Kähler-Einstein Bergman metrics on pseudoconvex domains of dimension two

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November 8, 2022

Bergman kernels in microlocal analysis and mathematical physics (CIRM)

Riemann mapping theorem

Let $\mathbb{D}^n := \{z \in \mathbb{C}^n | \, |z| < 1\} \subset \mathbb{C}^n$.

Theorem (Riemann mapping theorem 1851)

Let $U \subset \mathbb{C}$ open, proper, simply connected. Then $U \cong \mathbb{D}^1$ (biholomorphic)

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Naive higher dimensional generalization is false

Theorem (Poincare 1907)

In $\mathbb{C}^2$, the polydisk and disk $\mathbb{D}^1 \times \mathbb{D}^1 \ncong \mathbb{D}^2$ (not biholomorphic).

Poincare's proof computes: Aut $(\mathbb{D}^n) = PSU(n, 1)$

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Questions:

-find generalization of RMT in higher dimensions?

-biholomorphism classification of domains in higher dimensions

-more robust biholomorphism invariants?

Bergman kernel & metric

Let $U \subset \mathbb{C}^n$ open, connected (often smoothly bounded..)

Bergman projection: $\Pi_U : L^2(U) \to L^2(U) \cap \mathcal{O}(U)$

Bergman (Schwartz) kernel: $\Pi_U(z, z') \in L^2(U \times U)$

Properties: i) $\Pi_U(z, z')$ hol./antihol. in z/z', smooth in the interior
ii) $\Pi_U(z, z)$ smooth & positive in the interior

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Bergman metric: $g^U_{j\bar{k}} = -\partial_{z_j}\partial_{\overline{z_k}}[\ln \Pi_U(z, z)]$

Is a biholomorphism invariant!

Computations of Π_U

1. $U = \mathbb{D}^n$ disk

$$\Pi_{\mathbb{D}^n}\left(z, z'\right) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{(\alpha, n)!}{\pi^n} z^\alpha \bar{z}^\alpha = \frac{n!}{\pi^n} \frac{1}{\left(1 - z\overline{z'}\right)^{n+1}}$$

$g^{\mathbb{D}^n}$ is the Poincare hyperbolic disk

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2. (D'Angelo '78)
$U = E_p := \left\{ |z_1|^2 + |z_2|^{2p} \leq 1 \right\}$ ellipsoid

$$\Pi_{E_p}\left(z, z'\right) = \frac{2}{\pi^2} \frac{1}{p} \frac{\left(1 - z_1\overline{z_1'}\right)^{\frac{2}{p}-2}}{\left[\left(1 - z_1\overline{z_1'}\right)^{\frac{1}{p}} - z_2\overline{z_2'}\right]^3} + \frac{2}{\pi^2} \frac{p-1}{p} \frac{\left(1 - z_1\overline{z_1'}\right)^{\frac{2}{p}-2}}{\left[\left(1 - z_1\overline{z_1'}\right)^{\frac{1}{p}} - z_2\overline{z_2'}\right]^2}$$

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Computing curvatures: $\mathbb{D}^1 \times \mathbb{D}^1 \ncong \mathbb{D}^2$, $E_p \ncong E_{p'}$ for $p \neq p'$ (not biholomorphic).

Fefferman's theorem

Another strategy for $\mathbb{D}^1 \times \mathbb{D}^1 \ncong \mathbb{D}^2$:
Look for behaviour of biholomorphisms near boundary (∂polydisk is non-smooth)

Folklore conjecture: Let $U_1, U_2 \subset \mathbb{C}^n$ smoothly bounded.
Then any biholomorphism $F : U_1 \to U_2$ extends smoothly to the boundary.

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Theorem (Fefferman '74)

1. Let $U_1, U_2 \subset \mathbb{C}^n$ smoothly bounded and strongly pseudoconvex.
Then $U_1 \cong U_2$ (biholomorphic) $\iff \exists$CR diffeomorphism $\partial U_1 \cong \partial U_2$.
2. Let $U = \{\rho < 0\} \subset \mathbb{C}^n$ be strongly pseudoconvex.
Then the Bergman kernel satisfies

$$\Pi_U(z,z) = a(z)\,\rho^{-n-1} + b(z)\ln(-\rho)$$

(II $\implies$ I) Study geodesic flow for Bergman metric near boundary.

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Ramadanov Conjecture ('81):
Let $U \subset \mathbb{C}^n$ be strongly pseudoconvex. If $b = O(\rho^\infty)$ then ∂U is locally spherical.
Known for $n = 2$: Burns-Graham ('87), Boutet de Monvel ('88).

