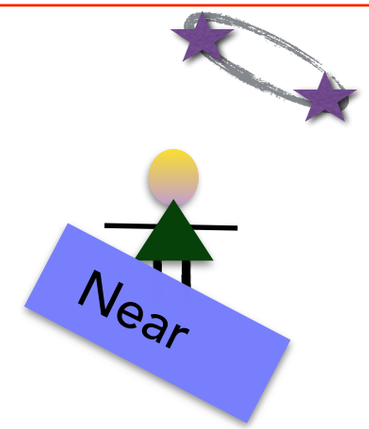

$$f_s = (1+z_s)f_o$$

$$d_p = a_s r = d_L/(1+z_s)$$

$$\phi_s(\tau_s) = \phi_o(\tau_o(\tau_s))$$

because the phase is constant along null geodesics

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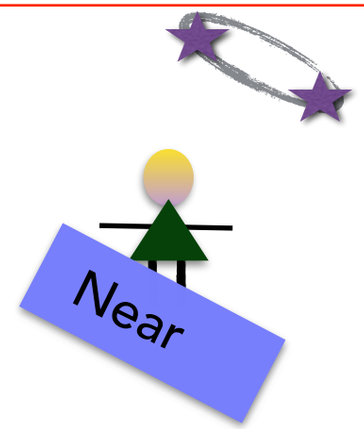

$M_c(1+z)$

$$h_+(\tau_o) = \frac{4}{d_L}(G\mathcal{M}_c(z))^{5/3}(\pi f_o(\tau_o))^{2/3}\frac{1+\cos^2\theta}{2}\cos(\phi_o(\tau_o))$$

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Which parameters can be extracted

$M_c(1+z)$

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observed waveform

We see that:

- from ratio between two polarisation: extract **inclination angle**
- what can be extracted looking at the **evolution of frequency with time**?

$$\frac{df_o}{dt_o} = C((1+z_s)M_c)f_o^{11/3}$$

redshift degenerate with chirp mass

depends on redshifted chirp mass

- from what is left we extract the **luminosity distance**

Effect of redshift evolution on frequency and phase

Waveform and cosmology: our goal

Goal: study waveform at observer taking into account redshift evolution: is there way to extract redshift information?

We study evolution of frequency with time. We proceed in two steps

no info on redshift can be extracted

- We neglect the evolution of redshift during the time of observation

- We take into account the evolution of redshift
  - background contribution
  - contribution of perturbations

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these two contributions to redshift evolution compete….

redshift degenerate with acceleration source!

Case 1: redshift evolution neglected

I want to find the explicit expression of $f_o(\tau_o)$ and $\phi_o(\tau_o)$

We know

$$\frac{df_s}{dt_s} = C(M_c) f_s^{11/3}$$

$$dt_o = (1+z_s) dt_s$$

$$f_s = (1+z_s) f_o$$

$$(1+z_s)\frac{d((1+z_s)f_o)}{dt_o} = C(M_c) f_o(t_o)^{11/3} (1+z_s)^{11/3}$$

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$$(1+z_s)\frac{d((1+z_s)f_o)}{dt_o} = C(M_c) f_o(t_o)^{11/3} (1+z_s)^{11/3}$$

Neglecting the evolution of redshift over the time of observation

dependence on redshift only through $M_c(1+z)$

$$\frac{df_o}{dt_o} = C((1+z_s)M_c) f_o^{11/3}$$

redshift degenerate with chirp mass ⟶ redshift info cannot be extracted

Case 2: evolution of redshift taken into account

We relax the hypothesis that the redshift is constant during the time of observation

$$(1+z_s)\frac{d((1+z_s)f_o)}{dt_o} = C(M_c)f_o(t_o)^{11/3}(1+z_s)^{11/3}$$

evolves during the time of observation

$$[(1+z_s(t_o))f_o]^{-8/3} = \frac{8C(M_c)}{3}\int_{t_o}^{t_c} dt'_o[1+z_s(t'_o)]^{-1}$$

Case 2: evolution of redshift

We need to evaluate the following integral

$$[(1+z_s(t_o))f_o]^{-8/3} = \frac{8C(M_c)}{3}\int_{t_o}^{t_c} dt'_o[1+z_s(t'_o)]^{-1}$$

$$1+z_s = \frac{a_o}{a_s}\left[1+\mathbf{v}_s\cdot\mathbf{n}-\mathbf{v}_o\cdot\mathbf{n}+\psi_o-\psi_s-\int_{t_s}^{t_o} dt(\dot{\phi}+\dot{\psi})\right]$$

background contribution

perturbations

We have to consider the evolution of the background redshift and of perturbations.
The idea is to expand $(1+z_s(t'_o))$ around $t_o = t'_o + \delta t$

Exercise (see Bonvin et al. 2016)

Evolution of redshift: final results

By doing so, we find

$$f(\tau_o) = \frac{1}{\pi}\left(\frac{5}{256\tau_o}\right)^{3/8}(G\mathcal{M}_c(z_s))^{-5/8}\left[1+\frac{3}{8}Y(z_s)\tau_o\right]$$

$$Y(z_s) = X(z_s) + \delta X(z_s)$$

$$X(z_s) = \frac{1}{2}\left(H_0 - \frac{H_s}{1+\bar{z}_s}\right)$$

$$\delta X(z_s) = \frac{1}{2}\left(\frac{\dot{\mathbf{v}}_s\cdot\mathbf{n}}{1+\bar{z}_s} - \dot{\mathbf{v}}_o\cdot\mathbf{n} + \frac{\dot{\phi}_s}{1+\bar{z}_s} - \dot{\phi}_o\right)$$

background contribution

contribution of perturbations

Evolution of redshift: final results

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background contribution

contribution of perturbations

already taken into account in GW experiments

small

Evolution of redshift: final results

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background contribution

$$\delta X(z_s) = \frac{1}{2}\left(\frac{\dot{\mathbf{v}}_s\cdot\mathbf{n}}{1+\bar{z}_s} - \dot{\mathbf{v}}_o\cdot\mathbf{n} + \frac{\dot{\phi}_s}{1+\bar{z}_s} - \dot{\phi}_o\right)$$

contribution of perturbations

$$H_0\phi_s \sim 10^{-5}H_0$$

small

Some numerical estimates

$$\delta X(z_s) = \frac{1}{2}\left(\frac{\dot{\mathbf{v}}_s \cdot \mathbf{n}}{1+\bar{z}_s} - \dot{\mathbf{v}}_o \cdot \mathbf{n} + \frac{\dot{\phi}_s}{1+\bar{z}_s} - \dot{\phi}_o\right)$$

This leading order term can be split into the sum of three contributions

-1- velocity cluster with respect CMB frame

-2- velocity galaxy inside the cluster

-3- velocity source inside galaxy

modulus velocity

$$\frac{\delta X}{H_0}(z_s) = \frac{\alpha}{2}\frac{v_s^2}{r}\frac{\mathbf{e}\cdot\mathbf{n}}{H_0(1+\bar{z})}$$

$$\dot{v}_s \sim v_s^2/r$$

angle between acceleration and line of sight

distance from the center galaxy/ cluster

Some numerical estimates

$$\frac{Y(z_s)}{H_0} = \frac{X(z_s)}{H_0} + \frac{\delta X(z_s)}{H_0}$$

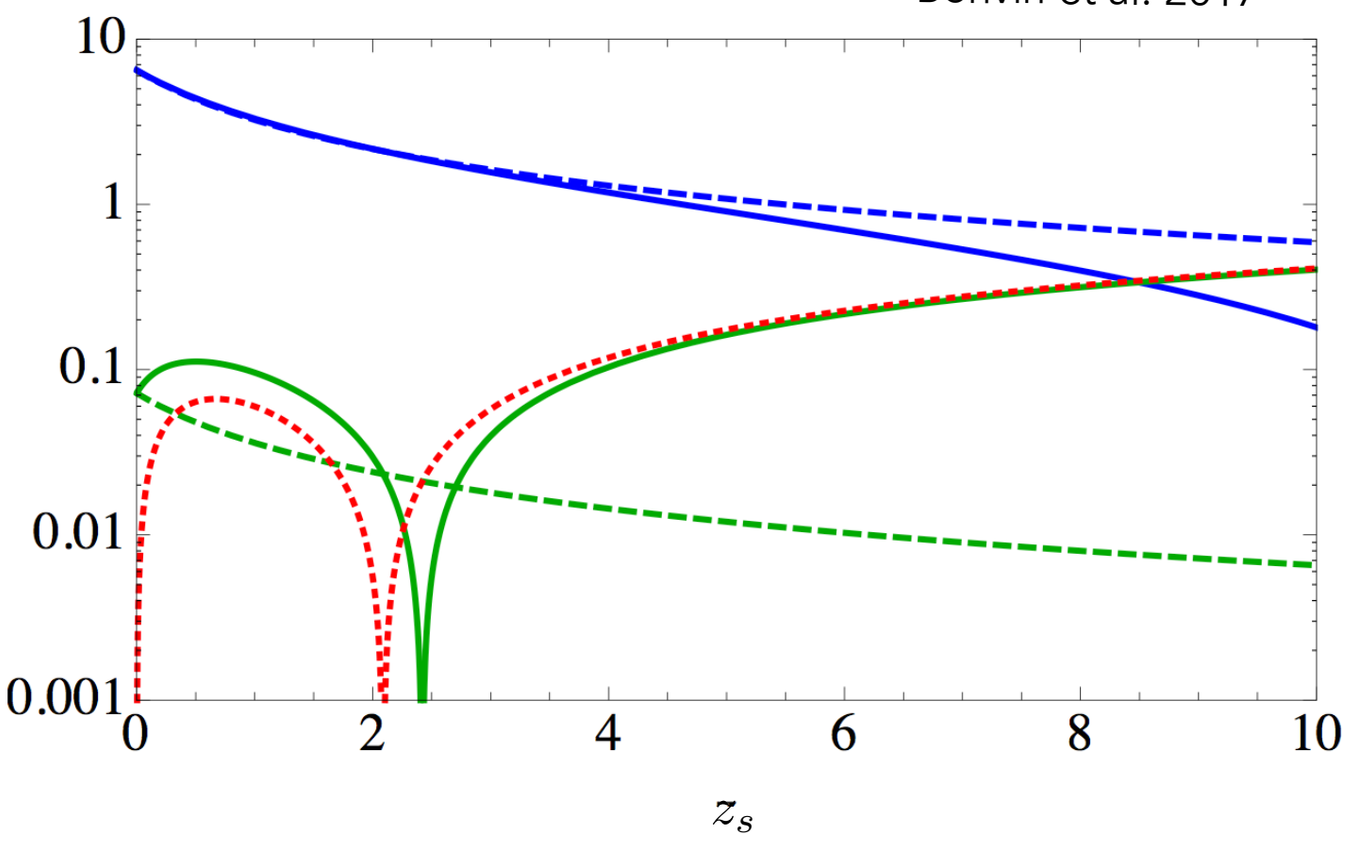
$$\frac{\delta X}{H_0}(z_s) = \frac{\alpha}{2} \boxed{\frac{v_s^2}{r} \frac{\mathbf{e}\cdot\mathbf{n}}{H_0} \frac{1}{(1+\bar{z})}} \epsilon$$

Some numerical estimates

$$\frac{Y(z_s)}{H_0} = \frac{X(z_s)}{H_0} + \frac{\delta X(z_s)}{H_0}$$

$$\frac{\delta X}{H_0}(z_s) = \frac{\alpha}{2}\boxed{\frac{v_s^2}{r}\frac{\mathbf{e}\cdot\mathbf{n}}{H_0}\frac{1}{(1+\bar{z})}}\ \epsilon$$

We compare the two dotted lines (two different value of velocity)



Bonvin et al. 2017

background contribution

- $|\frac{X(z)}{H_0}|$
- $|\frac{Y(z)}{H_0}|$ (ϵ=3)
- $|\frac{\delta X(z)}{H_0}|$ (ϵ=3)
- $|\frac{Y(z)}{H_0}|$ (ϵ=270)
- $|\frac{\delta X(z)}{H_0}|$ (ϵ=270)

→ binary at the edge of the cluster $r = 100\mathrm{kpc}$ with average velocity $v_2 = 1000\mathrm{km/s}$

→ binary with very high peculiar velocity $v_s = 3000\mathrm{km/s}$ very close to the center of the cluster $r = 10\mathrm{kpc}$

Can we extract an info on redshift?

