A Brief Introduction to Massive Gravity

PART I

Andrew J. Tolley
Imperial College London

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Beyond Einstein Theories of Gravity

Type I: UV Modifications:
eg. Quantum Gravity, string theory, extra dimensions, branes, supergravity

At energies well below the scale of new physics Λ , gravitational effects are well incorporated in the language of Effective Field Theories

$$S = M_{\rm Planck}^2 \int d^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{a}{\Lambda^2}R^2 + \frac{b}{\Lambda^2}R_{\mu\nu}^2 + \dots + \frac{c}{\Lambda^4}R_{abcd}R^{cd}_{ef}R^{efab} + \dots + \mathcal{L}_{\rm matter}\right]$$

$$+\frac{d}{\Lambda^6}(R_{abcd}R^{abcd})^2 + \dots$$

eg Cardoso et al 2018

Addition of Higher Dimension, (generally higher derivative operators), **no failure of well-posedness/ghosts** etc as all such operators should be treated perturbatively (rules of EFT)

Type 2: IR Modifications:

Why modify gravity (in the IR)?

Principle Motivation is Cosmological:

Dark Energy and Cosmological Constant

I: Old cosmological constant problem:

Why is the universe not accelerating at a gigantic rate determined by the vacuum energy?

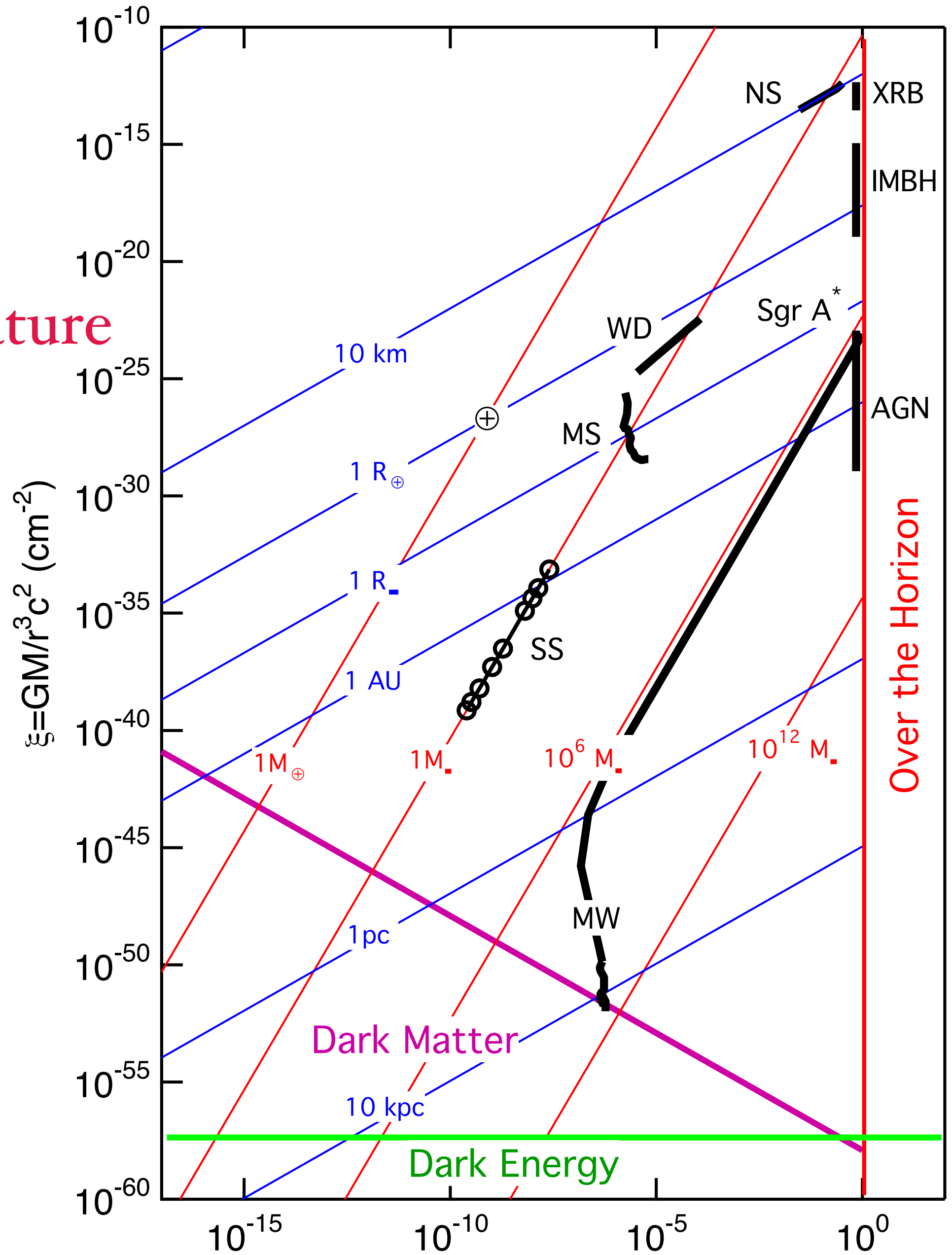
II: New cosmological constant problem:

Assuming I is solved, what gives rise to the remaining vacuum energy or dark energy which leads to the acceleration we observe?

Why modify gravity (in the IR)?

III: Because it allows us to put better constraints on Einstein gravity!

Curvature



Potential ε=GM/rc^2

Figure 1: A parameter space for quantifying the strength of a gravitational field. The x-axis measures the potential $\epsilon \equiv GM/rc^2$ and the y-axis measures the spacetime curvature $\xi \equiv GM/r^3c^2$ of the gravitational field at a radius r away from a central object of mass M. These two parameters provide two different quantitative measures of the strength of the gravitational fields. The various curves, points, and legends are described in the text.

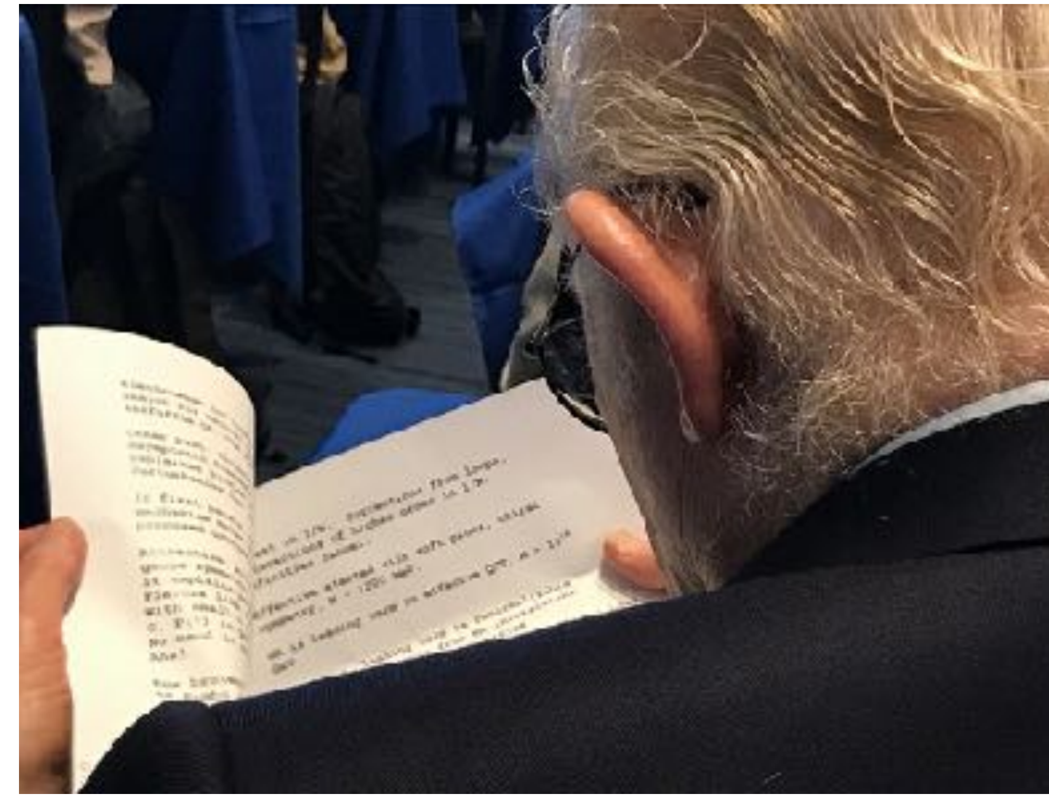
Gravity has only been tested over special ranges of scales and curvatures

e.g. Weinberg's nonlinear Quantum Mechanics-constructing to test linearity of QM

D. Psaltis, Living Reviews

Guiding Principle

Theorem: General Relativity is the Unique local and Lorentz invariant theory describing an interacting single massless spin two particle that couples to matter



Weinberg, Deser, Wald, Feynman, ...

