Why are reversible septenary NCCAs so simple? A preliminary inquiry

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Abstract

Little is known about the dynamics of k-ary (binary, ternary, quaternary, quinary, etc.) reversible number-conserving cellular automata. Here, we present some preliminary results in the case of seven states. We examine one of the most complex seven-state reversible and number-conserving rules and provide a full description of its dynamics.

Keywords: multi-state cellular automata, number conservation, reversibility

1 Introduction

A popular class of multi-state cellular automata (CAs) is the class of so-called k-ary CAs, which have $\{0, 1, 2, \ldots, k-1\}$ as state set, for some natural number k greater than 1. Since each state is a nonnegative integer, it can, for instance, be interpreted as the number of particles occupying a given cell and for this reason, these CAs are readily used for modeling various physical phenomena. Unfortunately, for the same reason, we cannot equip the state set $\{0, 1, 2, \ldots, k-1\}$ with a nice mathematical structure as, for example, in the case of the ring $\mathbb{Z}_k$, to resort to methods that have been developed so far.

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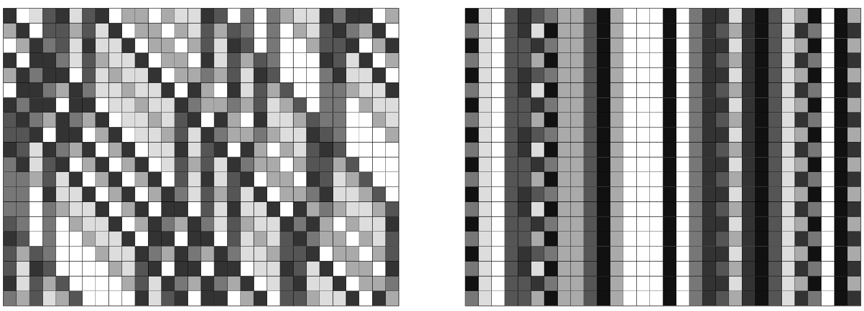
When physical phenomena governed by some conservation law (for example, of mass or energy) are simulated, a special subclass of CAs is used: number-conserving CAs (NCCAs), *i.e.* CAs that preserve the sum of the states of all the cells upon every update (see, for example, [1]). The second very desirable property of CAs used for modeling physical phenomena is reversibility, which ensures preservation of information. As a result, from the modeling point of view, k-ary CAs that are both number-conserving and reversible are the most interesting ones.

The dynamics of reversible k-ary NCCAs has been very little studied, even in the one-dimensional case. However, when the smallest possible radius of the neighborhood is considered, *i.e.* radius 1/2, it is known that all reversible k-ary NCCAs can be seen as shift-identity product cellular automata (see [2] for details). Such CAs, of course, have very simple dynamics and therefore their computing ability is seriously limited. On the other hand, it has been shown that when the radius is increased to 3/2, then it is possible to find a reversible k-ary NCCA that is computationally universal (see [3]).

The study of the dynamics of reversible k-ary NCCAs with radius 1 has gained momentum in recent years, in particular thanks to establishment of complete lists of such CAs for $k \in \{5, 6, 7\}$ (previously, complete lists were known only for $k \leq 4$). The investigation carried revealed, *inter alia*, that for $k = 7$ all reversible k-ary NCCAs (septenary NCCAs), except for shifts, have a finite order. Moreover, each of them repeats each configuration in a 60-cycle (see [4]). Hence, their dynamics is definitely simpler than that of reversible k-ary NCCAs for $k = 6$, as most of the latter (306 out of 471) do not have a finite order. For example, this means that in a particle interpretation of a reversible septenary NCCA, each particle moves in a limited space, while for reversible senary NCCAs the particles can move arbitrarily far (see Figure 1a and Figure 1b to compare the dynamics of reversible senary and septenary NCCAs).



(a) (b)

Figure 1: A sample space-time diagram of a reversible senary (a) and a reversible septenary (b) NCCA.

It is not known why the reversible septenary NCCAs are so simple, but intuition says that it has to do with the fact that 7 is a prime number. Indeed, if k is a composite number, then there exist methods (see, for example, [4]) that allow to construct reversible k-ary NCCAs with radius 1 having infinite order (other than shifts). Hence, such a simple dynamics can happen only for prime k. But does it happen for every prime k? In an attempt to answer this question, we initiated the study of reversible septenary NCCAs and present our first observations.