Summarizing: because of redshift evolution over the observation time, we found

$$f(\tau_o) = \frac{1}{\pi}\left(\frac{5}{256\tau_o}\right)^{3/8}(G\mathcal{M}_c(z_s))^{-5/8}\left[1+\frac{3}{8}Y(z_s)\tau_o\right]$$

$$Y(z_s) = X(z_s) + \delta X(z_s)$$

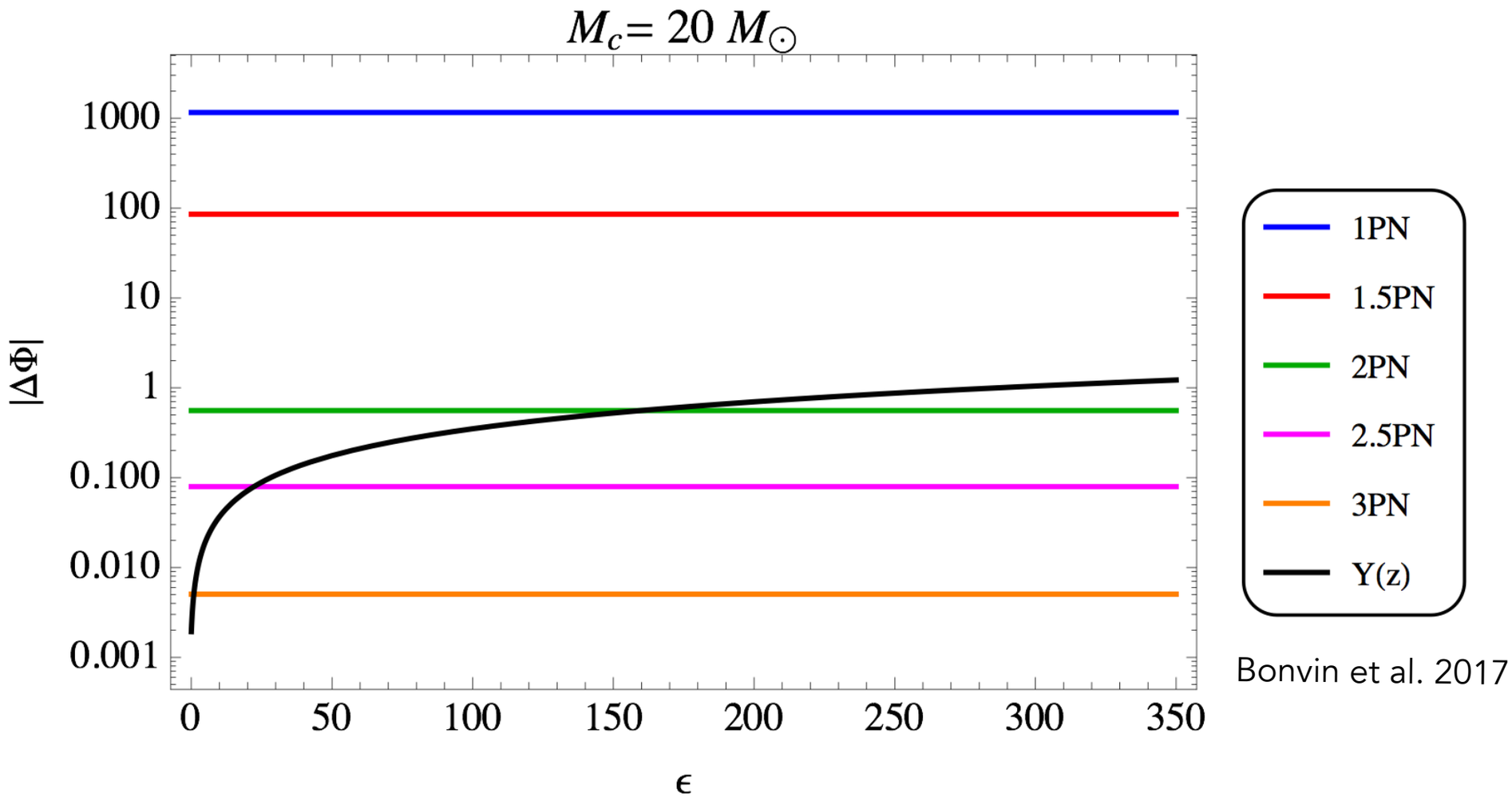
depends on redshift and cosmological parameters

depends also on the **unknown acceleration** of the binary along the line of sight

We need separate redshift determination

We expect the contribution from the peculiar acceleration to seriously degrade the possibility of using binaries as standard candles

Corrections to the phase from redshift evolution



Bonvin et al. 2017

Equal mass binaries with $\mathcal{M}_c = 20 M_\bullet$ entering the LISA detector at $f_{\rm min} = 0.004$ and staying in band for 5 years

Summarizing

The dependance of the waveform on the **binary acceleration** along the line of sight make it difficult to extract a redshift information from the signal

The shift in the phase induced by the evolution of redshift over the time of observation has a size observable by future LISA

The binary parameters are biased by the acceleration effect. To remove this bias: **add new search parameters** \epsilon to templates

Summarizing

The dependance of the waveform on the **binary acceleration** along the line of sight make it difficult to extract a redshift information from the signal

The shift in the phase induced by the evolution of redshift over the time of observation has a size observable by future LISA

The binary parameters are biased by the acceleration effect. To remove this bias: **add new search parameters** \epsilon to templates

this reduces the precision with which parameters binary are recovered but possibly measure binary acceleration

Summarizing

but if an optical counterpart is present…

The dependance of the waveform on the **binary acceleration** along the line of sight make it difficult to extract a redshift information from the signal

The shift in the phase induced by the evolution of redshift over the time of observation has a size observable by future LISA

The binary parameters are biased by the acceleration effect. To remove this bias: **add new search parameters** \epsilon to templates

Gravitational standard sirens

In (not so good) analogy with type Ia Supernovae (SNIa)

Supernovae: standard candles

they emit the same energy when exploding: we know their luminosity

by measuring the flux, we infer $d_L = \sqrt{L/(4\pi F)}$

we have a separate measurement of redshift

Binary systems: standard sirens

energy emitted depends on the system

by measuring the two polarizations, we can extract d_L

Good! We do not rely on the similarity of objects

redshift?

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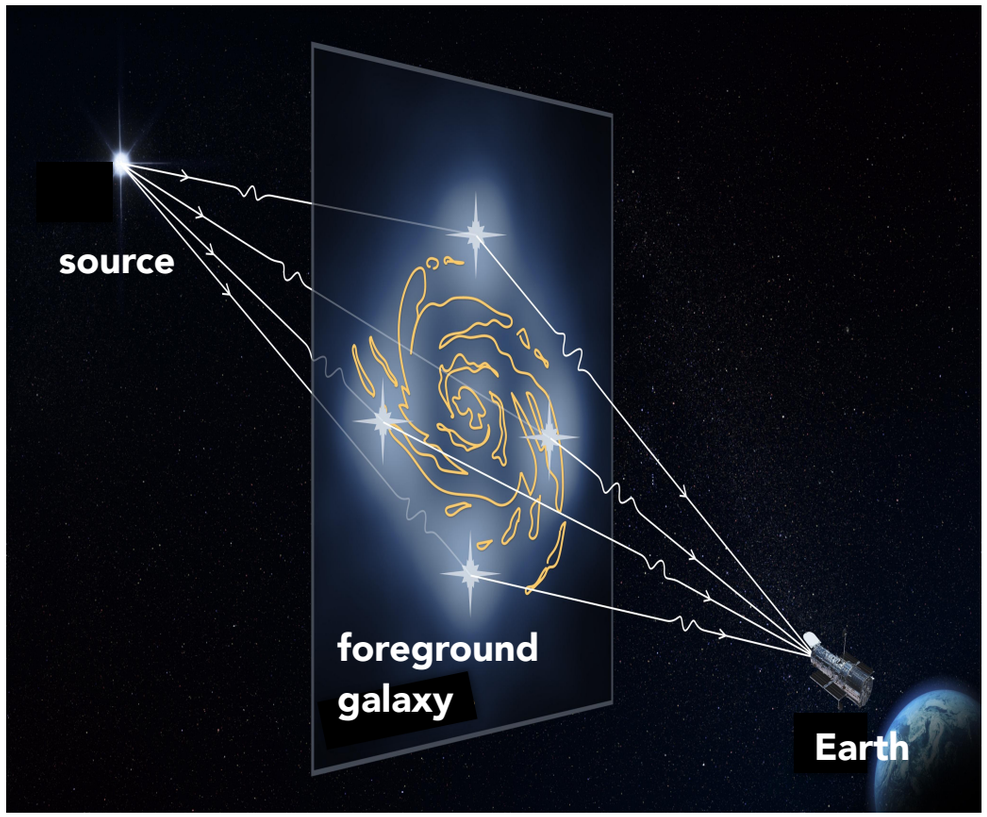
redshift?

we need optical counterpart

Effect of lensing on the waveform

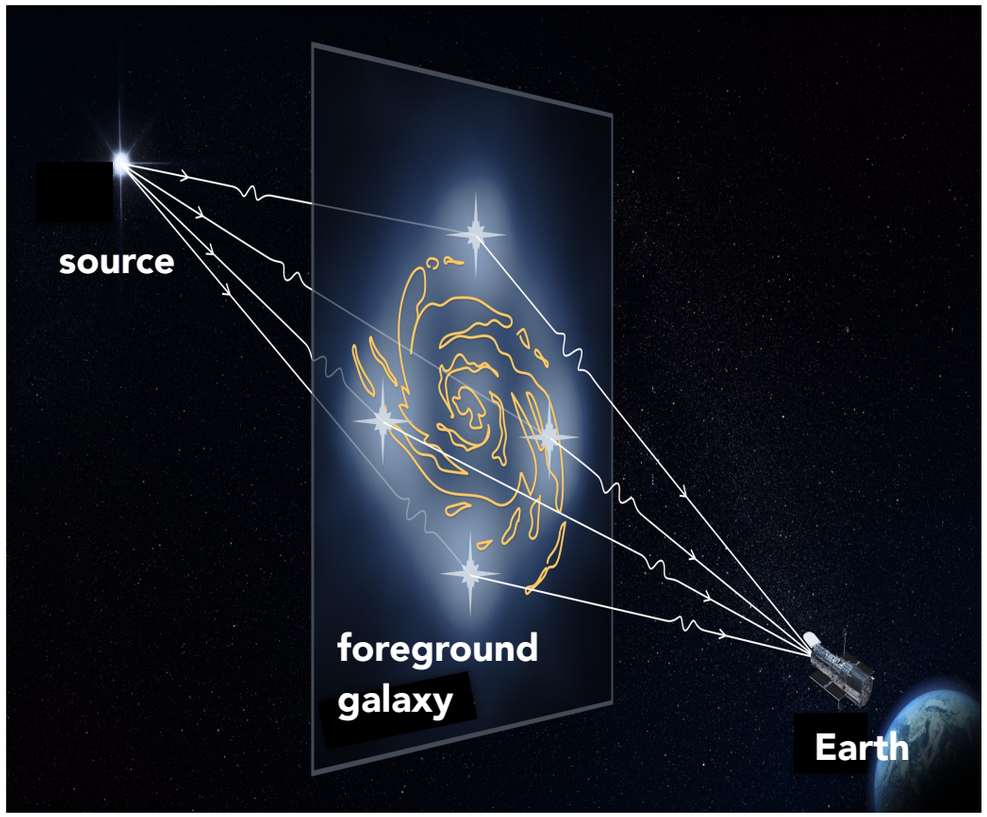
Lensing of gravitational waves



Distortion of bundle of “rays” due to gravitational potential of structures. Effects:

1) **deflection:** apparent angular position source biased
2) **magnification:** observed brightness biased
3) **time delay** : between multiple images, if produced

Lensing of gravitational waves



Distortion of bundle of “rays” due to gravitational potential of structures. Effects:

1) **deflection:** apparent angular position source biased
2) **magnification:** observed brightness biased
3) **time delay** : between multiple images, if produced

Differences with respect to the electromagnetic case:

1) Different detection methods (not limited by angular resolution detector -> time resolution)

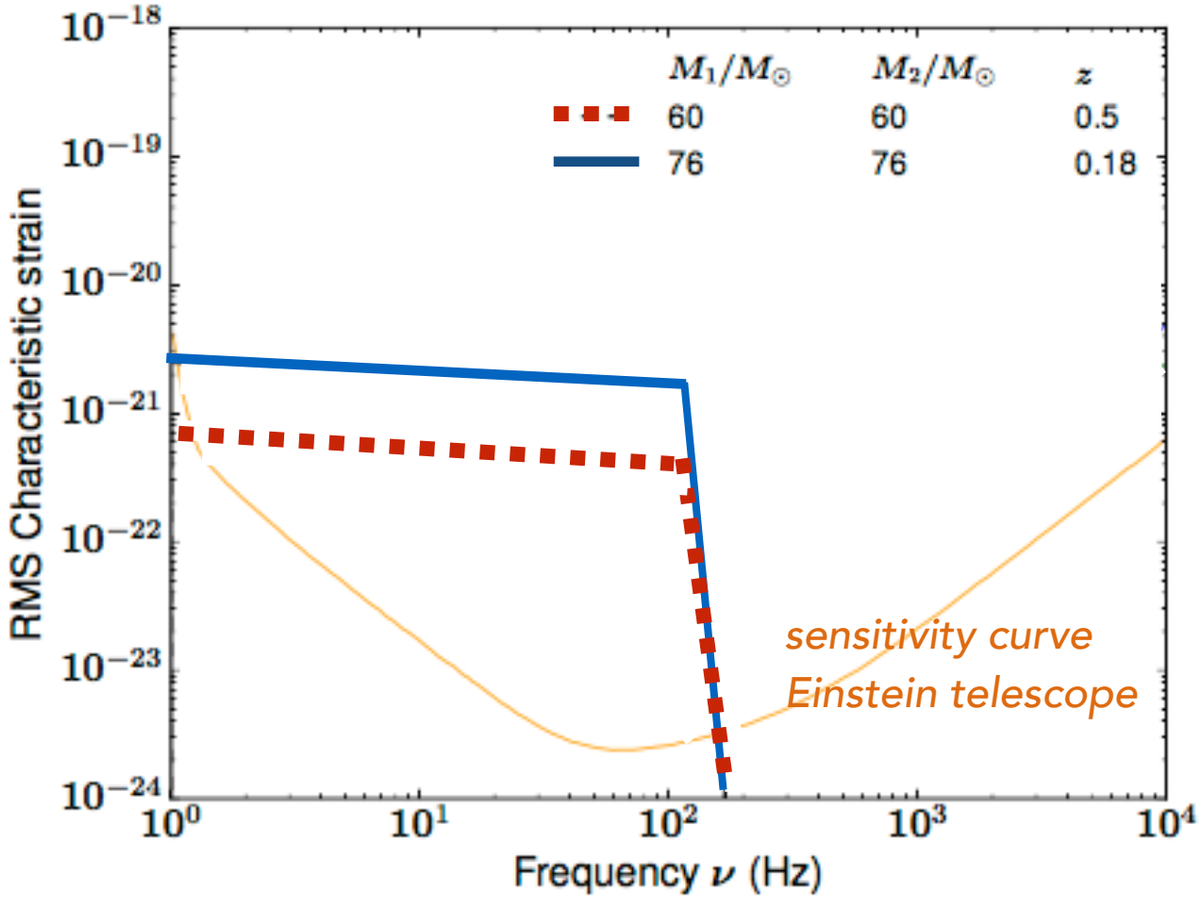
2) $\lambda_{\rm GW} \gg \lambda_{\gamma}$ beyond ray-optics effects (**wave effects**) important for wavelength of the size of deflectors i.e. astrophysical scale (e.g. LISA, PTA) vs electromagnetic case