Locality

Massless ? Lorentz Invariant

Single Spin 2

Guiding Principle

Corollary: Any theory which preserves Lorentz invariance and Locality leads to new degrees of freedom!

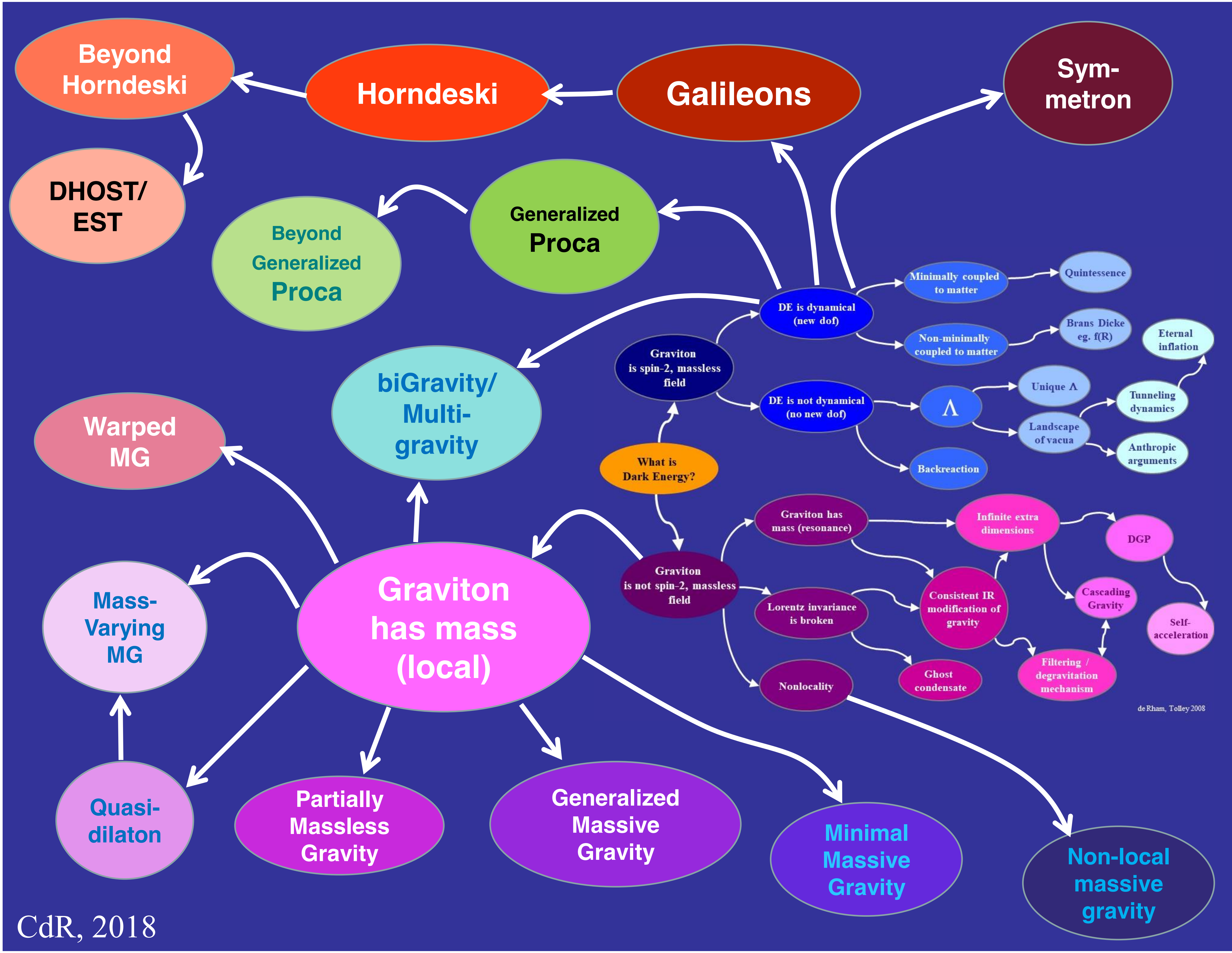
Locality ✓

Massless ✗

?

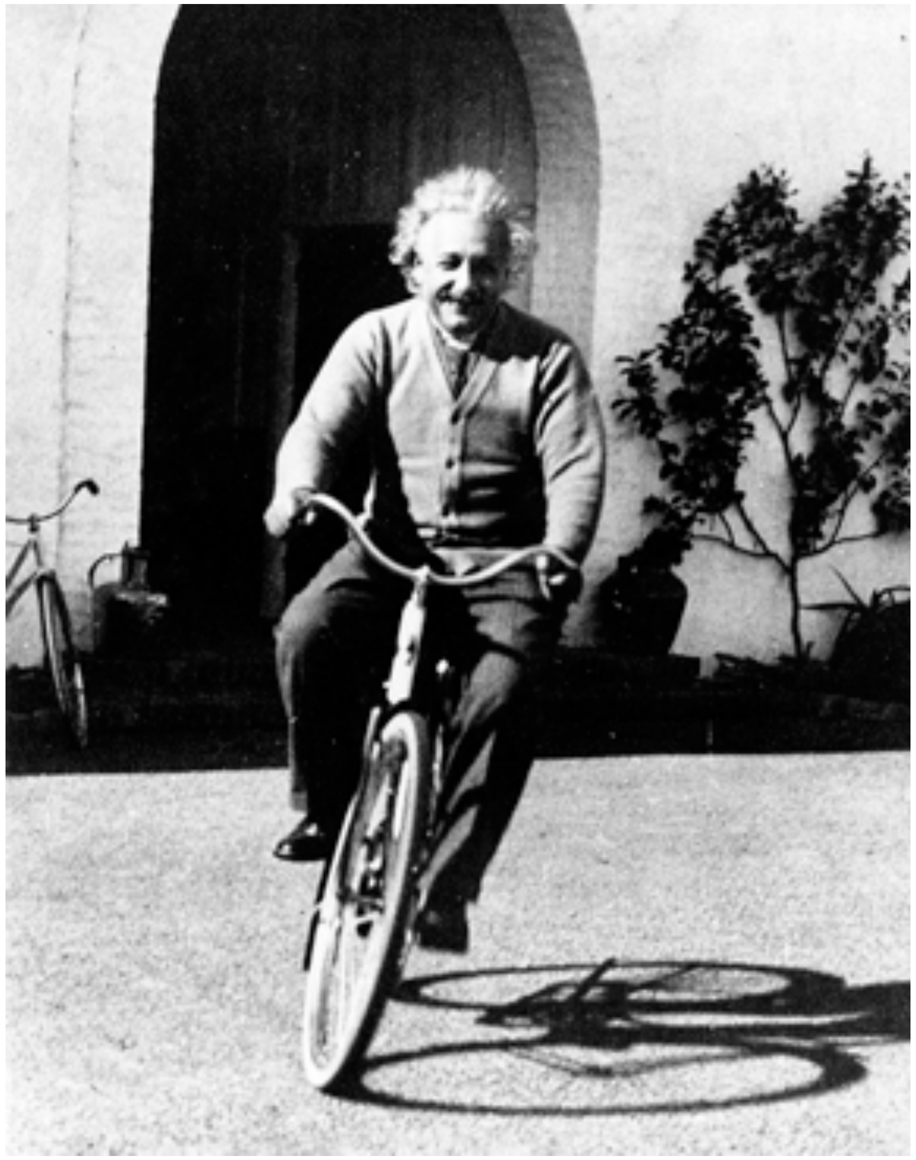
Lorentz Invariant ✓

Single Spin 2 ✗



Beyond Horndeski
Horndeski
Galileons
Sym-metron
DHOST/ EST
Beyond Generalized Proca
Generalized Proca
biGravity/ Multi-gravity
Warped MG
Minimally coupled to matter
Quintessence
DE is dynamical (new dof)
Non-minimally coupled to matter
Brans Dicke eg. f(R)
Eternal inflation
Graviton is spin-2, massless field
DE is not dynamical (no new dof)
Λ
Unique Λ
Tunneling dynamics
Landscape of vacua
Anthropic arguments
What is Dark Energy?
Backreaction
Graviton has mass (resonance)
Infinite extra dimensions
DGP
Graviton is not spin-2, massless field
Lorentz invariance is broken
Consistent IR modification of gravity
Cascading Gravity
Self-acceleration
Nonlocality
Ghost condensate
Filtering / degravitation mechanism
de Rham, Tolley 2008
Mass-Varying MG
Graviton has mass (local)
Quasi-dilaton
Partially Massless Gravity
Generalized Massive Gravity
Minimal Massive Gravity
Non-local massive gravity
CdR, 2018

Why is General Relativity so special?