CR manifold, Pseudoconvexity, Finite type

CR manifold $\left(X^{2n+1}, T^{1,0}X \subset T_{\mathbb{C}}X\right)$ integrable, n dimensional, non-deg
eg. $X := \partial U$, $U \subset \mathbb{C}^n$ domain with $T^{1,0}X = T^{1,0}\mathbb{C}^n \cap T_{\mathbb{C}}X$.

Levi distribution: $HX = \mathrm{Re}\ \left[T^{1,0}X \oplus T^{0,1}X\right]$
Levi form: $\mathscr{L} \in (HX^*)^{\otimes 2} \otimes (T_xX/H_xX)$
$\mathscr{L}(U,V) := [[U,V]] \in T_xX/H_xX$

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$x \in X$ is weakly/strongly pseudoconvex $\iff \mathscr{L}_x$ is pos. def./semi-def.
$x \in X$ is finite type $\iff HX$ is bracket generating at x

type of a point $x \in X$:
$r(x) = 1 + \min$ # brackets in HX necessary to generate TX

type of strongly pseudoconvex point $x \in X$ is $r(x) = 2$.

Tangential CR complex & Szegő kernel

Tangential CR operator: $\bar{\partial}_b : C^{\infty}\left(\Lambda^q T^{0,1*}X\right) \rightarrow C^{\infty}\left(T^{0,1*}X\right)$,
Defined via: $\left(\bar{\partial}_b f\right)\left(\bar{Z}\right) = \bar{Z}\left(f\right)$, $f \in C^{\infty}\left(X\right)$, $\overline{Z} \in T^{0,1}X$.

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Szegő projector: $\Pi_b : L^2\left(X\right) \to \ker\ \left(\bar{\partial}_b\right)$
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Theorem (Kohn '65)

Let $\left(X^{2n+1}, T^{1,0}X\right)$ strongly pseudoconvex. Furthermore assuming $\bar{\partial}_b$ has closed range, $\exists G : H^s\left(T^{0,1*}X\right) \to H^{s+\frac{1}{2}}\left(X\right)$ such that $\Pi_b = I - G\bar{\partial}_b$

Szegő parametrix

Interesting to describe the singularities of its Schwartz kernel.

Theorem (Boutet de Monvel-Sjöstrand '75)

Let $\left(X^{2n+1}, T^{1,0}X\right)$ *be strongly pseudoconvex CR manifold. Assume* ∂_b *has closed range. Then near each* $x \in X$ *there exist coordinates* $\left(\underbrace{x_1 \ldots, x_{2n}}_{=x'}, x_{2n+1}\right)$ *such that*

$$\begin{aligned}\Pi_b(x,y) &= \int_0^\infty dt e^{it\Psi(x,y)} a(t;x,y) \\ &= \int_0^\infty dt e^{it\left(x_{2n+1}-y_{2n+1}\right)} \underbrace{e^{it\Phi\left(x',y\right)} a(t;x,y)}_{=b(t;x,y)}\end{aligned}$$

Phase: $Im\Phi(x,y) \geq C\left|x'-y'\right|^2$, $x = y \iff \Psi = 0$.
Amplitude: $a(t;x,y) \in S^n_{t,cl}$, $a \sim \sum_{j=0}^\infty t^{n-j} a_j(x,y)$

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Note: regrouped amplitude $b(t;x,y) \in S^n_{t,\frac{1}{2},\mathrm{cl}}\left(\mathbb{R}^{2n+1}_x\right)$

(lies in a more general class; $\partial_x^\alpha \partial_t^\gamma b = O\left(t^{n+\frac{1}{2}|\alpha|-\gamma}\right)$)

Bergman asymptotics

Specialize to $X = \partial U$.

Consider Poisson operator

$$P : C^{\infty}(X) \to C^{-\infty}(U)$$
$$\Box_U P = 0, \quad \gamma P = I.$$

P approximately relates the Bergman-Szegő projectors

$$\Pi_U \text{"} = \text{"} P (P^* P)^{-1} \Pi_b P$$
$$\text{(at highest order)}$$

Parametrices for P described by Boutet de Monvel '71.

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Using the above recover/refine Fefferman's theorem

Greens function, pointwise bounds

Consider $\left(X^3, T^{1,0}X\right)$ weakly pseudoconvex, finite type.
Let maximum number of brackets be $r := \max_{x\in X} r\left(x\right)$.

Theorem (Christ '89)

Let $\left(X^3, T^{1,0}X\right)$ weakly pseudoconvex, finite type. Assume $\overline{\partial}_b$ has closed range. There exists (microlocal) $G : H^s\left(X\right) \to H^{s+\frac{1}{r}}\left(X\right)$ such that $\Pi_b = I - G\overline{\partial}_b$.