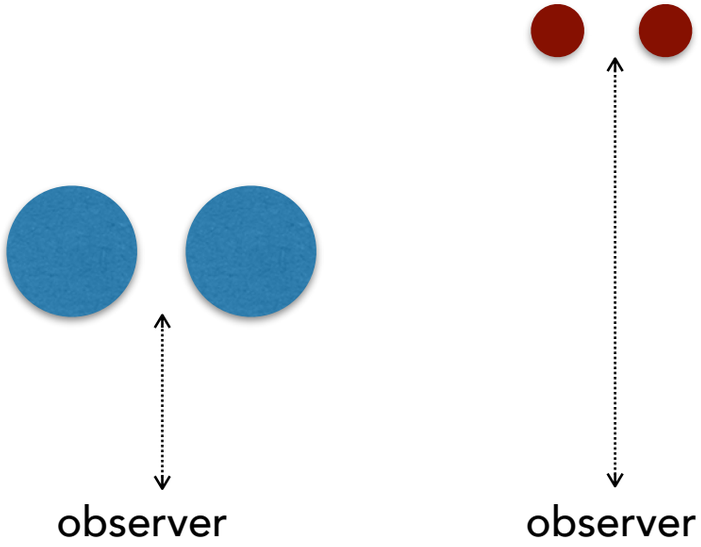
↘

New mathematical techniques to describe propagation need to be developed

Lensing effects on the waveform: degeneracies introduced

Lensing magnification: source brightness is **(de)-magnified** by the gravitational potential of massive structures along the line of sight





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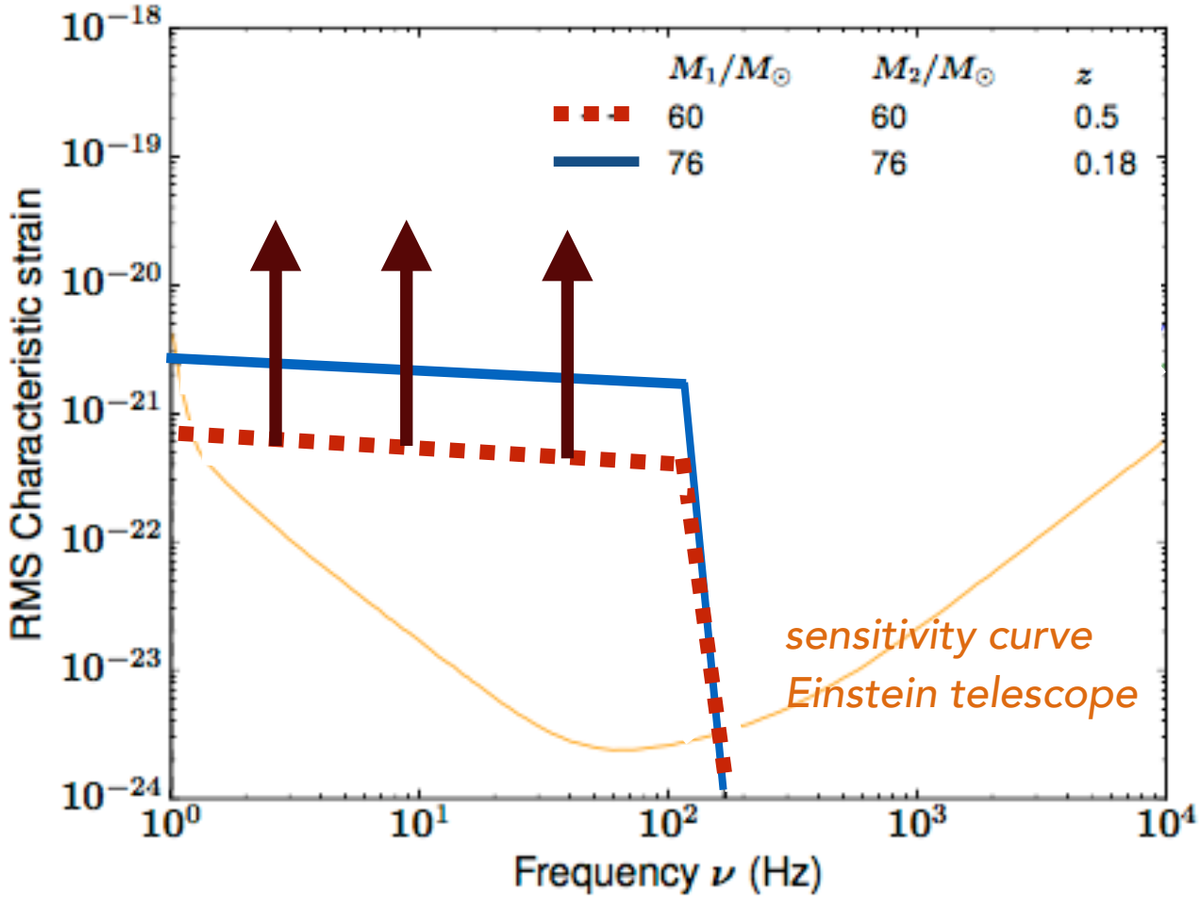
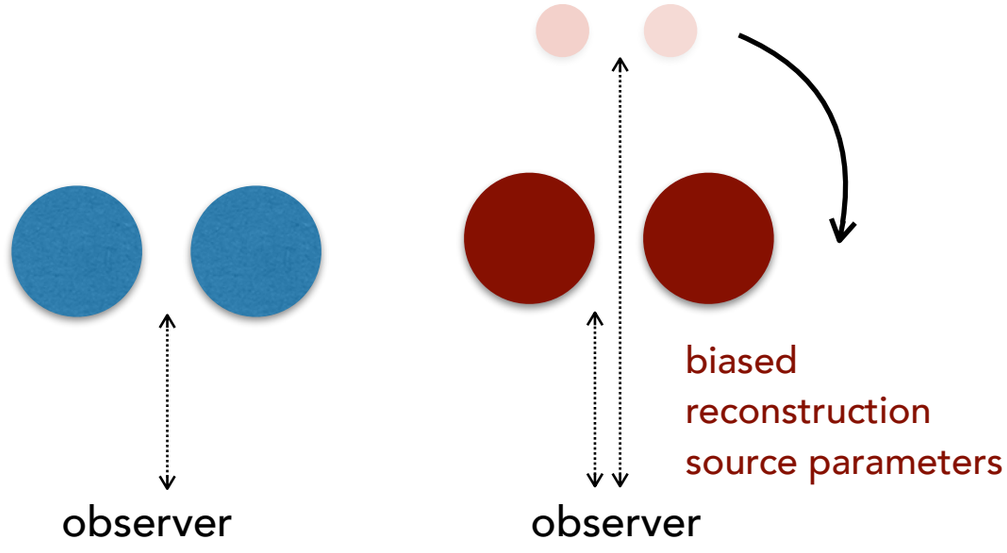


if the fainter signal is magnified, and we neglect lensing, we interpret it as GW signal from sources with **higher masses** and at **lower distance**



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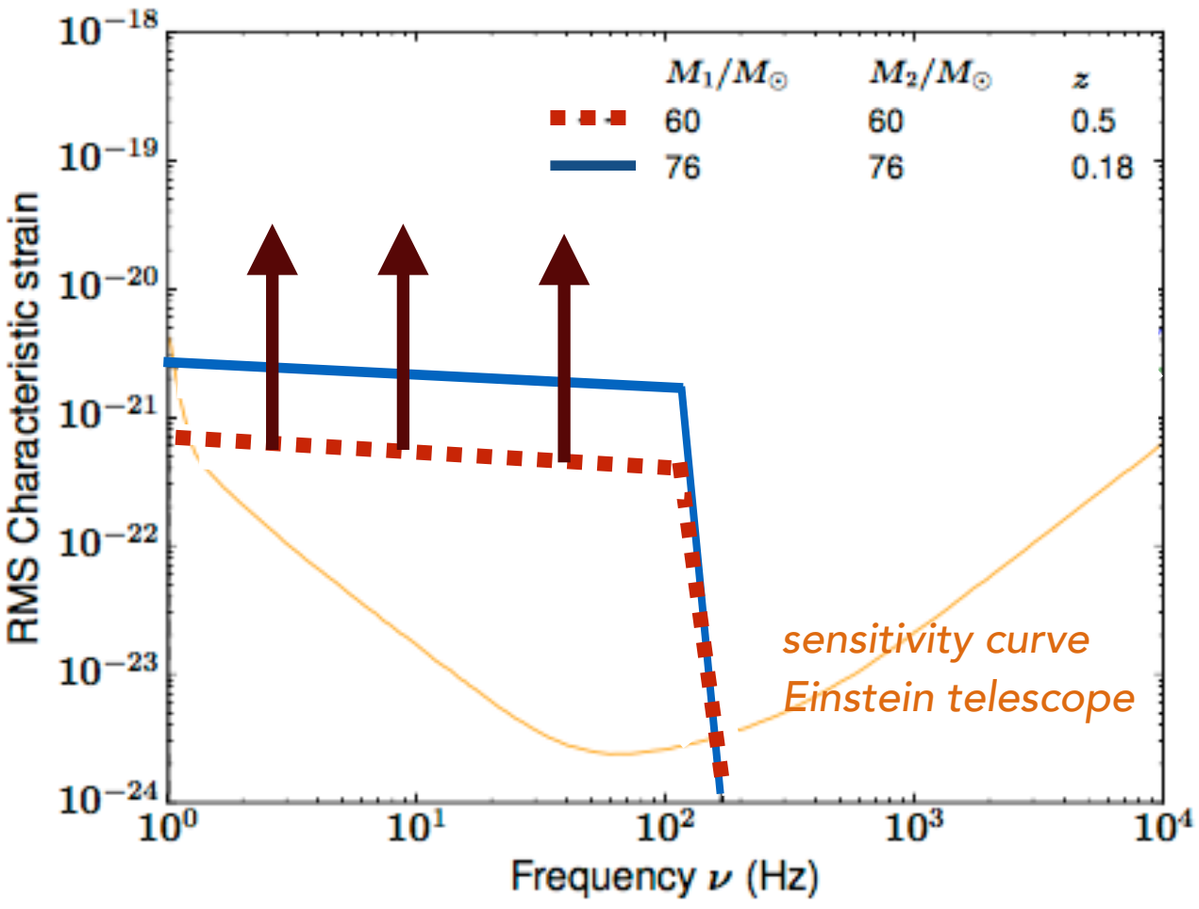


Case-study

The two signals are perfectly degenerate if the fainter is amplified of $\mu = 10$

$$h_{\rm obs} = \sqrt{\mu}\, h \qquad h_{\rm obs} \propto \frac{1}{D_{\rm obs}}$$

Reconstruction of luminosity distance and chirp mass is biased. If source is magnified:

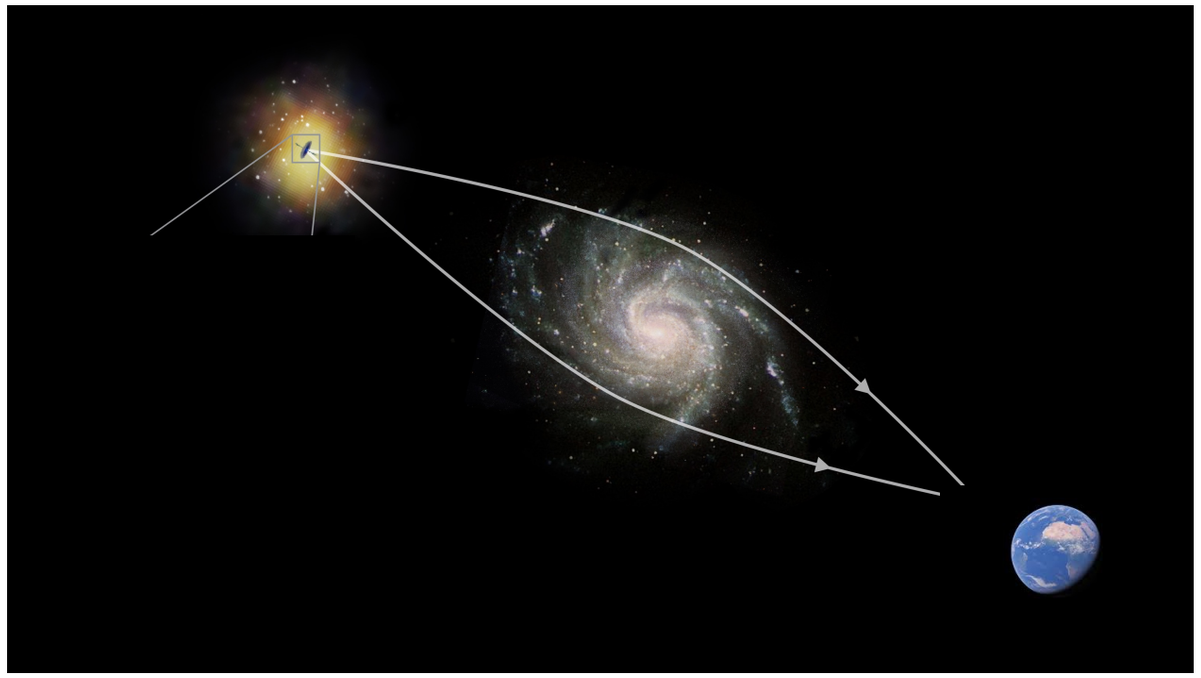
$$D_{\rm obs} = \frac{D}{\sqrt{\mu}} \ll D$$ understimate distance

$$\mathcal{M}^z_{\rm obs} = \mathcal{M}(1 + z_{\rm obs})$$ overstimate chirp mass