1. GR is Diffeomorphism Invariant

i.e. it exhibits 4 local symmetries - General Coordinate Transformations

$$x^\mu \to x^\mu(x')$$

Every theory can be written in a coordinate invariant way, but there is usually a preferred system of coordinates/frame of reference

- in GR there is no preferred system in the absence of matter
  - in the presence of matter there is a preferred reference frame, e.g. the rest frame of the cosmic microwave background

2. In GR Gravity is described by the curvature of spacetime

Einsteins equations take the form:

Curvature of spacetime $\propto$ Energy Momentum Density

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

radius of curvature^2 $\propto$ 1/energy density

3. GR is locally Lorentz Invariant

Every geometry is locally Minkowski -

GR can be rewritten as spin-two perturbations around Minkowski

E.o.Ms for GR are Lorentz invariant to all orders

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

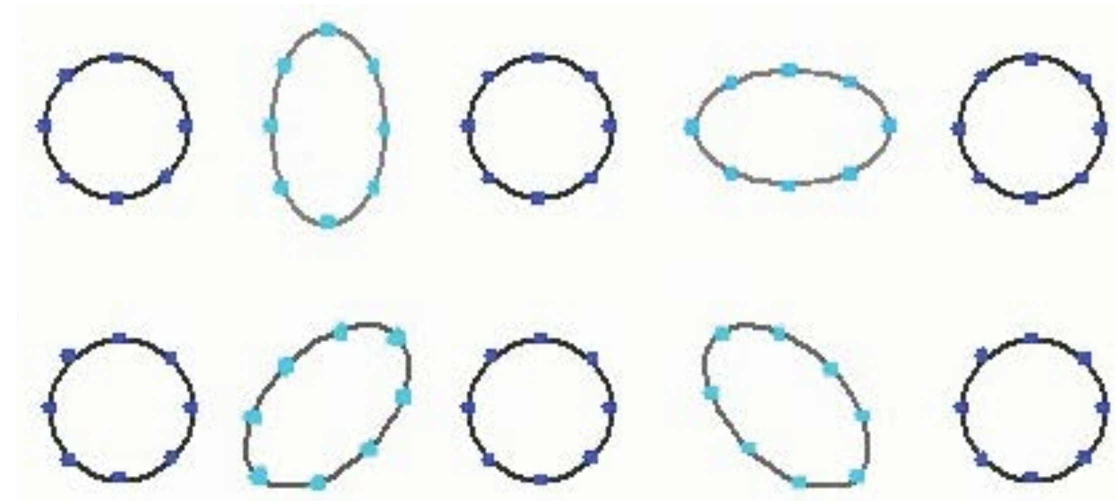
$$h_{\mu\nu} \sim R_{\mu\nu\alpha\beta}(x_P)(x^\alpha - x_P^\alpha)(x^\beta - x_P^\beta) + \mathcal{O}((x - x_P)^3)$$

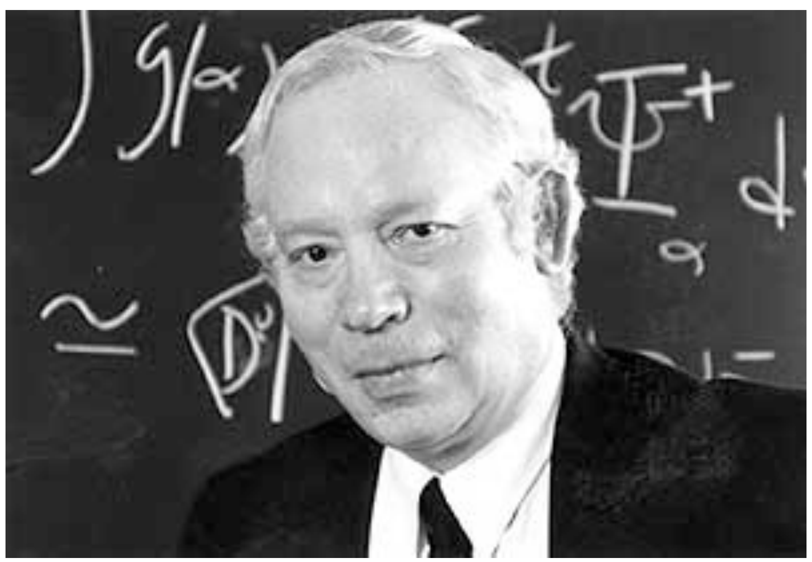
Essentially a different phrasing of the equivalence principle - ability to choice locally inertial frames

4. GR is unique theory of a massless spin-two field

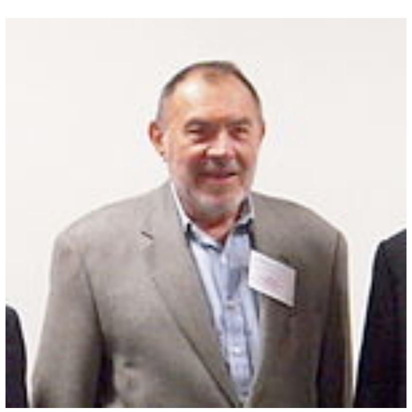
Metric perturbations transform as massless fields of spin 2!!

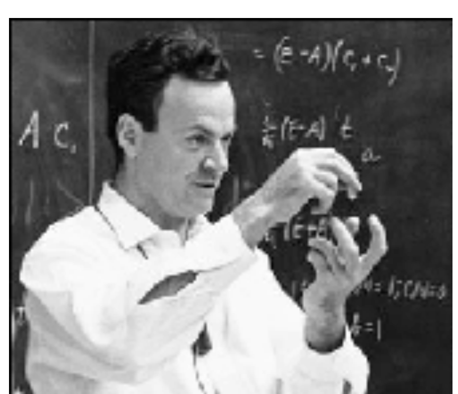
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$



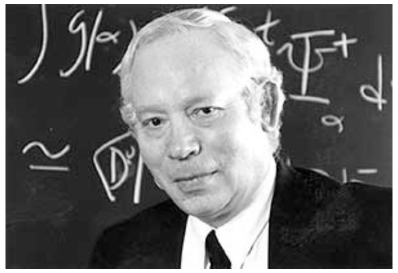


There are only two physical polarizations of gravitational waves!