2 Preliminaries

2.1 Basic definitions

We decided to only sketch the notations needed, as we hope that the readers are familiar with cellular automata.

Let k be some natural number greater than 1 and let $[0..k)$ denote the set $\{0, 1, 2, \ldots, k-1\}$. A one-dimensional k-ary CA with radius one is defined by a function $f : [0..k)^3 \to [0..k)$, referred as the local rule. As cell space we will consider regular linear grids $\mathcal{C}_N = \{0, 1, \ldots, N-1\}$ with periodic boundary conditions.

A configuration is any mapping from the grid $\mathcal{C}_N$ to $[0..k)$. The set of all possible configurations on the grid C_N is denoted by X_N and is identified with $[0..k)^N$. For a given configuration $\mathbf{x} \in X_N$, the value of cell $n \in \mathcal{C}_N$ is denoted by x_n and the sum of the states in $\mathbf{x}$ is denoted by $\#(\mathbf{x}) = \sum_{n \in \mathcal{C}_N} x_n$. The set of all finite configurations is denoted by X^*, *i.e.*,

$$X^* = \bigcup_{N=1}^{\infty} X_N = \bigcup_{N=1}^{\infty} [0..k)^N .$$

A given local rule f generates a global rule $F : X^* \to X^*$ (which we identify with the cellular automaton) in the usual way: if $\mathbf{x} \in X_N$, then $F(\mathbf{x}) \in X_N$ and $F(\mathbf{x})_n = f(x_{n-1}, x_n, x_{n+1})$, where, in view of periodic boundary conditions, all operations on the indices are performed modulo N.

In our investigation, we focus on k-ary CAs that have two important (from the point of view of applications) properties. The first one is number conservation, which means that the sum of all states in any configuration remains constant throughout the evolution of the automaton.

Definition 2.1. *A global rule $F : X^* \to X^*$ is number conserving if for all $\mathbf{x} \in X^*$, it holds that $\#(F(\mathbf{x})) = \#(\mathbf{x})$.*

The second property of k-ary CAs we are interested in is reversibility. Since X^* is a disjunctive sum of finite sets X_N, bijectivity of F is equivalent to injectivity. We thus use this property as a definition.

Definition 2.2. *A global rule $F : X^* \to X^*$ is reversible if F is an injection,* i.e., *for any $\mathbf{x}_1, \mathbf{x}_2 \in X^*$ such that $\mathbf{x}_1 \neq \mathbf{x}_2$, it holds that $F(\mathbf{x}_1) \neq F(\mathbf{x}_2)$.*

Of course, for any $k > 1$ there are at least three global rules that are both number conserving and reversible: the identity rule, the left-shift rule and the right-shift rule. We will call them *trivial*. For $k < 4$, there is no other reversible k-ary NCCA (even in a multi-dimensional case – see [5] and [6]). However, for $k \geq 4$, there are some non-trivial ones. Until advances were made recently, the complete lists of reversible k-ary NCCAs were known only for $k \leq 4$. First, the theory introduced in [7] made it possible to find all quinary NCCAs ($k = 5$) and then check which of them are reversible. Next, using the method described in [4] it was possible to enumerate all reversible senary and septenary NCCAs ($k = 6$ and $k = 7$), without listing all the NCCAs first. The cardinality of the obtained sets are shown in Table 1 and the mentioned lists can be found in the dataset [8].

k	all k-ary CAs	k-ary NCCAs	reversible k-ary CAs	reversible k-ary NCCAs
2	$2^{2^3} = 256$	5	6	3
3	$3^{3^3} \approx 7.6 \cdot 10^{12}$	144	1800	3
4	$4^{4^3} \approx 3.4 \cdot 10^{38}$	89 588	?	21
5	$5^{5^3} \approx 2.4 \cdot 10^{87}$	1 876 088 314	?	21
6	$6^{6^3} \approx 1.2 \cdot 10^{168}$	?	?	471
7	$7^{7^3} \approx 7.4 \cdot 10^{289}$	?	?	1669

Table 1: Numbers of specific types of one-dimensional k-ary CAs with radius 1.