Lensing amplification on waveform

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

We need to study GW propagation on the lens background



$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$\Box h_{\mu\nu} + R_{\mu\alpha\beta\nu} h^{\alpha\beta} = 0$$

for a vacuum solution

Lensing amplification on waveform

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

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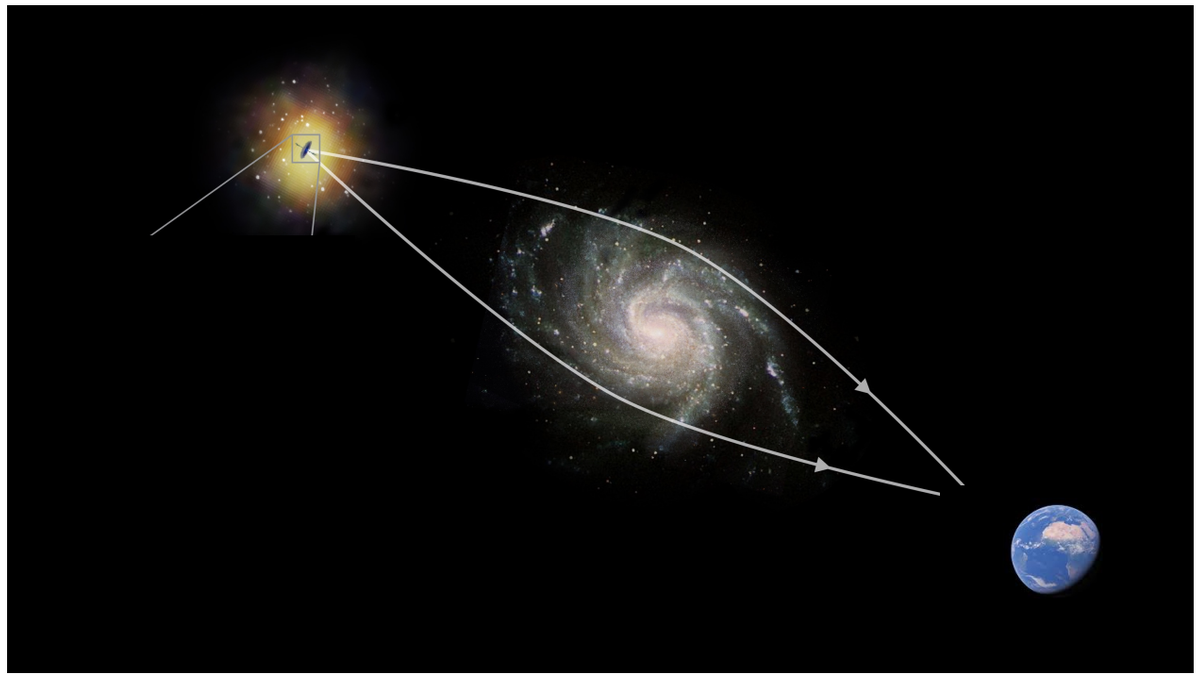


$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$\Box h_{\mu\nu} + \cancel{R_{\mu\alpha\beta\nu} h^{\alpha\beta}} = 0$$

for a vacuum solution

For the moment we treat wave as scalar wave, neglecting interaction spin curvature

blackboard

Wave effects in lensing: amplification factor

$$\Box h_{\mu\nu} + \cancel{R_{\mu\alpha\beta\nu}} h^{\alpha\beta} = 0$$

Traditional approach: GW treated as scalar wave

$$\tilde{H}^L(f) = F(f)\tilde{H}(f)$$

$$F(f) = \frac{4M_\ell G f}{i} \int d^2\mathbf{x} \, \exp\left[2\pi i f t_d(\mathbf{x}, \mathbf{y})\right]$$

amplification factor

integral over all possible image positions

time-delay function

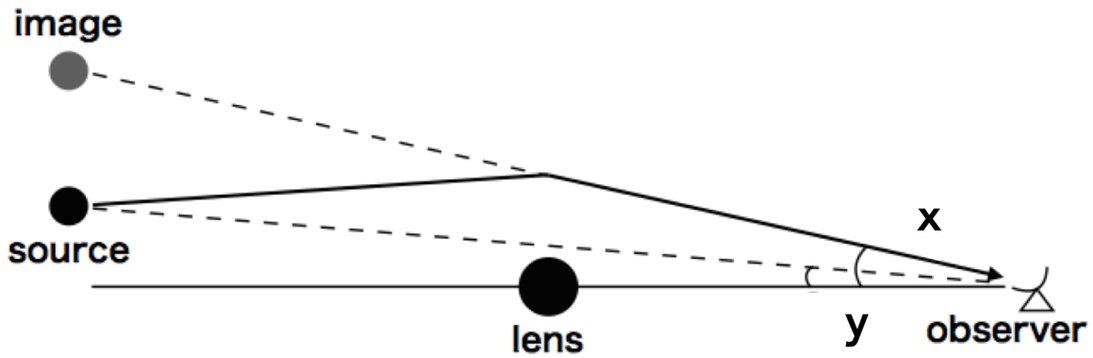
$$t_d = \frac{1}{2}|\mathbf{x} - \mathbf{y}|^2 - \Psi(\mathbf{x})$$

difference in length wrt unperturbed path

integral lens potential along path

x = (rescaled) image position

y = (rescaled) source position



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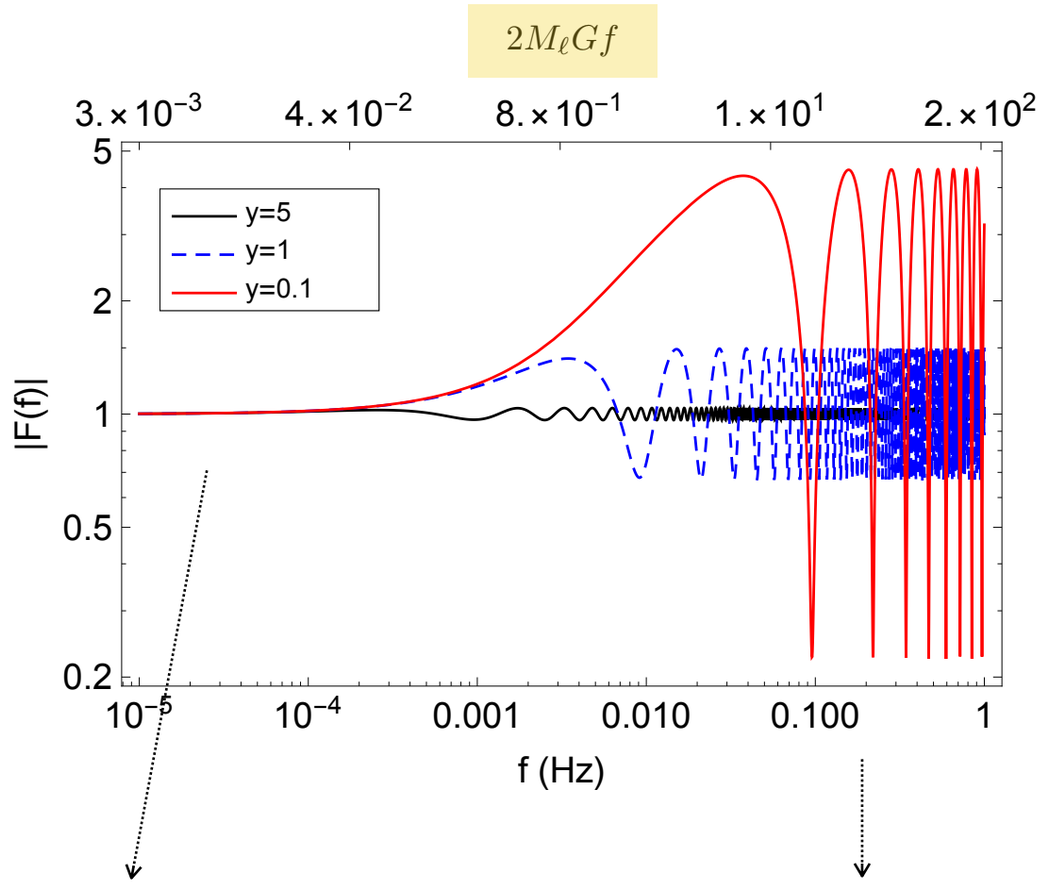
amplification factor

integral over all possible image positions

time-delay function

$\sim 8\pi G M_\ell (1+z_\ell)$



diffraction efficient: infinite number of rays contribute destructively

geometric optics: finite number of images, stationary points of time-delay

$$\frac{\partial t_d(\mathbf{x},\mathbf{y})}{\partial \mathbf{x}} = 0$$ *Fermat's principle*

How can we take spin into account

$$\Box h_{\mu\nu} + R_{\mu\alpha\beta\nu} h^{\alpha\beta} = 0$$

Traditional approach: GW treated as scalar wave

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amplification factor

integral over all possible image positions

time-delay function

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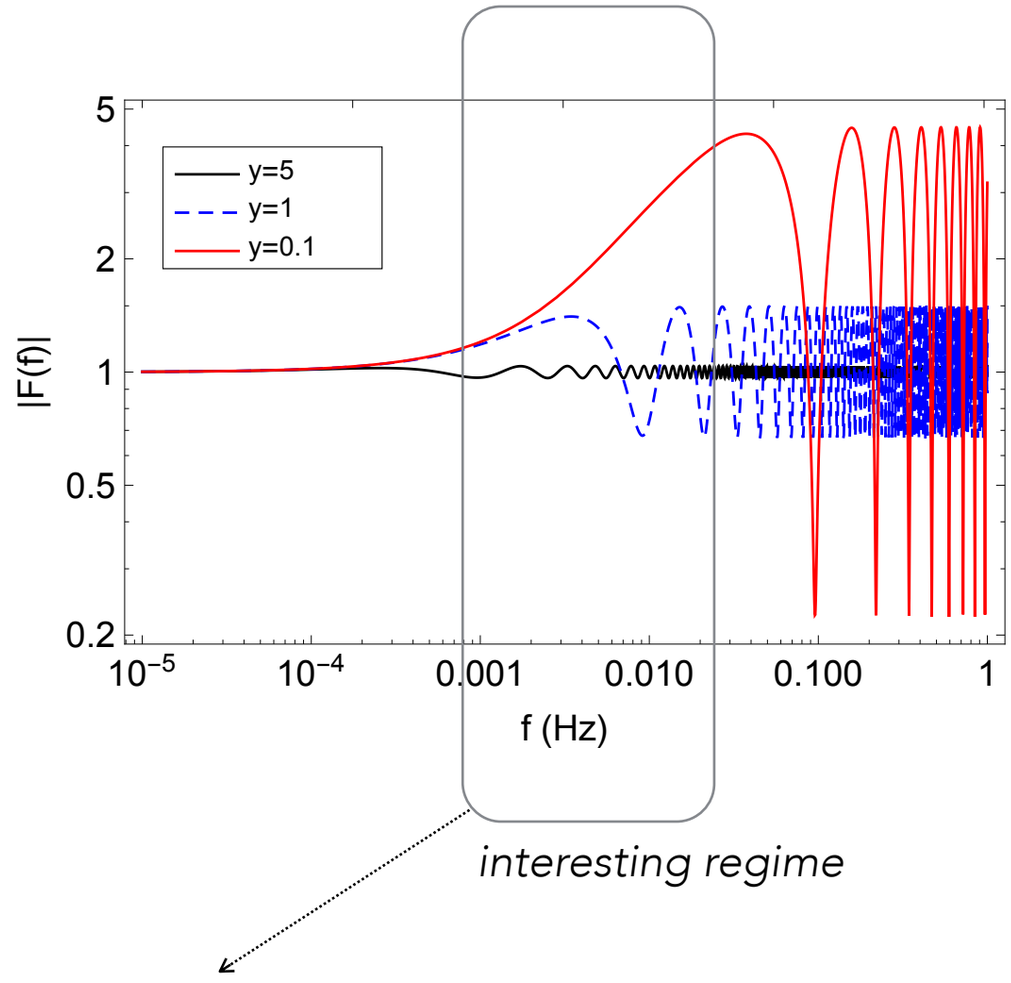