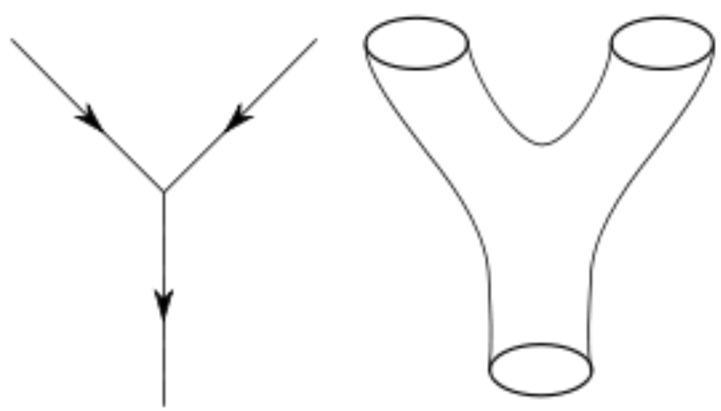


4. GR is unique theory of a massless spin-two field



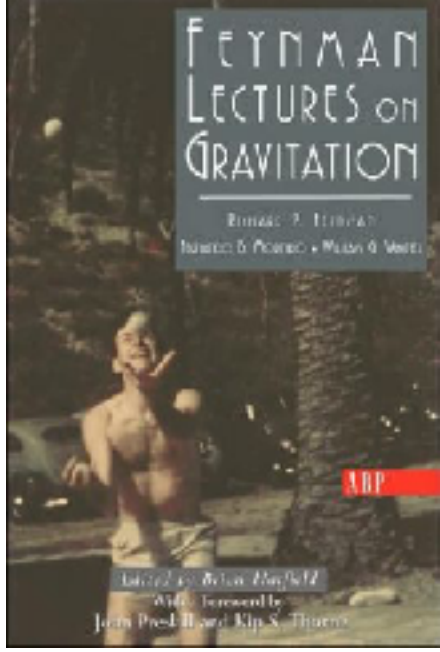
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

It is this argument that has played a more significant role in the particle physics community



String Theory is a theory of Quantum Gravity BECAUSE in the spectrum of oscillations of a quantized string is a m=0 s=2 state

Field theory approach to GR



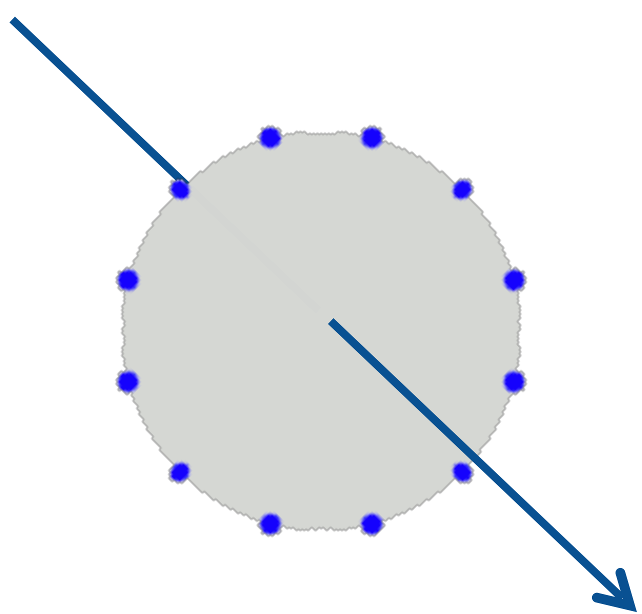
Gupta, Feynman, Weinberg, Deser, Boulware, Wald …

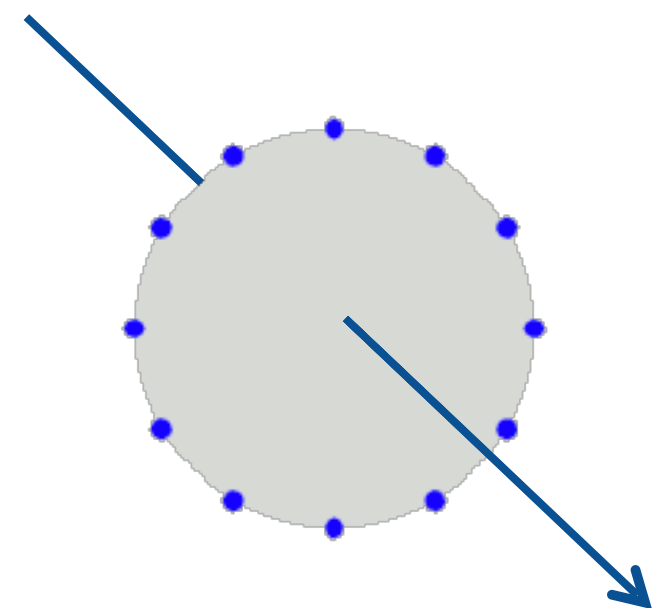
- Gravity is a force like EM propagated by a massless spin-2 particle
- GR (with a cosmological constant) is the unique Lorentz invariant low energy effective theory of a single massless spin 2 particle coupled to matter
- Diffeomorphism invariance is a derived concept
- Equivalence Principle is a derived concept (Weinberg ``Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass~1964)
- Form of action is derived by principles of LEEFT

General Relativity

$$S = \int \sqrt{-g}\frac{M_{\rm Pl}^2}{2} R$$

2 ‘tensors’ = helicity-2 modes





Sketch of proof

Spin 2 field is encoded in a 10 component symmetric tensor

$$h_{\mu\nu}$$

But physical degrees of freedom of a massless spin 2 field are
d.o.f. = 2

We need to subtract 8 = 2 x 4

This is achieved by introducing 4 local symmetries

Every symmetry removes one component since 1 is pure gauge and the other is fixed by associated first class constraint (Lagrangian counting)

Sketch of proof

Lorentz invariance demands that the 4 symmetries form a vector (there are only 2 possible distinct scalar symmetries) and so we are led to the **unique** possibility

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

We can call this linear Diff symmetry but its really just 4 U(1) symmetries, its sometimes called **spin 2 gauge invariance**

Quadratic action

Demanding that the action is **local** and starts at lowest order in derivatives (two), we are led to a unique quadratic action which respects linear diffs

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$$S = \int d^4x \frac{M_P^2}{8} h^{\mu\nu} \Box (h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}) + \dots$$

Where … are terms which vanish in de Donder/harmonic gauge. It has an elegant representation with the Levi-Civita symbols …..

$$S \propto \int d^4x \epsilon^{ABCD} \epsilon^{abcd} \eta_{aA} \partial_c h_{bB} \partial_C h_{dD}$$

Invariance of kinetic term

$$S = \int d^4x \; \varepsilon^{abcd} \varepsilon^{ABCD} \eta_{aA} h_{bB} \partial_c \partial_C h_{dD}$$

$$\delta S \sim 2\int d^4x \; \varepsilon^{abcd} \varepsilon^{ABCD} \eta_{aA} h_{bB} \partial_c \partial_C \delta h_{dD}$$

$$\delta h_{dD} = \partial_d X_D + d \leftrightarrow D$$

$$\delta S \sim 4\int d^4x \; \varepsilon^{abcd} \varepsilon^{ABCD} \eta_{aA} h_{bB} \partial_c \partial_C \partial_d X_D = 0$$

Nonlinear theory

In order to construct the nonlinear theory we must have a **nonlinear completion** of the linear Diff symmetry to ensure that nonlinearly the degrees of freedom are

$$10 - 2 \times 4 = 2$$

So the relevant question, and what all the proofs in effect rely on is, what are the nonlinear extensions of the symmetry which are consistent (i.e. form a group)

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Nonlinear theory

The nonlinear symmetry should preserve Lorentz invariance so

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

becomes schematically

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + h_\mu^\alpha h_\nu^\beta (\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha) + h^n (\partial h) \xi + h^m \partial \xi$$

$$+\text{higher derivatives}$$

but the form of the transformation is **strongly constrained** by the requirement that it forms a group

Unique result

Most complete proof Wald 1986

There are **only two** nonlinear extensions of the linear Diff symmetry, (assumption over number of derivatives)

1. Linear Diff -> Linear Diff

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

2. Linear Diff -> Full Diffeomorphism

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \xi^\omega \partial_\omega h_{\mu\nu} + g_{\mu\omega} \partial_\nu \xi^\omega + g_{\omega\nu} \partial_\mu \xi^\omega$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Metric emerges as derived concept

Punch Line

Spin 2 + Nonlinear Gauge Symmetry

$\downarrow$

Metric + Diffeomorphism Invariance

$\downarrow$

Geometry!!!!

Case 1: Coupling to matter

1. Linear Diff -> Linear Diff

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

The coupling to matter must respect this symmetry, e.g. if we consider

$$\int d^4x \frac{1}{2} h_{\mu\nu}(x) J^{\mu\nu}(x)$$ then we must have

performing transformation:

$$\int d^4x \partial_\mu \xi_\nu J^{\mu\nu} \longrightarrow \partial_\mu J^{\mu\nu}(x) = 0$$

Case 1: Coupling to matter

$$\int d^4x \frac{1}{2} h_{\mu\nu}(x) J^{\mu\nu}(x)$$ then we must have

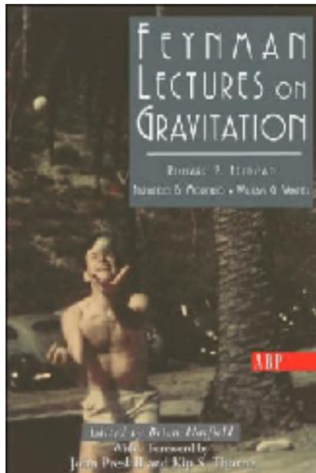
$$\partial_\mu J^{\mu\nu}(x) = 0$$

The problem is that this must hold as an IDENTITY!!