The following definition is formulated in the language of group theory.

Definition 2.3. *Let F be a global rule. If there exists a natural number m such that the function F^m is the identity on X^*, then we say that F has finite order and we define the order of F as the smallest natural number with this property.*

If a global rule F has a finite order, then it is not very interesting for applications as it repeats each configuration in an m-cycle, where m is the order of F. In particular, such a CA cannot be computationally universal (in any reasonable sense).

2.2 The description of reversible quinary NCCAs

Although our investigation concerns reversible septenary NCCAs, we start by recalling the description of the dynamics of reversible quinary NCCAs (since $k = 5$ is also a prime number). The full description of all 21 such CAs is given in [4], but we decided to present it here for two reasons: for the readers' convenience and to have the possibility to introduce the language of *swaps*.

Since the state set $[0..5)$ is sufficiently rich, we can easily design a quinary CA that is both number conserving and reversible. Indeed, if we pick some swap $(ab) \leftrightarrow (cd)$, where a, b, c, d are different elements from $[0..5)$ such that $a + b = c + d$, then we can define a CA by a simple relation: if in a given time step in a configuration there is the pattern 'ab', then in the next time step it will be replaced by the pattern 'cd', while each pattern 'cd' will be replaced by the pattern 'ab'. Such a CA is obviously number conserving (since we assume that $a + b = c + d$) and reversible (since it has order 2, *i.e.* is self-inverse). In the case of $[0..5)$, there are exactly twelve possible swaps, thus we can easily design 12 reversible quinary NCCAs using this method.

F	Description of F	F	Description of F	F	Description of F
1	the right-shift rule	8	(31)↔(40), (32)↔(41)	15	(04)↔(13), (14)↔(23)
2	(12)↔(30)	9	the identity rule	16	(04)↔(31)
3	(12)↔(30), (14)↔(32)	10	(32)↔(41)	17	(03)↔(12)
4	(21)↔(30), (31)↔(40)	11	(23)↔(41)	18	(03)↔(12), (04)↔(13)
5	(21)↔(30)	12	(14)↔(23)	19	(03)↔(21)
6	(13)↔(40)	13	(14)↔(32)	20	(03)↔(21), (23)↔(41)
7	(31)↔(40)	14	(04)↔(13)	21	the left-shift rule

Table 2: The list of all reversible quinary NCCAs with radius 1. The global rule of a given CA is described in terms of swaps, where a swap $(ab) \leftrightarrow (cd)$ means that every pattern 'ab' in the subsequent time step is replaced by the pattern 'cd' and vice versa.

We will call two swaps $(a_1b_1) \leftrightarrow (c_1d_1)$ and $(a_2b_2) \leftrightarrow (c_2d_2)$ *grade-separated* if both a_1, c_1, b_2, d_2 are different elements and b_1, d_1, a_2, c_2 are different elements. Note that two grade-separated swaps $(a_1b_1) \leftrightarrow (c_1d_1)$ and $(a_2b_2) \leftrightarrow (c_2d_2)$ can coexist in the same CA, because their patterns cannot overlap in any configuration. For example, $(30) \leftrightarrow (12)$ and $(14) \leftrightarrow (32)$ can coexist, *i.e.*, we can define a reversible NCCA by the relation: each pattern '30', '12', '14', '32' is replaced in the next time step by the pattern '12', '30', '32', '14', respectively. Since in the case of the set $[0..5)$ there are exactly six pairs of such swaps, we can design an additional six reversible quinary NCCAs.

It turned out (see [4]) that there exists no other non-trivial quinary reversible NCCA apart from the ones described above. Moreover, each of the non-trivial quinary reversible NCCAs acts as follows: each configuration is repeated every two time steps. The list of all reversible quinary NCCAs and their description in the language of swaps is given in Table 2.

3 An exploration of the dynamics of reversible septenary NCCAs

We have been able to find all reversible NCCAs with state set [0..7) and radius 1 (see [4]). It turns out that there are as many as 1669 of them. It is not possible to list all of them in this paper, but the complete list can be found in the dataset [8]. A computational study of the obtained CAs has shown that all reversible septenary NCCAs have a very limited dynamical behavior: for each global rule F, except for two shifts, there exists $m \in \{1, 2, 3, 4, 6, 12, 30, 60\}$ such that F^m is the identity rule (see Table 3), *i.e.*, all of them have finite order.

m	1	2	3	4	6	12	30	60
	1	634	72	324	540	8	60	28

Table 3: The number of reversible septenary NCCAs that have order m (Table 6 in [4]).