Let us start from the full covariant equation for GW propagation and look for a perturbative expansion around the ray optics solution

WKB expansion for GW propagation

$$\Box h_{\mu\nu} + R_{\mu\alpha\beta\nu} h^{\alpha\beta} = 0$$

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$$\tilde{H}^L(f) = F(f)\tilde{H}(f)$$

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amplification factor

integral over all possible image positions

time-delay function

$\sim 8\pi G M_\ell (1 + z_\ell)$

Idea: WKB expansion for spin-2 field

$\omega \gg 1$ *(perturbative parameter)*

$$h_{\mu\nu} = \Re\left(\left(\varepsilon^{(0)}_{\mu\nu} + \omega^{-1}\varepsilon^{(1)}_{\mu\nu} + \dots\right) e^{i\omega\Phi}\right)$$

complex polarization tensor

fast varying phase

Let us start from the full covariant equation for GW propagation and look for a perturbative expansion around the ray optics solution

Cusin, *Lagos, Phys.Rev.D101 (2020)*

Dalang, Cusin, Lagos (2021)

Lensing of spin-2 field beyond ray optics

WKB for spin-2 field

$$h_{\mu\nu} = \Re\left(\left(\varepsilon^{(0)}_{\mu\nu} + \omega^{-1}\varepsilon^{(1)}_{\mu\nu} + \ldots\right) e^{i\omega\Phi}\right) \qquad k_\mu \equiv \nabla_\mu \Phi$$

$$\omega \gg 1$$

geometric optics

order ω^2 $\quad k^\mu \nabla_\mu k^\nu = 0 \qquad k^\mu k_\mu = 0$

gravitons follow null geodesic

order ω $\quad 2k^\beta \varepsilon^{(0)}_{\mu\nu;\beta} + k^\beta{}_{;\beta}\varepsilon^{(0)}_{\mu\nu} = 0$

$$k^\mu \varepsilon^{(0)}_{\mu\nu} = 0$$

polarization tensor is transverse and parallel transported and flux is conserved

Geometric optics equations

$$2k_\beta \varepsilon^{(0);\beta}_{\mu\nu} + k^\beta{}_{;\beta}\varepsilon^{(0)}_{\mu\nu} = 0$$

from equation of motion

$$k^\mu \varepsilon^{(0)}_{\mu\nu} = 0$$

from Hilbert gauge

$$\varepsilon^{(0)}_{\mu\nu} = AA_{\mu\nu} \longrightarrow k^\alpha A_{\mu\nu;\alpha} = 0$$

polarisation parallel transported

$$(A^2 k^\alpha)_{;\alpha} = 0$$

flux of gravitons conserved

graviton number current density conserved
(hence number gravitons in a ray bundle is conserved, via Gauss theorem)

Lensing of spin-2 field beyond ray optics

WKB for spin-2 field

$$h_{\mu\nu} = \Re\left(\left(\varepsilon^{(0)}_{\mu\nu} + \omega^{-1}\varepsilon^{(1)}_{\mu\nu} + \ldots\right) e^{i\omega\Phi}\right) \qquad k_\mu \equiv \nabla_\mu \Phi$$

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$$k^\mu \varepsilon^{(0)}_{\mu\nu} = 0$$

polarization tensor is transverse and parallel transported and flux is conserved

beyond geometric optics

order ω^0 $\quad 2k^\beta \varepsilon^{(1)}_{\mu\nu;\beta} + k^\beta{}_{;\beta}\varepsilon^{(1)}_{\mu\nu} = -i\left[2\varepsilon^{\alpha\beta}_{(0)} R_{\alpha\mu\nu\beta} - \Box\varepsilon^{(0)}_{\mu\nu}\right]$

$$k^\mu \varepsilon^{(1)}_{\mu\nu} = i\nabla^\nu \varepsilon^{(0)}_{\mu\nu}\,.$$

new transport properties

Beyond geometric optics the polarization tensor is ***not parallel transported*** *along a ray and develops a* ***longitudinal component***

Polarizations of a GW in full generality: a step back

$$\{k^\mu, m^\mu, \ell^\mu, n^\mu\}$$

Tetrad of vectors: defines an observer

define polarisation screen

$$\ell^{\mu *} = m^\mu\,, \quad g_{\mu\nu} m^\mu \ell^\nu = 1\,, \quad g_{\mu\nu} k^\mu n^\nu = -1$$

Orthogonality properties

Newman Penrose scalars (projection of the GW Riemann tensor on the tetrad). 6 dofs in total in a generic theory of gravity

$$\Psi_2 = -\frac{1}{6}\mathscr{C}_{\mu\nu\alpha\beta} k^\mu n^\nu k^\alpha n^\beta\,,$$ spin 0

$$\Psi_3 = -\frac{1}{2}\mathscr{C}_{\mu\nu\alpha\beta} n^\mu k^\nu n^\alpha \ell^\beta\,,$$ spin 1

$$\Psi_4 = -\mathscr{C}_{\mu\nu\alpha\beta} n^\mu \ell^\nu n^\alpha \ell^\beta\,,$$ spin 2 the only present in general relativity

$$\Phi_{22} = \mathscr{C}_{\mu\nu\alpha\beta} n^\mu m^\nu \ell^\alpha n^\beta\,,$$ spin 0

Polarizations of a GW in full generality: a step back

Effect of GW on test particles can be described by

$$\frac{d^2\xi^i}{dt^2} = \mathcal{R}_{0i0j}\xi^j$$ geodesic deviation equation (ξ^i vector between two nearby rays)

$$S_{ij}(t) \equiv \mathcal{R}_{0i0j}$$ driving force matrix

Polarizations of a GW in full generality: a step back

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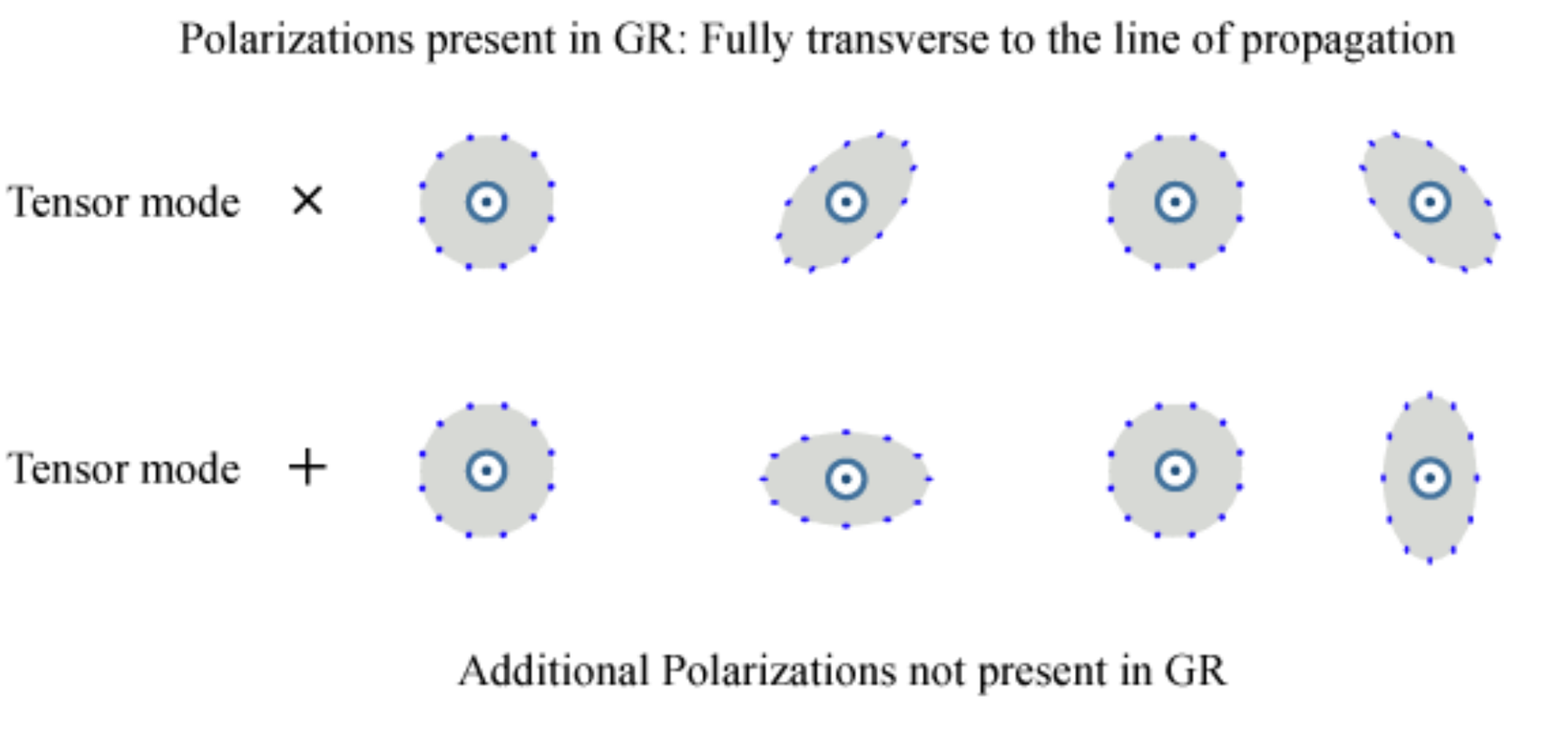
For a wave propagating along the z direction, the driving force matrix can be written as

$$S_{ij}(t) = \begin{pmatrix} -\mathrm{Re}\Psi_4 - \Phi_{22} & \mathrm{Im}\Psi_4 & -2\sqrt{2}\mathrm{Re}\Psi_3 \\ \mathrm{Im}\Psi_4 & \mathrm{Re}\Psi_4 - \Phi_{22} & 2\sqrt{2}\mathrm{Im}\Psi_3 \\ -2\sqrt{2}\mathrm{Re}\Psi_3 & 2\sqrt{2}\mathrm{Im}\Psi_3 & -6\Psi_2 \end{pmatrix},$$

general relativity
(geometric optics)

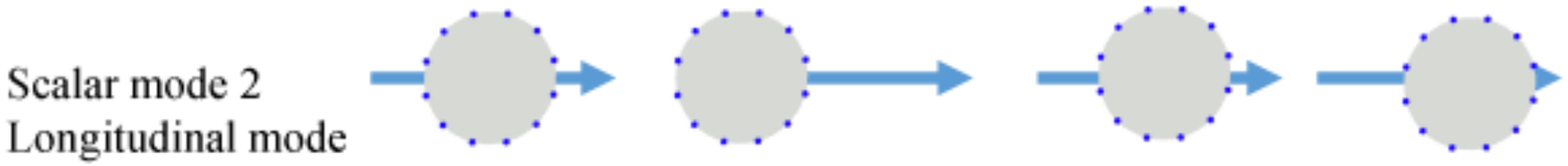
Ψ_4



Ψ_3

Φ_{22}

Ψ_2



C de Rham "Massive gravity"

Lensing of spin-2 field beyond ray optics

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polarization tensor is transverse and parallel transported and flux is conserved

beyond geometric optics

order ω^0

$$2k^\beta \varepsilon^{(1)}_{\mu\nu;\beta} + k^\beta{}_{;\beta}\varepsilon^{(1)}_{\mu\nu} = -i\left[2\varepsilon^{\alpha\beta}_{(0)} R_{\alpha\mu\nu\beta} - \Box\varepsilon^{(0)}_{\mu\nu}\right]$$

$$k^\mu \varepsilon^{(1)}_{\mu\nu} = i\nabla^\nu \varepsilon^{(0)}_{\mu\nu}\,.$$

new transport properties

Lensing of spin-2 field beyond ray optics

geometric optics

standard helicity-2 modes
(cross and plus)

beyond geometric optics

scalar breathing mode

scalar longitudinal mode

***Case study**: GW propagating along z at emission, lensed by point-like lens. Deformation of particle sphere described by:*

GW Riemann tensor

$$S_{ij}(t) \equiv \mathcal{R}_{0i0j}$$ driving force matrix

$$S_{ij}(t) = \begin{pmatrix} -\mathrm{Re}\Psi_4 - \Phi_{22} & \mathrm{Im}\Psi_4 & 0 \\ \mathrm{Im}\Psi_4 & \mathrm{Re}\Psi_4 - \Phi_{22} & 0 \\ 0 & 0 & -6\Psi_2 \end{pmatrix}$$

cross and plus polarisations

new scalar modes

New polarization modes are excited by the interaction spin-curvature

Cusin, Lagos, Phys.Rev.D101 (2020)
Dalang, Cusin, Lagos (2021)