We cannot couple h to the stress energy of matter which is conserved in the absence of the coupling because as soon as we add the interaction, the equations of motion for matter are modified in such a way that the stress energy is no longer conserved

$$J^{\mu\nu} \neq T^{\mu\nu}$$

e.g. Feynman goes through expample of a point particle in his book …



Case 1: Non-gravitational spin 2 theory

$$\partial_\mu J^{\mu\nu}(x) = 0$$

An interacting theory does exist in case 1, by taking J to be identically conserved

Example: `Galileon combinations'

$$J^{\mu\nu} = \epsilon^{\mu abc}\epsilon^{\nu ABC} A_{aA} A'_{bB} A''_{cC}$$

where each entry is either

$$A_{aA} = \partial_a \partial_A \pi \text{ or } \eta_{aA}$$

Precisely these terms arise in the Decoupling Limit of Massive Gravity de Rham, Gabadadze 2010

Case 2: Coupling to matter

2. Linear Diff -> Full Diffeomorphism

$$h_{\mu\nu} \to h_{\mu\nu} + \xi^{\omega}\partial_{\omega}h_{\mu\nu} + g_{\mu\omega}\partial_{\nu}\xi^{\omega} + g_{\omega\nu}\partial_{\mu}\xi^{\omega}$$

The coupling to matter must respect this symmetry, but this is now easy, we just couple matter covariantly to

$$g_{\mu\nu}$$

any such coupling is perturbatively equivalent to

$$\int d^4x\, h_{\mu\nu}T^{\mu\nu}$$

and so is a theory of gravity!

Kinetic Terms

Case 1: Non-Gravitational Spin 2.

Since nonlinear symmetry is linear Diff, existing kinetic term is leading term at two derivative order (however there is a second term ….)

$$S \propto \int d^4x \epsilon^{ABCD} \epsilon^{abcd} \eta_{aA} \partial_c h_{bB} \partial_C h_{dD}$$

Zero in de Donder/harmonic gauge

$$\partial^\mu (h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h) = 0$$

$$S = \int d^4x \frac{M_P^2}{8} h^{\mu\nu} \Box (h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}) + \dots$$

Kinetic Terms

Case 2: Gravitational Spin 2

Since nonlinear symmetry is nonlinear Diff, kinetic term must be leading two derivative diffeomorphism invariant operator

$$S = \int d^4x \frac{M_P^2}{2} \sqrt{-g} R$$

HENCE GR!!!!

Basic Question

What happens if we repeat this arguments starting with the assumption of a **massive spin 2 field?**

i.e. suppose that the graviton is massive, are we inevitably led to the Einstein-Hilbert action (plus mass term)?

One argument says no

In a Massive theory of Gravity Diffeomorphism invariance is completely broken. Thus **superficially** it appears that everything that makes GR nice is completely lost

For instance, already at 2 derivative order we can imagine an infinite number of possible kinetic terms which are schematically

$$S = \int d^4x - \frac{M_P^2}{2} \left(\partial h \partial h + \cdots \sum \alpha_n h^{n-2} \partial h \partial h \right)$$

Fortunately this is wrong

If we really allowed for such a completely general form, then we would be at risk that all 10 components of metric are dynamical

$$\mathcal{L} = \frac{1}{2} h_{\mu\nu} \Box h^{\mu\nu} + \dots$$

Even if we ensure that $h_{0\mu}$ is not dynamical, we are at risk that the 6 remaining spatial components are dynamical

h_{ij} which is one two many

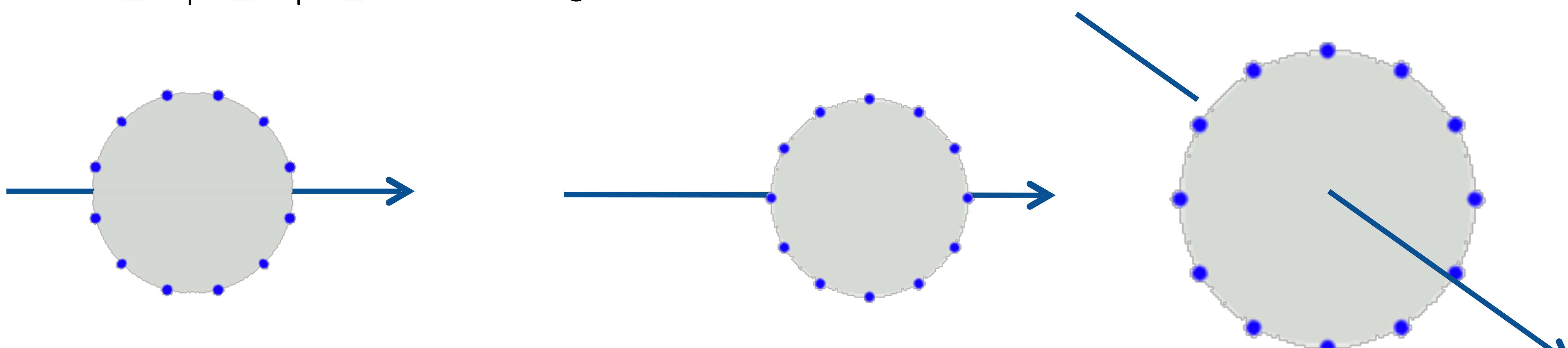
6 = 5 + Ostrogradski ghost

Lorenz Invariant Massive Gravity

$$S = \int \sqrt{-g}\frac{M_{\rm Pl}^2}{2}\left(R - \text{Mass Term}\right)$$

Constraint means only
one scalar propagates

$$2+2+2-1=5$$



2 ‘vectors’= helicity-1 modes

2 ‘scalars’= helicity-0 modes

5 propagating degrees of freedom
5 polarizations of gravitational waves!!!!

A toy example, Proca theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}m^2 A_\mu A^\mu$$

Unitary gauge formulation for massive spin-1 particle

In canonical/phase space form...

$$\mathcal{L} = \pi^i \partial_0 A_i + A_0 \partial^i \pi_i - \frac{1}{2}\pi_i^2 - \frac{1}{4}F_{ij}^2 \quad -\frac{1}{2}m^2 A_i^2 + \frac{1}{2}m^2 A_0^2$$

A toy example, Proca theory

$$\mathcal{L} = \pi^i \partial_0 A_i + A_0 \partial^i \pi_i - \frac{1}{2}\pi_i^2 - \frac{1}{4}F_{ij}^2 \quad -\frac{1}{2}m^2 A_i^2 + \frac{1}{2}m^2 A_0^2$$

In massless case A_0 is a Lagrange multiplier for a **1st class constraint**

$$\partial_i \pi^i = 0$$

1st class $\longleftrightarrow$ gauge symmetry

In massive case A_0 is a non-dynamical/auxiliary field

2nd class constraint $\qquad \pi_{A_0} = \dfrac{\partial \mathcal{L}}{\partial \partial_0 A_0} = 0$

2nd class constraints come in pairs in L.I. theory

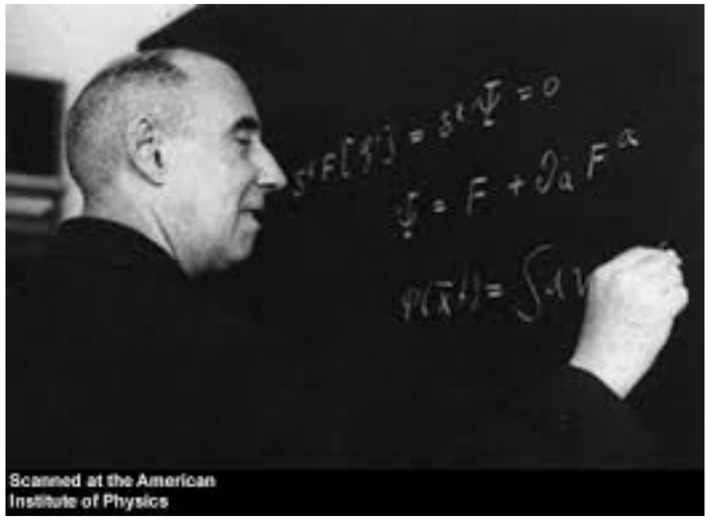
Upgrading from Second to First Class

In many systems it is more natural to formulate a system with two second class constraints as a system with one first class constraint

$$2 \times 1 \text{ second class} = 1 \text{ local symmetry } =$$
$$1 \text{ first class constraint} + 1 \text{ gauge choice}$$