Additionally, 1249 reversible septenary NCCAs have an inverse CA with radius 1 and only 420 rules have an inverse CAs with radius 2. Moreover, if F is the global rule of a CA with radius 1, then its mth power F^m can even have radius m. For non-trivial reversible septenary NCCAs, however, we have found that the actual radius of any power is at most 2 (thanks to this property, the calculation of F^{60} was possible).

Unfortunately, the language of swaps used for quinary CAs proved to be insufficient to describe the rules with seven states: only 634 of them allow for a description using swaps only. These are exactly the ones having order 2.

In [0..7) we can consider longer pattern cycles than just swaps (one can see a swap $(ab) \leftrightarrow (cd)$ as a pattern 2-cycle $(ab) \to (cd) \to (ab)$). There are 72 new global rules that can be described by using pattern 3-cycles $(ab){\to}(cd){\to}(ef){\to}(ab)$, where $\{a, b, c, d, e, f\} = \{0, 1, 2, 3, 4, 5, 6\}$ and $a+b = c + d = e + f \in \{6, 7, 8\}$. Obviously, these global rules have order 3 and the inverse CA has radius 1, since it can be described by the reverse pattern 3-cycles (the reverse pattern 3-cycle to $(ab){\to}(cd){\to}(ef){\to}(ab)$ is, of course, $(ab){\to}(ef){\to}(cd){\to}(ab)$).

As we join the swaps with pattern 3-cycles (taking into account grade separation), we obtain another set of 540 global rules. These rules have order 6 and their inverse rules are to be found among them (in particular the inverse is of radius 1).

There are 420 remaining rules that do not allow for a simple description as above. They need to be more complicated as their orders are 4, 12, 30 or 60. Moreover, we know that their inverses have radius 2. We set forth to create another description of these rules in order to understand them better and explain their order.

It is known that there always exists a particle representation of an NCCA, however, it is usually not unique (see, for example, [9] or [10]). We decided to base our particle representation on 2-cell patterns, which could be seen as a generalization of the descriptions in the language of swaps and pattern 3-cycles.

Let a global rule F be given. We will write the arrow $a \xrightarrow{x} b$, where $a, b \in [0..7)$ and $x > 0$, to indicate that F acts as follows: exactly x particles from state a move to the right, whenever the cell on the right is in state b (and analogously $a \xleftarrow{x} b$). For example, a swap $(ab) \leftrightarrow (cd)$, where $a > c$, can be described as a pair of arrows $a \xrightarrow{x} b$ and $c \xleftarrow{x} d$, where $x = a - c$. Similarly, when $a < c$, we get $c \xrightarrow{x} d$ and $a \xleftarrow{x} b$, where $x = c - a$. The set of all arrows for the global rule F will be denoted as $\alpha(F)$. Note that the corresponding local rule f can be obtained by the following simple formula:

$$f(x, y, z) = y + a + b - c - d,$$

where in $\alpha(F)$ there are arrows $x \xrightarrow{a} y, y \xleftarrow{b} z, x \xleftarrow{c} y, y \xrightarrow{d} z$ (if any of these arrows does not occur in $\alpha(F)$, we put 0 as the appropriate term). This approach allows us also to introduce a partial order on the set of all reversible septenary NCCAs: $F_1 \leq F_2$ if and only if $\alpha(F_1) \subseteq \alpha(F_2)$. Although this partial order is quite sparse (as there are as many as 416 maximal elements), it seems to be a decent measure of the complexity of the considered global rules.