Lensing of spin-2 field beyond ray optics

geometric optics

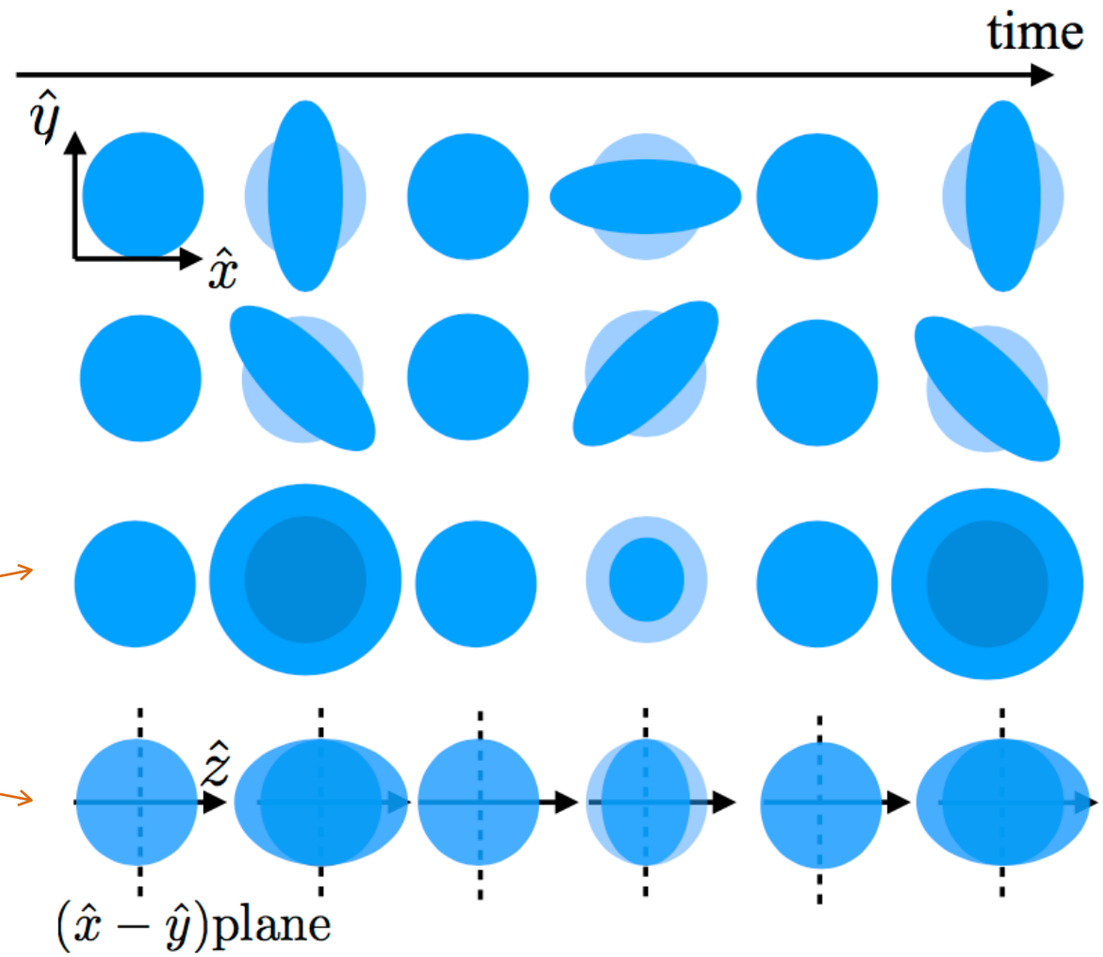
standard helicity-2 modes
(cross and plus)

beyond geometric optics

scalar conformal mode

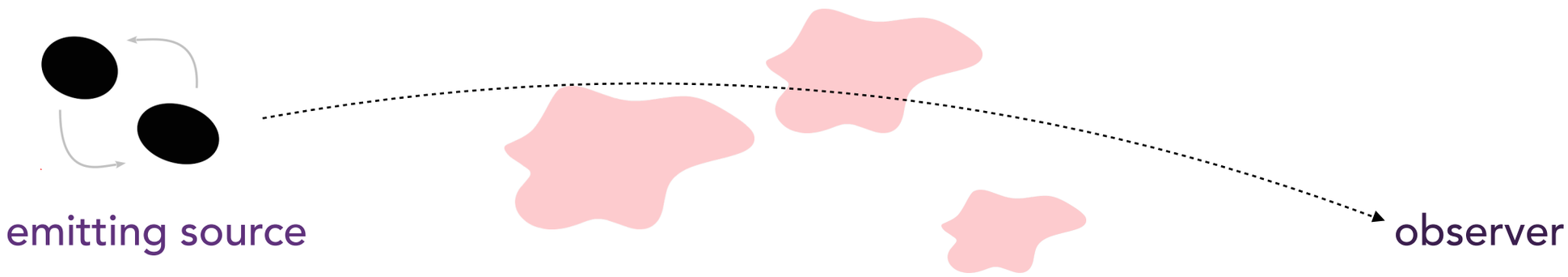
scalar longitudinal mode

Deformation of sphere of particles



These new modes can be misinterpreted as smoking gun of modified gravity

Summary of this course



(1) accurate description of the **emitted signal** + (2) description of the evolution of the signal during **propagation** = **observed signal**

Different effects, e.g.

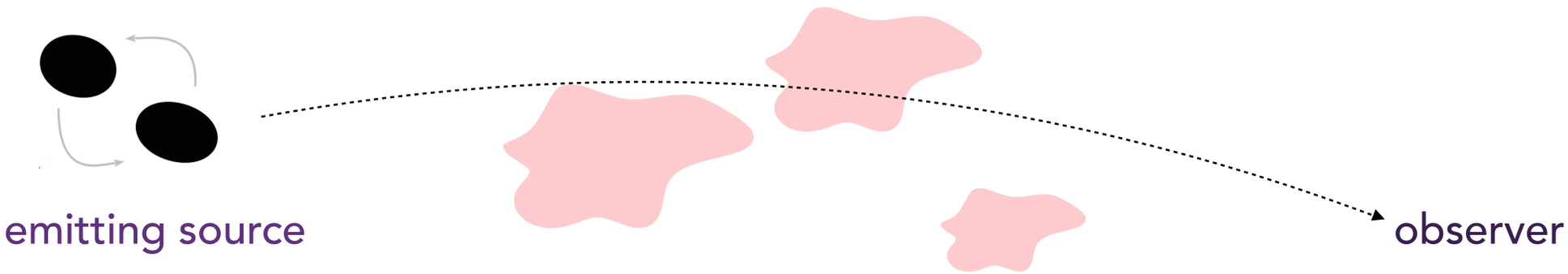
1) effects due to peculiar velocities

2) evolution of redshift during the observation time

3) lensing by massive structures

4) wave effects beyond ray optics

Summary of this course



we can’t extract redshift from waveform

if we don’t take velocities into account: biased reconstruction source parameters

Different effects, e.g.

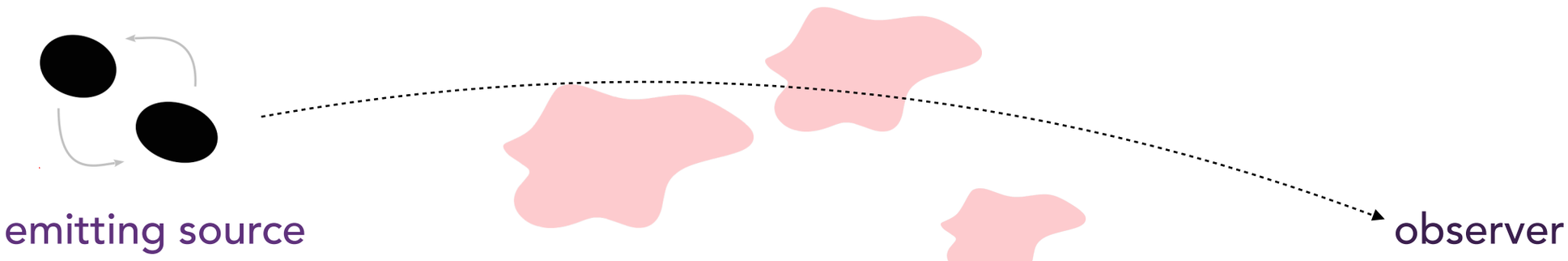
1) effects due to peculiar velocities and acceleration

2) evolution of redshift during the observation time

3) lensing by massive structures

4) wave effects beyond ray optics

Summary of this course



a magnified source appears as source with higher masses and from lower distance

Different effects, e.g.

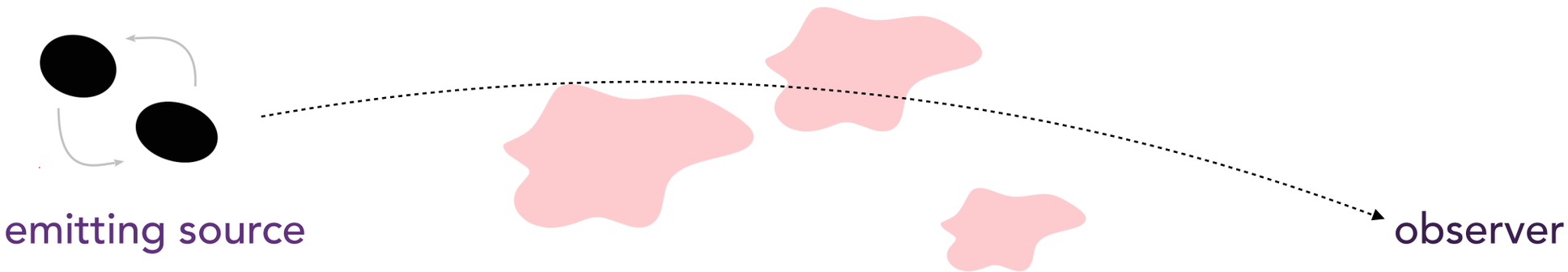
1) effects due to peculiar velocities and acceleration

2) evolution of redshift during the observation time

3) lensing by massive structures

4) wave effects beyond ray optics

Summary of this course



beyond geometric optics new polarisation modes appear which could be misinterpreted as smoking gun modified gravity

Different effects, e.g.

1) effects due to peculiar velocities and acceleration

2) evolution of redshift during the observation time

3) lensing by massive structures

4) wave effects beyond ray optics

The End

Contribution 2: For a virialised cluster of mass M one has, at distance r from the centre,

$$\frac{5GM}{3r} \simeq \frac{1}{2}v_S^2 \,. \tag{47}$$

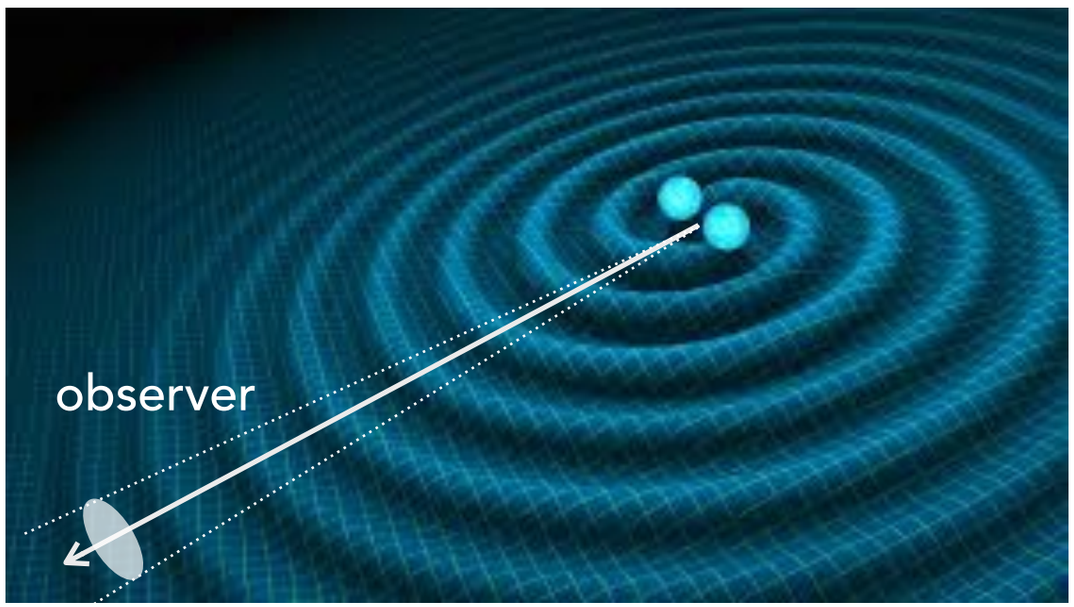
Combining this with Newton's law we find for the peculiar acceleration

$$\dot{v}_S \simeq \frac{3}{10}\frac{v_S^2}{r} \,. \tag{48}$$

Contribution 3 : For a circular motion of the binary around the centre of the galaxy one has