Unitary gauge (Proca) picture emphasises second class form
Stuckelberg picture emphasises first class form

Stuckelberg picture



All of this is much easier to understand in the Stuckelberg picture in which reintroduce gauge invariance

$$A_\mu \to A_\mu + \partial_\mu \chi$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}m^2(A_\mu + \partial_\mu \chi)^2$$

Massive theory is now gauge invariant

$$A_\mu \to A_\mu + \partial_\mu \xi \, , \; \chi \to \chi - \xi$$

Therefore number of degrees of freedom are

2 A_μ + 1 χ

Proca theory + non-minimal kinetic term

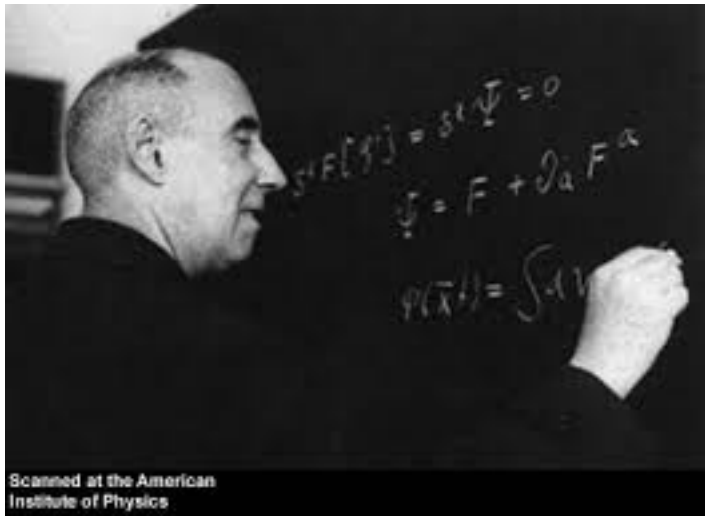
For a massive spin 1 field, we break gauge invariance, so we may think that we can allow non-gauge invariant kinetic terms of the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \alpha(\partial_\mu A^\mu)^2$$

However this would lead to 4 propagating degrees of freedom, instead of 2s+1 = 3

The key point is that A^0 must remain non-dynamical to impose a second class constraint

Stuckelberg picture



Again this is much easier to understand in the Stuckelberg picture

$$A_\mu \to A_\mu + \partial_\mu \chi$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \alpha(\Box\chi + \partial_\mu A^\mu)^2$$

Massive theory is now gauge invariant

$$A_\mu \to A_\mu + \partial_\mu \xi \,, \; \chi \to \chi - \xi$$

But is now clearly higher derivative for χ

Therefore number of degrees of freedom are

2 A_μ + 1 χ+ 1 χ Ostrogradski

Now to massive spin 2

The general principle is the same in the spin 2 case

Although the massive theory breaks the 4 nonlinear gauge symmetries, we still need that at least one second class constraint to ensure 5 degrees of freedom

Equivalently, if we Stuckelberg back the symmetries of the massless theory then we must demand that the Stuckelberg fields do not admit Ostrogradski instabilities

However, how we do this depends on whether we are looking at non-gravitational (SPIN 2 MESONS) or gravitational spin 2 fields (GRAVITONS)

Case 1. Non-gravitational massive spin 2

In this case we should Stuckelberg the linear Diff symmetry

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

If we choose the massless kinetic term, Stuckelberg fields do not enter

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \chi_\nu + \partial_\nu \chi_\mu$$

There is a unique quadratic mass term

$$\mathcal{L}^{\text{unitary}}_{\text{mass}} \sim m^2 \, \varepsilon^{abcd} \, \varepsilon^{ABCD} \, \eta_{aA} \, \eta_{bB} \, h_{cC} \, h_{dD}$$

Reason: Part quadratic in X

$$\sim m^2 \varepsilon^{abcd} \varepsilon^{ABCD} \eta_{aA} \eta_{bB} \underbrace{\partial_c X_d}_{\text{antisymmetric}} \underbrace{\partial_C X_D}$$

$$X_\mu = \frac{1}{m} A_\mu$$

$$\sim F_{\mu\nu}^2 \; !$$

Kinetic term for helicity 1.

Part linear in X

$$\sim m^2 \varepsilon \varepsilon \eta \eta h \, \partial X$$

$$\sim m \, \varepsilon^{abcd} \varepsilon^{ABCD} \eta_{aA} \eta_{bB} h_{cC} \underbrace{\partial_d A_D}_{\text{Not U(1) gauge invariant}}$$

$$A_\mu \to A_\mu + \underbrace{\frac{1}{m} \partial_\mu \pi}_{\text{helicity zero mode}}$$

Final term

$$\sim \varepsilon\,\varepsilon\,\eta\,\eta\,h\,\partial\,\partial\,\Pi$$

$$\equiv \varepsilon\,\varepsilon\,\eta\,\eta\,\underbrace{\Pi\,\partial\,\partial\,h}_{\text{Mixing term}}$$

Removed with de-mixing $\quad h_{\mu\nu} \to h_{\mu\nu} + \#\,\Pi\,\eta_{\mu\nu}$

Summary $\equiv$ Fierz-Pauli

Unitary gauge

$$\mathcal{L} = \varepsilon\,\varepsilon\,\eta\, h\,\partial\partial h + m^2 \varepsilon\,\varepsilon\,\eta\,\eta\, h\, h$$

Stuckelberg

$$h_{\mu\nu} \to \underbrace{h_{\mu\nu}}_{\text{helicity } 2} + \underbrace{\frac{\partial_\mu A_\nu}{m} + \frac{\partial_\nu A_\mu}{m}}_{\text{helicity } 1} + \underbrace{\frac{\partial_\mu \partial_\nu \pi}{m^2}}_{\text{helicity } 0}$$

Case 1. Non-gravitational massive spin 2 - kinetic term

Remarkably there is a unique extension to the kinetic term already at two derivative level which is cubic

Hinterbichler 2013
Folkerts, Pritzel, Wintergerst 2011

$$S_{(3)} = \int d^4x \epsilon^{ABCD} \epsilon^{abcd} h_{aA} \partial_c h_{bB} \partial_C h_{dD}$$

Thus for Case 1 theories, linearized E-H kinetic term, i.e. Fierz-Pauli kinetic term is not unique!!!

Note this is NOT a limit of a Lovelock term as seen by counting derivatives

Case 2. Gravitational massive spin 2

In this case we should Stuckelberg the nonlinear Diff symmetry

$$h_{\mu\nu} \to h_{\mu\nu} + \xi^\omega \partial_\omega h_{\mu\nu} + g_{\mu\omega}\partial_\nu \xi^\omega + g_{\omega\nu}\partial_\mu \xi^\omega$$

This is done explicitly by replacing h with a tensor

$$h_{\mu\nu} = g_{\mu\nu} - \partial_\mu \phi^a \partial_\nu \phi^b \eta_{ab} \qquad \phi^a = x^a + \frac{A^a}{mM_P} + \frac{\partial^a \pi}{m^2 M_P}$$

In this case we are led (after much calculation) to a unique kinetic term in four dimensions (up to total derivatives), i.e. Einstein-Hilbert kinetic term

$$S = \int d^4x \frac{M_P^2}{2}\sqrt{-g}R$$

Case 2. Gravitational massive spin 2

de Rham, Matas, Tolley,
``New Kinetic Interactions for Massive Gravity?,''
1311.6485

I'm leaving out all the details of the proof which is complicated but what it means is there is no `graviton' analogue of the spin-2 meson kinetic term ….

$$S_{(3)} = \int d^4x \epsilon^{ABCD} \epsilon^{abcd} h_{aA} \partial_c h_{bB} \partial_C h_{dD}$$

$$S = \int d^4x \frac{M_P^2}{2} \sqrt{-g} R$$

Thus **all of the key features of Einstein gravity** emerge equally from the assumption that the graviton is massive even though Diffeomorphism invariance is strictly broken

Coupled with the uniqueness of the mass terms this means the theory of a massive spin 2 particle is unique!

de Rham, Gabadadze, Tolley (2010)

This is remarkable!