We decided to choose one of the most complex rules to carry out a more comprehensive study and the choice fell on `Rule2` (we use this name because this rule has number 2 in the dataset [8]), since (i) it is a maximal element in the partial order, (ii) it is one of 28 global rules having order 60 and (iii) its inverse has radius 2. Below we present the lookup table of `Rule2` as the sequence of 7^3 values $f(0,0,0), f(0,0,1), f(0,0,2), \ldots, f(6,6,6)$ with additional spaces after each 7 values for the convenience:

```
0000000 1131313 2222622 1353135 4444444 1555355 2666666
0000000 1131313 0000400 1353135 2222222 1555355 0444444
0000000 1131313 2222622 1353135 0000000 1555355 2666666
2222222 1131313 0000400 1353135 6666666 1555355 0444444
0000000 1131313 2222622 1353135 4444444 1555355 2666666
4444444 1131313 2222622 1353135 6666666 1555355 2666666
4444444 1131313 2222622 1353135 4444444 1555355 2666666
```

A detailed study on `Rule2` allows to explain where the order 60 originates from and fully describe its dynamics. More precisely, the dynamics of `Rule2` can be easily described in the terms of rows of the following seven matrices:

$$\mathcal{A} = \begin{bmatrix} 3 & 6 & 0 \\ 5 & 0 & 4 \\ 1 & 4 & 4 \\ 3 & 2 & 4 \\ 5 & 4 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 1 & 6 & 0 \\ 3 & 0 & 4 \\ 1 & 2 & 4 \\ 3 & 4 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 3 & 2 \\ 5 & 0 \\ 1 & 4 \end{bmatrix}, \tag{1}$$
$$\mathcal{D} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} 1 & 6 \\ 3 & 4 \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} 2 & 4 \\ 6 & 0 \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} 3 & 6 \\ 5 & 4 \end{bmatrix}.$$

Indeed, let any initial configuration $\mathbf{x} \in X^*$ be given. We can cut it into small pieces (with a length of at most three) according to the following procedure. First, we find and mark all positions in $\mathbf{x}$ where there is some row of $\mathcal{A}$ or $\mathcal{B}$. Next, in parts of $\mathbf{x}$ not marked for this time, we find and mark all places where there is some row of $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ or $\mathcal{G}$. Finally, all unmarked parts of $\mathbf{x}$ are cut into one-cell pieces. It turns out that `Rule2` acts on each of the obtained pieces separately. The pieces with length one remain unaltered. The pieces with length two or three change cyclically, according to the cycle given by the appropriate matrix. Below we describe this more formally.

For $M \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}\}$, let $|M|$ denote the number of rows of M and for $i \in \{1, 2, \ldots, |M|\}$, let M_i denote the ith row of M. Then in each time step, `Rule2` acts as follows:

$$M_i \xrightarrow{(\texttt{Rule2})^t} M_{\mathrm{mod}(i+t,|M|)} ,$$

where $\mathrm{mod}(t, m)$ denotes the remainder of the division of t by m. Since 60 is the least common multiple of 5, 4, 3 and 2, this must be an order of `Rule2`. Figure 2 illustrates the above description on a sample configuration.

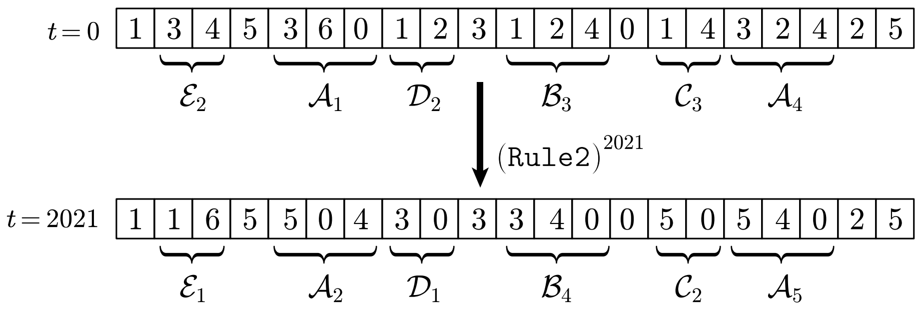


Figure 2: Explanation on a sample initial configuration with the periodic boundary conditions how `Rule2` works in terms of the rows of the matrices listed in (1).

We are able to theoretically prove all the facts about `Rule2` presented above, but space limitations prevent us to include these demonstrations.

Moreover, at this moment, our proofs are strongly based on properties of `Rule2`'s lookup table, thus they cannot be easily adapted to other rules. In the near future, we hope to find more general methods for unraveling the dynamics of reversible septenary NCCAs.

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