$$\dot{v}_S = \frac{v_S^2}{r} \,. \tag{49}$$

Power radiated

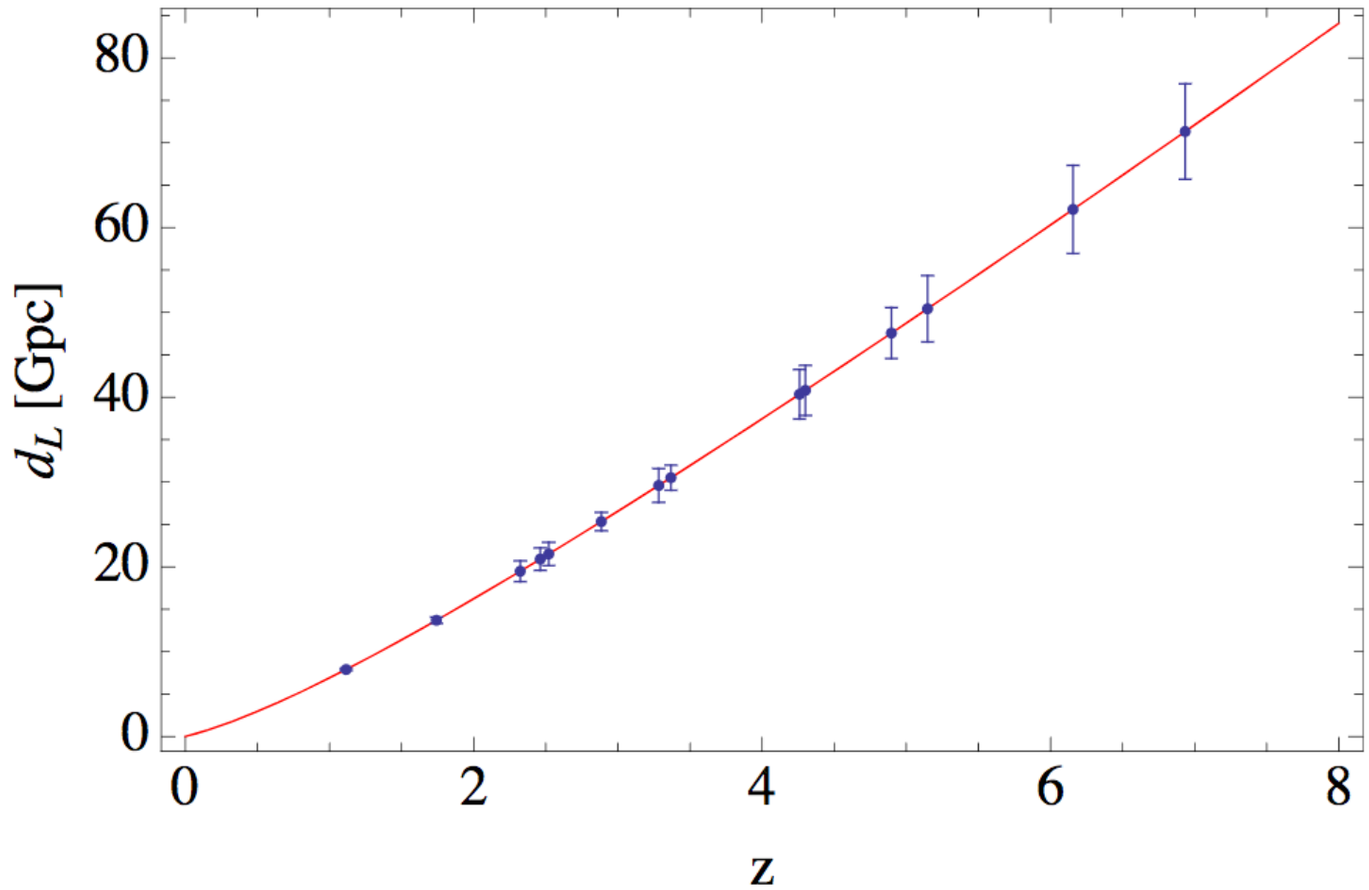


$$\left(\frac{dP}{d\Omega}\right)_{\rm quad} = \frac{r^2c^3}{32\pi G}[\dot{h}^{TT}_{ij}\dot{h}^{TT}_{ij}] = \Lambda_{ij,kl}(\mathbf{n})[\dddot{M}_{ij}\dddot{M}_{kl}]$$

Power radiated per units of solid angle in the quadrupole approximation

Probing the expansion of the universe with GW

Binaries with electromagnetic counterpart



luminosity distance extracted from the waveform

Tamanini et al. 2016

optical/ radio emission allows the identification of redshift

Testing cosmological models: eLISA & sirens

Tamanini et al. 2016

Capability of different eLISA configurations to test expansion universe tested

Simulations of catalogues of standard sirens composed by MBH binaries whose GW radiation is observable by eLISA and supposed to have EM counterpart

Different two and three arms configurations are considered

For each configuration and each simulated catalogue, the capability to constrain cosmological models is tested via Fisher-matrix analysis

Fisher matrix analysis

Fisher matrix analysis to test the capability of different configurations of eLISA to constrain cosmological models

Tamanini et al. 2016

$$F_{ij}^{C} = \sum_{n} \frac{1}{\sigma_n^2 + \sigma_{\rm lens}^2(z_n) + \sigma_v^2(z_n)} \frac{\partial d_L(z_n)}{\partial \theta_i}\Big|_{\rm fid} \frac{\partial d_L(z_n)}{\partial \theta_j}\Big|_{\rm fid}$$

weak lensing error

$$\theta_i = (\Omega_M, \Omega_\Lambda, h, \omega_0, \omega_a)$$

experimental error

$$\sigma_n = \begin{cases} (\Delta d_L)_n & \text{spectroscopic z} \\ \sqrt{(\Delta d_L)_n^2 + \left(\frac{\partial d_L}{\partial z}(\Delta z)_n\right)^2} & \text{photometric z} \end{cases}$$

error due to peculiar velocities GW sources

eLISA and standard sirens: what can be done

- 3 arms configuration have the potential to provide a test of the expansion of the universe up to $z \sim 8$, complementary to other cosmological probes based on EM counterpart

- Constraint on Hubble at 1-2% depending on the mission configuration (with all other parameters fixed to their best-fit value)

- $(\Omega_M, \Omega_\Lambda)$ can be constrained to a level competitive with present SNIa results

- The lack of MBH standard sirens at low redshift allows to constraint dark energy only at the level of few percent.

see Tamanini, Caprini 2017 for early time dark energy with standard sirens

Conclusions

- Coalescing binaries give a measurement of the luminosity distance

- If they have an electromagnetic counterpart, an independent measurement of redshift is possible

- We can extract an Hubble diagram, to constrain cosmological models

- Forecasts: by using standard sirens eLISA will be able to put constrains on the Hubble factor and on $(\Omega_M, \Omega_\Lambda)$ competitive & complementary to the ones coming from SNIa (see Tamanini et al. 2017)

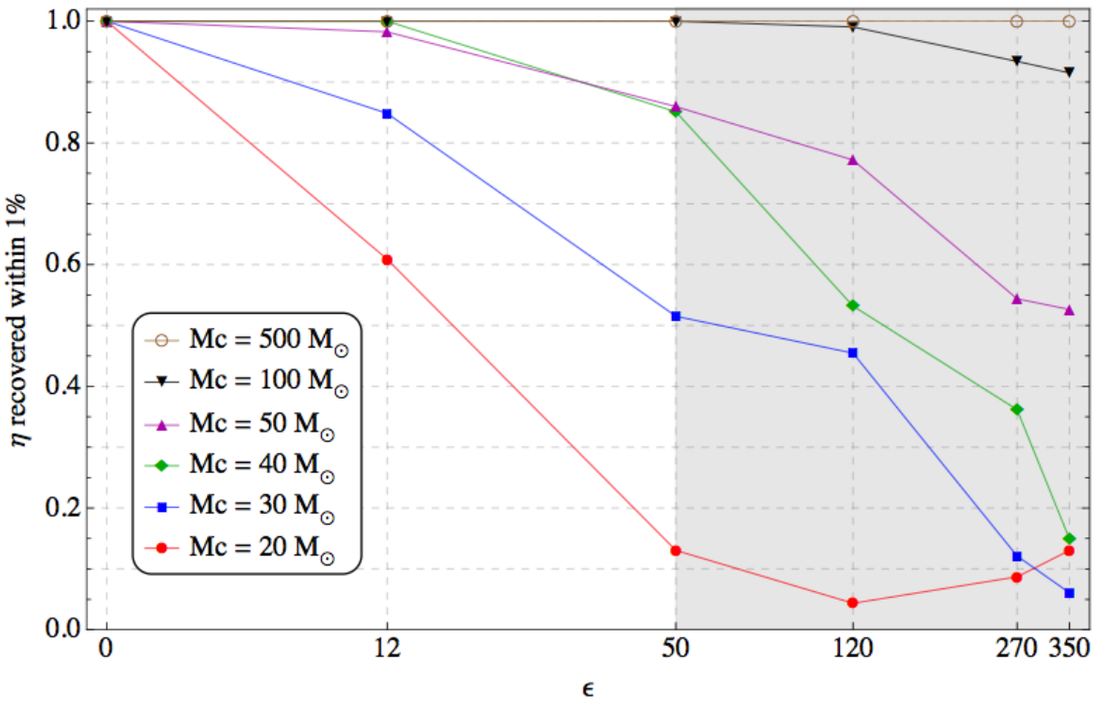
Bias in the parameters

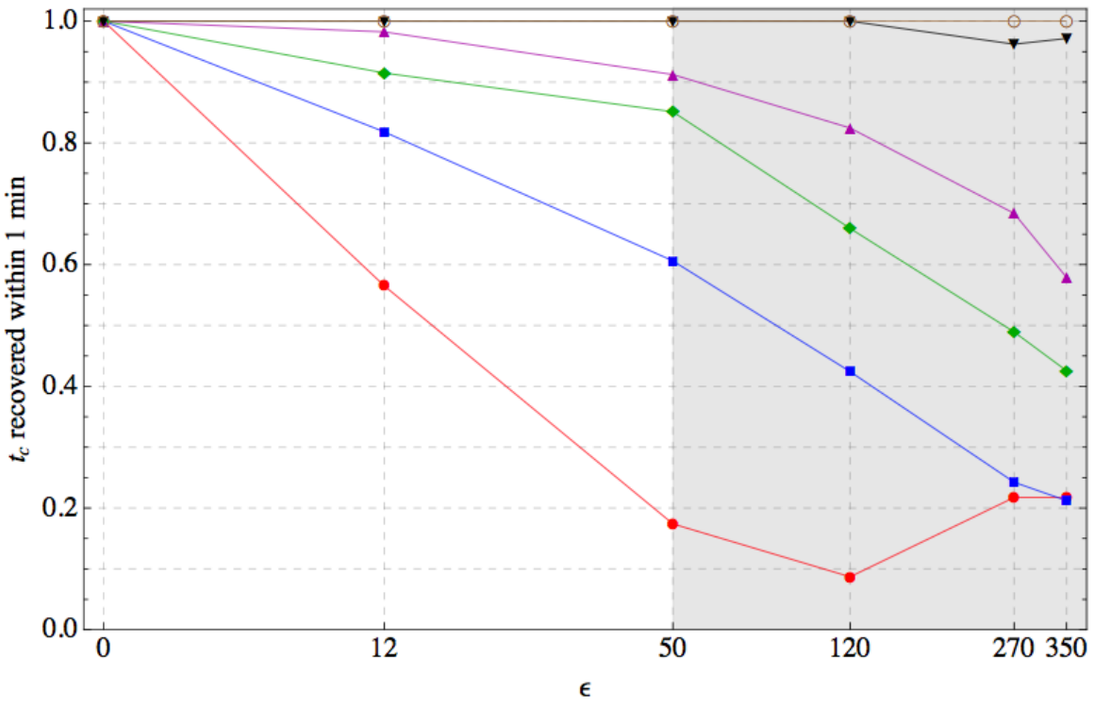
symmetric mass ratio

$$\eta = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

time to coalescence

$$t_c$$



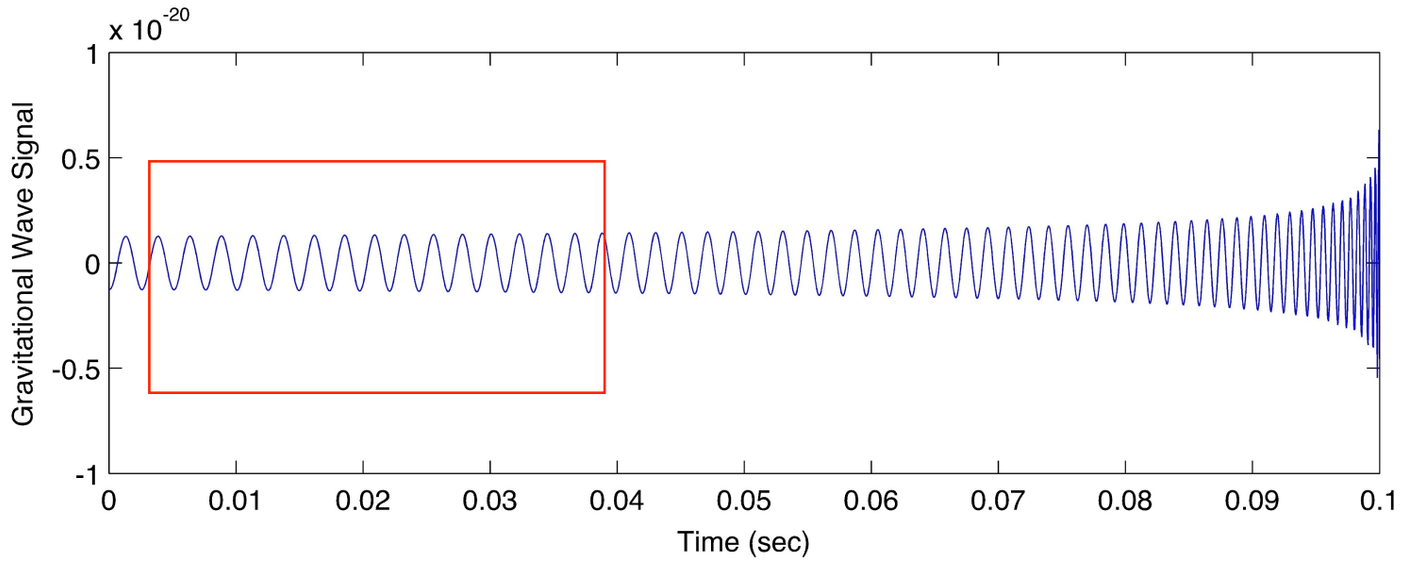


Bonvin et al. 2017

Fraction of events for which the injected symmetric mass ration (time to coalescence) is recovered better than 1% (within 1 min) . For $\epsilon > 50$ the results for low mass $M < 50 M_\odot$ can be affected by the Monte Carlo finite precision.