SOFT AND HARD MASSIVE GRAVITY

Massive Gravity: Hard or Soft?

Hard

A generic local, Lorentz invariant theory at the linearized level gives the following interaction between two stress energies

$$A \sim \frac{1}{M_{\rm Pl}^2}\int \frac{d^4k}{(2\pi)^4}\, T^{ab}(k)^* \left[\frac{P_{abcd}}{k^2} + \sum_{\rm pole} Z^{(2)}_{\rm pole}\frac{\mathcal{P}_{abcd}}{k^2+m^2_{\rm pole}} + \sum_{\rm pole} Z^{(0)}_{\rm pole}\frac{\eta_{ab}\eta_{cd}}{k^2+m^2_{\rm pole}}\right] T^{cd}(k)$$

$$+\frac{1}{M_{\rm Pl}^2}\int \frac{d^4k}{(2\pi)^4}\, T^{ab}(k)^* \left[\int d\mu\, \rho^{(2)}(\mu)\frac{\mathcal{P}_{abcd}}{k^2+\mu^2} + \rho^{(0)}(\mu)\frac{\eta_{ab}\eta_{cd}}{k^2+\mu^2}\right] T^{cd}(k)$$

Soft

$$P_{abcd} = \eta_{ac}\eta_{bd} + \eta_{bc}\eta_{ad} - \eta_{ab}\eta_{cd}$$

$$\mathcal{P}_{abcd} = \eta_{ac}\eta_{bd} + \eta_{bc}\eta_{ad} - \frac{2}{3}\eta_{ab}\eta_{cd}$$

Soft Massive Graviton is a **resonance**
Hard Massive Graviton is a **pole** (infinite lifetime)

Generating Function

$$e^{iW[J]} = \int D\phi \; e^{iS(\phi) + i\int d^4x\, J\phi}$$

eg

$$S(\phi) = \int d^4x\, \frac{1}{2}\phi\left[\Box - m^2\right]\phi$$

$$W[J] = \frac{1}{2}\int d^4x \;\; J(x)\left[-\frac{1}{[\Box - m^2]}\right]J(x)$$

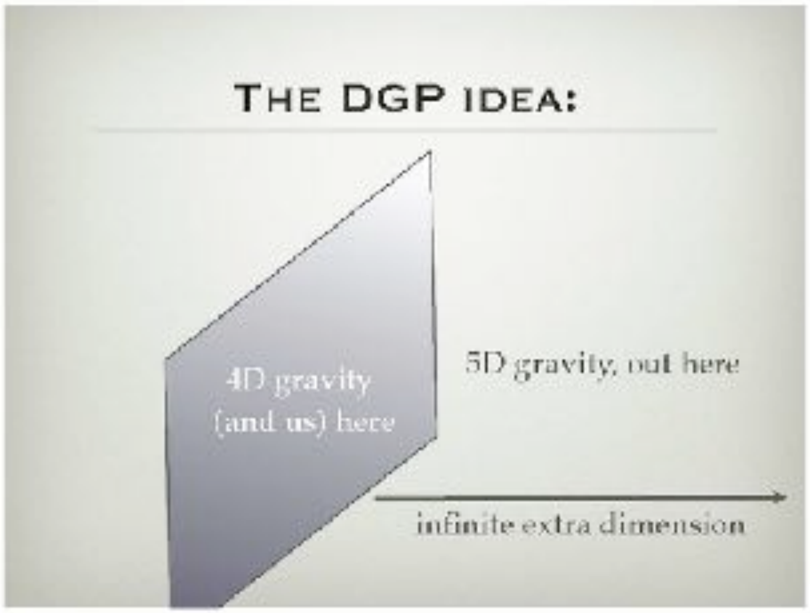
Generating Function

$$W[J] = \int \frac{d^4k}{(2\pi)^4} J(k)^* \left(\frac{1}{k^2+m^2}\right) J(k)$$

$$\Downarrow$$

$$\int_0^\infty d\mu \quad \rho(\mu) \int \frac{d^4k}{(2\pi)^4} J(k)^* \frac{1}{(\mu+k^2)} J(k)$$

Soft Massive Gravity: DGP Model



Soft Massive Gravity theories were constructed first! Naturally arise in Braneworld Models: **DGP**, **Cascading Gravity**: Soft Massive Graviton is a <u>Resonance State</u> localized on Brane

$$\Delta S \sim \frac{1}{M^2_{\text{Planck}}} \int \frac{d^4k}{(2\pi)^4} T^{\mu\nu}(k) \left[\int_0^\infty d\mu \frac{\rho(\mu) P_{\mu\nu\alpha\beta}(k)}{k^2+\mu} \right] T^{\alpha\beta}(k)$$

Soft

More irrelevant

More relevant

$$S = \int d^4x \sqrt{-g_4} \frac{M_4^2}{2} R_4 + \int d^4x \sqrt{-g_4} \mathcal{L}_M + \int d^5x \sqrt{-g_5} \frac{M_5^3}{2} R_5$$

Dominates in UV

Dominates in IR

Gravity in Higher Dimensions

In *4+n* dimensional spacetime, gravitational potential scales as

$$V(r) \sim \frac{1}{r^{1+n}}$$

weaker gravity

we want to achieve this in the IR

$$V(r) \sim \frac{1}{r} \longrightarrow V(r) \sim \frac{1}{r^{1+n}}$$

UV, small r

IR, large r

Gravity in Higher Dimensions

Form of potential

$$V(r) = \int_0^\infty ds^2 \rho(s^2) \frac{e^{-sr}}{r}$$

corresponds to propagator

$$G_F(k) = \int_0^\infty ds^2 \rho(s^2) \frac{1}{k^2 + s^2 - i\epsilon} = \frac{1}{k^2 + m^2(k) - i\epsilon}$$

for DGP $\qquad m^2(k) \propto \sqrt{-k^2}$

DISCOVERY OF HARD MASSIVE GRAVITY

What does massive gravity mean?

In SM, Electroweak symmetry
is spontaneously broken by the VEV of the Higgs field

$$SU(2) \times U(1)_{\rm Y} \to U(1)_{\rm EM}$$

Result, W and Z bosons become massive

Would-be-Goldstone-mode in Higgs field becomes
Stuckelberg field which gives boson mass

Higgs Vev Higgs Boson Stuckelberg field

$$\Phi = (v + \rho)e^{i\pi}$$

e.g. for Abelian Higgs

$$A_\mu \to A_\mu + \partial_\mu \chi \qquad \rho \to \rho + \chi$$

Symmetry Breaking Pattern

In **Massive Gravity** - Local Diffeomorphism Group and an additional global Poincare group is broken down the diagonal subgroup

$$Diff(M) \times \text{Poincare} \to \text{Poincare}_{\text{diagonal}}$$

In **Bigravity** - Two copies of local Diffeomorphism Group are broken down to a single copy of Diff group

$$Diff(M) \times Diff(M) \to Diff(M)_{\text{diagonal}}$$

Higgs for Gravity

Despite much *blood, sweat and tears* an explicit Higgs mechanism for gravity is not known

However if such a mechanism exists, we DO know how to write down the low energy effective theory in the spontaneously broken phase

For Abelian Higgs this corresponds to integrating out the Higgs boson and working at energy scales lower that the mass of the Higgs boson

Higgs Boson

Stuckelberg field

$$E \ll m_\rho \qquad \Phi = (v+\rho)e^{i\pi}$$

Stuckelberg formulation of massive vector bosons

Stuckelberg Formulation for Massive Gravity

Arkani-Hamed et al 2002
de Rham, Gabadadze 2009

Diffeomorphism invariance is spontaneously broken but maintained by introducing Stueckelberg fields