Corrections to the phase from redshift evolution

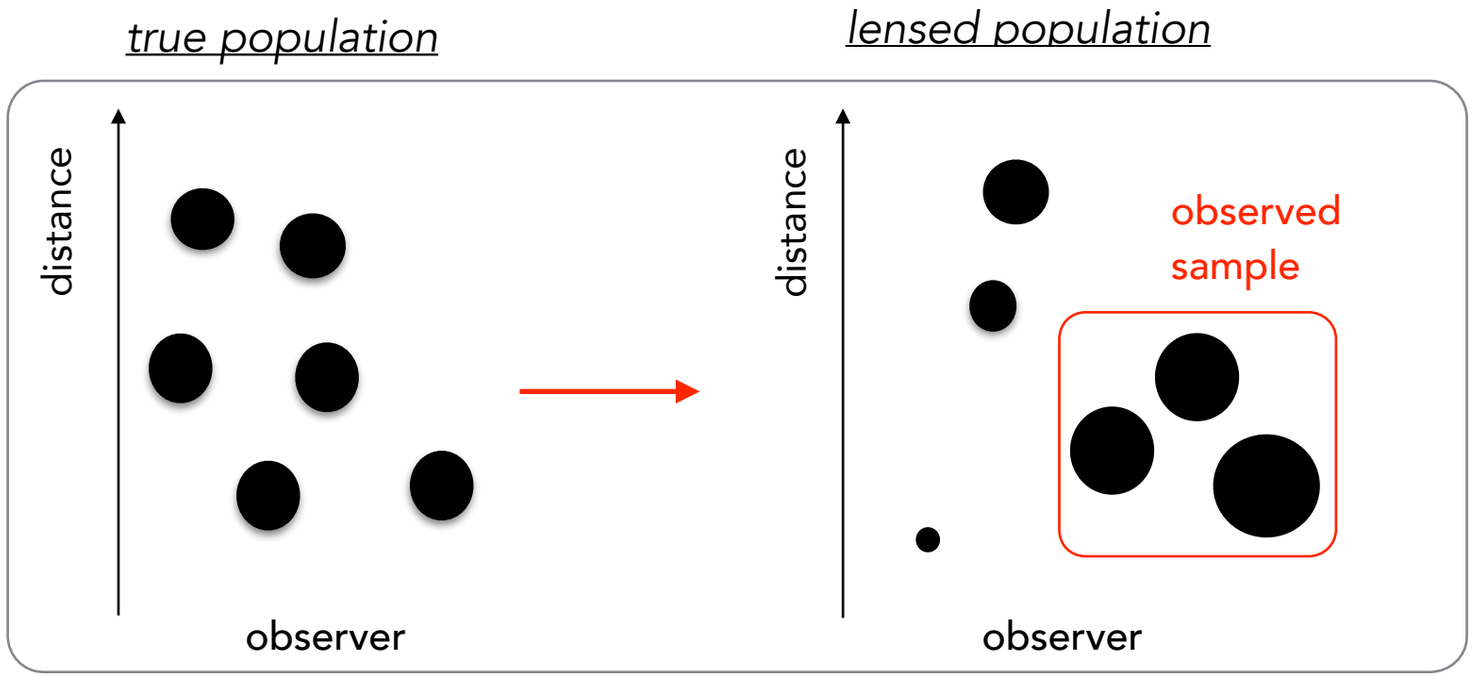
When deeply in the inspiral regime, GW is almost monochromatic (chirping negligible): phase shift can be **reabsorbed** in a shift of the binary parameters



effects of peculiar acceleration can be absorbed into a tiny shift in the intrinsic parameters

Other aspects of lensing

My recent work: statistical characterization of lensing/kinematic effects



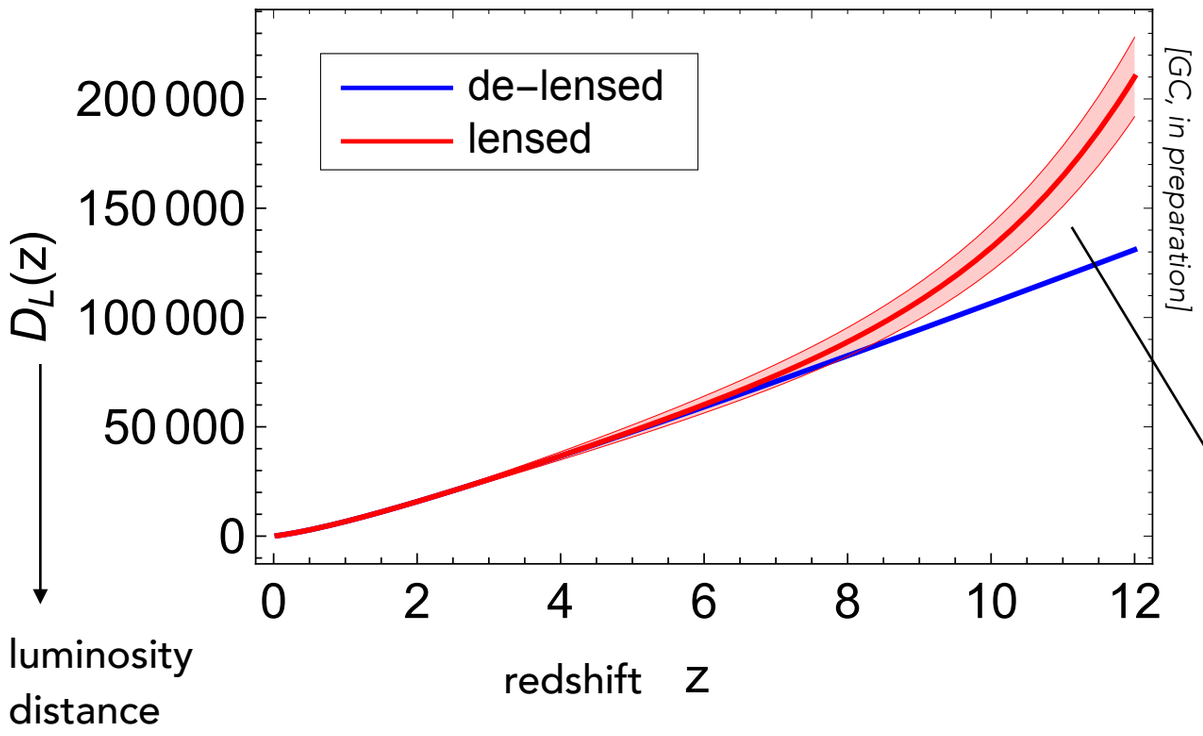
Realistic detector observes a biased sample of the population: *magnified sources are easier to detect than de-magnified ones*

Biased reconstruction of astrophysical and cosmological parameters (*e.g. mass and luminosity distance of a binary system*) *in population studies*

Future work: inclusion of selection biases in templates

Lensing: dominant source of uncertainty on distance reconstruction for **high-redshift** sources
(velocity effects important for low-redshift population)

Case study: Hubble diagram for LISA massive black hole binaries (pop III)



Lensing induced bias: *important for massive black hole binaries visible by LISA*
Velocity induced bias: *important for neutron star binaries visible by Einstein Telescope*

lensing bias important at high redshift:
impact on statistical methods with dark sirens

How NP depend on tetrad

The presence or absence of the components of various helicities depends upon the frame.

The most general class of transformations that preserve orthogonality properties of the tetrad is

$$k^{\mu'} = k^{\mu} + |q_1|^2 n^{\mu} + q_1^* m^{\mu} + q_1 \ell^{\mu}$$
$$m^{\mu'} = m^{\mu} + q_1 n^{\mu} + q_2 k^{\mu},$$
$$n^{\mu'} = n^{\mu} + |q_2|^2 k^{\mu} + q_2^* m^{\mu} + q_2 \ell^{\mu}$$

(helicity defined by pure rotation around direction propagation wave)

class II transformation (n is unchanged)

$$\Psi_2' = \Psi_2,$$
$$\Psi_3' = \Psi_3 + 3\Psi_2 q_2^*,$$
$$\Psi_4' = \Psi_4 + 4q_2^*\Psi_3 + 6q_2^{*2}\Psi_2,$$
$$\Phi_{22}' = \Phi_{22} + 2(q_2\Psi_3 + q_2^*\Psi_3^*) + 6|q_2|^2\Psi_2,$$

class I transformation (k is unchanged)

$$\Psi_2' = \Psi_2 + \frac{2}{3}(q_1\Psi_3 + q_1^*\Psi_3^*) + \frac{1}{6}(q_1^2\Psi_4 + q_1^{*2}\Psi_4^*)$$
$$\Psi_3' = \Psi_3 + \frac{1}{2}(q_1^*\Phi_{22} + q_1\Psi_4),$$
$$\Psi_4' = \Psi_4,$$
$$\Phi_{22}' = \Phi_{22}.$$

Constraining extra polarization modes

Signal from detector a of a network can be decomposed as

tensor modes

scalar modes

$$S_a = F_a^+ h_+ + F_a^\times h_\times + F_a^S h_S + F_a^L h_L + F_a^V h_V + F_a^W h_W + n_a,$$

vector modes

instrumental noise

The antenna patterns depend on the source position and detector geometry

Some examples: massive gravity (dRGT): scalar mode is breathing mode

massless scalar-tensor: breathing mode

massive scalar-tensor: combination of longitudinal and breathing modes

Typically amplitude of longitudinal mode is proportional to scalar mass (which has to be small not to have fast decay)

Geometric optics equations

$$2k_\beta \varepsilon^{(0);\beta}_{\mu\nu} + k^\beta{}_{;\beta}\varepsilon^{(0)}_{\mu\nu} = 0$$

from equation of motion

$$k^\mu \varepsilon^{(0)}_{\mu\nu} = 0$$

from Hilbert gauge

$$\varepsilon^{(0)}_{\mu\nu} = AA_{\mu\nu} \longrightarrow (A^2 k^\alpha)_{;\alpha} = 0$$

flux of gravitons conserved

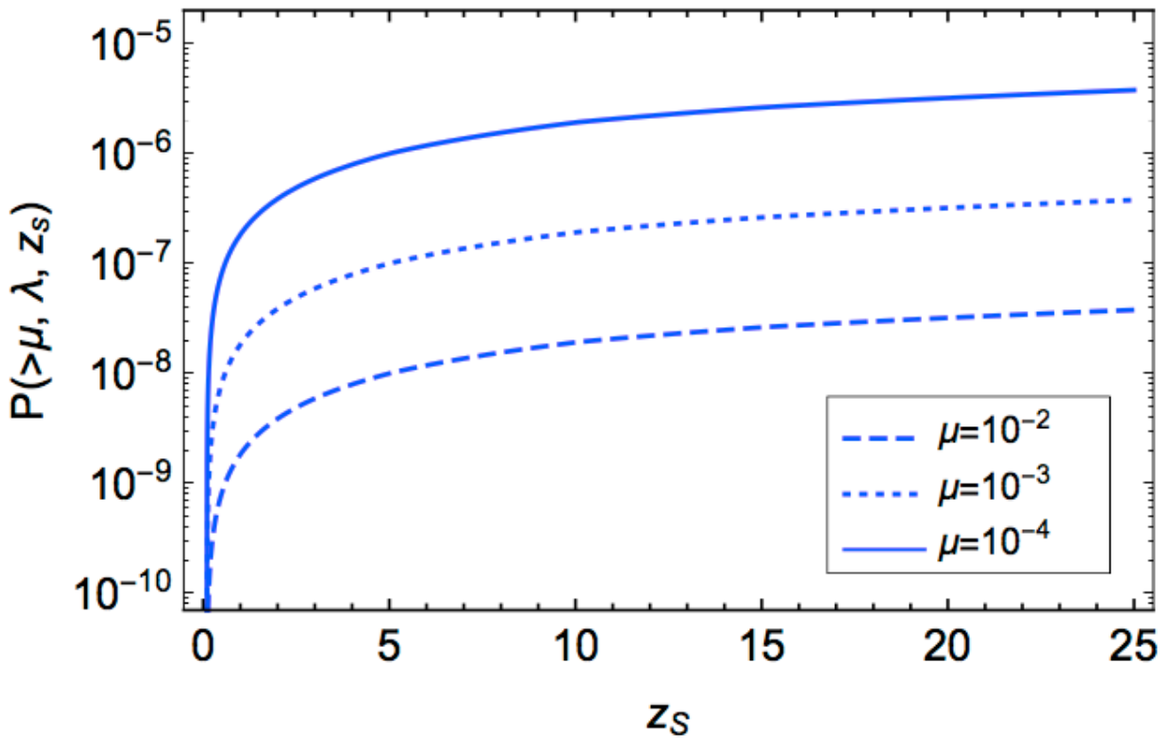
$$\searrow \quad k^\alpha A_{\mu\nu;\alpha} = 0$$

polarisation parallel transported

Probability of developing a scalar mode

$$\frac{A_s}{A_t} = \frac{NR_s\lambda}{||\boldsymbol{b}_{\max}||^2} = \mu \ll 1 \qquad \Longrightarrow ||\boldsymbol{b}_{\max}|| = \sqrt{\frac{NR_s\lambda}{\mu}}$$

defines area cross-section



If galaxies are lenses

$$\tau(\mu,\lambda,z_s) = \frac{N\pi\lambda}{\mu c}\int_0^{z_s} \mathrm{d}z \frac{(1+z)^2}{H(z)} \int_0^{M_{\max}} \mathrm{d}M (2MG) \frac{\mathrm{d}n}{\mathrm{d}M}$$