Vev of spin 2 Higgs field defines a 'reference metric'

$$f_{\mu\nu} = \langle \hat{O}_{\mu\nu} \rangle$$

reference metric

Stuckelberg fields

Dynamical Metric

$$g_{\mu\nu}(x)$$

$$F_{\mu\nu} = f_{AB}(\phi)\partial_\mu \phi^A \partial_\nu \phi^B$$

helicity-1 mode of graviton

$$\phi^a = x^a + \frac{1}{mM_P}A^a + \frac{1}{\Lambda^3}\partial^a \pi$$

$$\Lambda^3 = m^2 M_P$$

helicity-0 mode of graviton

Discovering how to square root

$$F_{\mu\nu} = f_{AB}(\phi)\partial_\mu \phi^A \partial_\nu \phi^B$$

$$\phi^a = x^a + \frac{1}{mM_P} A^a + \frac{1}{\Lambda^3}\partial^a \pi$$

Helicity zero mode enters reference metric **squared**

$$F_{\mu\nu} \approx \eta_{\mu\nu} + \frac{2}{\Lambda^3}\partial_\mu\partial_\nu\pi + \frac{1}{\Lambda^6}\partial_\mu\partial_\alpha\pi\partial^\alpha\partial_\nu\pi$$

To extract dominant helicity zero interactions we need to take a square root

$$\left[\sqrt{g^{-1}F}\right]_{\mu\nu} \approx \eta_{\mu\nu} + \frac{1}{\Lambda^3}\partial_\mu\partial_\nu\pi$$

Branch uniquely chosen to give rise to 1 when Minkowski

Helicity Zero mode = Galileon

The helicity zero mode $\pi(x)$ only enters in the combination

$$\Pi_{\mu\nu} = \partial_\mu \partial_\nu \pi(x)$$

This is invariant under the
global nonlinearly realized symmetry

$$\pi(x) \to \pi(x) + c + v_\mu x^\mu$$

$$\Pi_{\mu\nu} \to \Pi_{\mu\nu}$$

Whats a Galileon?

A Galileon is a theory nonlinearly realizes the 'non-relativistic' limit (Wigner-Inonu contraction) of the Five Dimensional Poincare group that preserves Four Dimensional Poincare

de Rham, AJT 2010 - DBI and Galileon Reunited

Example - a scalar field $\pi(x)$ with the symmetry

$$\pi(x) \to \pi(x) + c + \boxed{v_\mu x^\mu}$$

Nicolis, Rattazzi, Trincherini 2009

Exact Galileon Symmetry

If we view a Galileon as a scalar theory then expect symmetry to broken by gravity - e.g. covariant Galileon does not respect symmetry and is therefore not really a covariant Galileon

But Galileons naturally arise in **Massive theories of Gravity**, like DGP and dRGT massive gravity, Bigravity and Multi-Gravity

Here the symmetry remains **exact** with gravity including quantum corrections because the scalar is not a scalar but is actually part of a massive spin-two field

Galilean Operators

$$\mathcal{L}_2 = \Pi \; \varepsilon \; \varepsilon \; \eta \; \eta \; \eta \; \partial\partial\Pi$$

$$\mathcal{L}_3 = \Pi \; \varepsilon \; \varepsilon \; \eta \; \eta \; \partial\partial\Pi \; \partial\partial\Pi$$

$$\mathcal{L}_4 = \Pi \; \varepsilon \; \varepsilon \; \eta \; \partial\partial\Pi \; \partial\partial\Pi \; \partial\partial\Pi$$

$$\mathcal{L}_5 = \Pi \; \varepsilon \; \varepsilon \; \partial\partial\Pi \; \partial\partial\Pi \; \partial\partial\Pi \; \partial\partial\Pi$$

Galileon - helicity 2 interactions

$$\mathcal{L}_{2}' = \varepsilon\ \varepsilon\ \eta\ \eta\ \partial\partial\pi\ h$$

$$\mathcal{L}_{3}' = \varepsilon\ \varepsilon\ \eta\ \partial\partial\pi\ \partial\partial\pi\ h$$

$$\mathcal{L}_{4}' = \varepsilon\ \varepsilon\ \partial\partial\pi\ \partial\partial\pi\ \partial\partial\pi\ h$$

'Characteristic polynomials'

$$\det\left(\alpha h + \beta\,\partial\partial\pi + \gamma\,\eta\right)$$

Discovering how to square root

$$F_{\mu\nu} = f_{AB}(\phi)\partial_\mu \phi^A \partial_\nu \phi^B$$

$$\phi^a = x^a + \frac{1}{mM_P} A^a + \frac{1}{\Lambda^3}\partial^a \pi$$

Helicity zero mode enters reference metric **squared**

$$F_{\mu\nu} \approx \eta_{\mu\nu} + \frac{2}{\Lambda^3}\partial_\mu \partial_\nu \pi + \frac{1}{\Lambda^6}\partial_\mu \partial_\alpha \pi \partial^\alpha \partial_\nu \pi$$

To extract dominant helicity zero interactions we need to take a square root

$$\left[\sqrt{g^{-1}F}\right]_{\mu\nu} \approx \eta_{\mu\nu} + \frac{1}{\Lambda^3}\partial_\mu \partial_\nu \pi$$

Branch uniquely chosen to give rise to 1 when Minkowski de Rham, Gabadadze, AJT 2010

Hard Λ_3 Massive Gravity

$$Diff(M) \times \text{Poincare} \to \text{Poincare}_{\text{diagonal}}$$

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(M_P^2\, R[g] - m^2 \sum_{n=0}^{4} \beta_n\, \mathcal{U}_n \right) + \mathcal{L}_M$$

$$K = 1 - \sqrt{g^{-1} f}$$

$$\text{Det}[1 + \lambda K] = \sum_{n=0}^{d} \lambda^n \mathcal{U}_n(K)$$

Characteristic Polynomials

Double epsilon structure!!!!!

Unique low energy EFT where the strong coupling scale is

$$\Lambda_3 = (m^2 M_P)^{1/3}$$

5 propagating degrees of freedom

5 polarizations of gravitational waves!!!!

Stuckelberg Formulation for Bigravity

Fasiello, AJT 1308.1647

$$Diff(M) \times Diff(M) \rightarrow Diff(M)_{\text{diagonal}}$$

Bigravity breaks the same amount of symmetry as massive gravity, need to introduce same number of Stuckelberg fields

Dynamical metric I

$$g_{\mu\nu}(x)$$

Dynamical metric II

$$F_{\mu\nu} = f_{AB}(\phi)\partial_\mu\phi^A\partial_\nu\phi^B$$

$$\phi^A = x^A + \frac{1}{\Lambda_3^3}\partial^A\pi$$

$=$

Fasiello, AJT 1308.1647

But there are two ways to introduce Stuckelberg fields!

Dynamical metric I

$$g_{\mu\nu}(x)$$

Dynamical metric II

$$F_{\mu\nu} = f_{AB}(\phi)\partial_\mu\phi^A\partial_\nu\phi^B$$

$$\tilde{x}^A = \phi^A(x) = x^A + \partial^A\boxed{\pi(x)}$$

OR

Dynamical metric I

$$\tilde{G}_{AB}(\tilde{x}) = g_{\mu\nu}(Z)\partial_A Z^\mu \partial_B Z^\nu$$

Dynamical metric II

$$f_{AB}(\tilde{x})$$

$$x^\mu = Z^\mu(\tilde{x}) = \tilde{x}^\mu + \partial^\mu\boxed{\tilde{\pi}(\tilde{x})}$$

Galileon Duality!!!!!

Hassan, Rosen 2011

Hard Massless plus Λ_3 Massive Gravity

$$\mathcal{L} = \frac{1}{2}\left(M_P^2 \sqrt{-g} R[g] + M_f^2 \sqrt{-f} R[f] - m^2 \sum_{n=0}^{d} \beta_n U_n(K) \right) + \mathcal{L}_M$$

$$\mathrm{Det}[1+\lambda K] = \sum_{n=0}^{d} \lambda^n \mathcal{U}_n(K)$$

$$K = 1 - \sqrt{g^{-1} f}$$

decoupling limit

$M_f \to \infty$

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left(M_P^2 R[g] - m^2 \sum_{n=0}^{4} \beta_n \mathcal{U}_n \right) + \mathcal{L}_M$$

Bigravity=
massless graviton (2 d.o.f.)
+ massive graviton (5 d.o.f.)

+decoupled massless graviton $f_{\mu\nu}$