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Let $\left(X^3, T^{1,0}X\right)$ weakly pseudoconvex, finite type. Assume $\bar{\partial}_b$ has closed range. Near any point $x' \in X$ of type $r\left(x'\right)$ there exists coordinates $\left(x_1, x_2, x_3\right)$ centered at x' such that

$$\left|\partial_x^\alpha \Pi\left(x,0\right)\right| \leq C_\alpha \left[d^H\left(x,0\right)\right]^{-2-r-\alpha_1-\alpha_2-r\alpha_3}$$

$$d^H\left(x,0\right) = \left|x_1\right| + \left|x_2\right| + \left|x_3\right|^{1/r\left(x'\right)}.$$

($d^H\left(x,0\right)$ is equivalent to the sub-Riemannian CC distance between $x, 0$).

Similar bounds for boundaries of weakly pseudoconvex finite type domains in $\mathbb{C}^2$:
McNeal '89, Nagel-Rosay-Stein-Wainger '89

Szegő parametrix

Consider $\left(X^3, T^{1,0}X\right)$ weakly pseudoconvex, finite type
$\left(HX = \mathrm{Re}\ \left[T^{1,0}X \oplus T^{0,1}X\right]\right.$ bracket generating).
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Theorem (Hsiao-S. '20)

Let $\left(X^3, T^{1,0}X\right)$ *weakly pseudoconvex, finite type. Assume* $\bar{\partial}_b$ *has closed range. Near any point* $x' \in X$ *of type* $r(x')$ *there exists coordinates* (x_1, x_2, x_3) *centered at* x' *such that*

$$\Pi(x, 0) = \int_0^\infty dt\, e^{itx_3} b(x; t) + C^\infty(X)$$

where $b \sim t^{\frac{2}{r}} \left[\sum_{j=0}^\infty t^{-\frac{j}{r}} b_j \left(t^{\frac{1}{r}} x_1, t^{\frac{1}{r}} x_2\right)\right] \in S^{\frac{2}{r}}_{\frac{1}{r}, cl}\left(\mathbb{R}^2_{x_1, x_2} \times \mathbb{R}_t\right)$, $b_j \in \mathcal{S}\left(\mathbb{R}^2\right)$.

Christ estimates are equivalent to $b \in S^{\frac{2}{r}}_{\frac{1}{r}}\left(\mathbb{R}^2_{x_1, x_2} \times \mathbb{R}_t\right)$

(i.e. $\partial_t^k \partial_x^\alpha b = O\left(t^{m-k+\frac{|\alpha|}{r}}\right)$ without classical expansion).

Bergman asymptotics

Specializations:

Theorem (Hsiao-S. '20)

Let $X^3 = \partial U$ *be boundary of weakly pseudoconvex, finite type domain* $U = \{\rho < 0\} \subset \mathbb{C}^2$. *For any point* $x' \in X = \partial U$ *on the boundary, of type* $r = r(x')$, *the Bergman kernel satisfies the asymptotics*

$$\Pi_U(z,z) \sim \sum_{j=0}^{\infty} \frac{1}{(-\rho)^{2+\frac{2}{r}-\frac{1}{r}j}} a_j + \sum_{j=0}^{\infty} b_j (-\rho)^j \log(-\rho),$$

as $z \to x'$ *for some set of reals* a_j, b_j.
Here $a_0 = B_0(0,0)$ *given by model Bergman of* $\left(\mathbb{C}, e^{-\Phi} dz_1\right)$ *where*

$$\rho \sim Rez_2 + \underbrace{\Phi(z_1, \bar{z}_1)}_{subharmonic\ homog.\ poly}, \qquad leading\ approx.$$

Fefferman '74 (strongly pseudoconvex case), D'Angelo '78 (ellipsoids), Boas-Straube-Yu '95 (h-extendible/semiregular domains), Kamimoto '98, '04 (tube domains, toric domains)...

Bergman asymptotics

Theorem (Marinescu-S. '18)

Let Y^2 Riemann surface. $\left(L, h^L\right)$ Hermitian, holomorphic, semi-positive. Assume that R^L vanishes to finite order at each point. The Bergman kernel admits on-diagonal expansion

$$\Pi_k\left(y, y\right) \sim k^{\frac{2}{r}}\left[b_0\left(y\right) + b_1\left(y\right) k^{-\frac{2}{r}} + b_2\left(y\right) k^{-\frac{4}{r}} \ldots\right]$$

where $r = r\left(y\right) = 2 + ord_y\left(R^L\right)$.

The above proved by local index theory method as in Bismut-Lebeau '91, Dai-Liu-Ma '06, Ma-Marinescu '07.

Proof Sketch

Step 1. Let

$$\begin{aligned} Z &= a_j(x)\,\partial_{x_j} && \text{CR vector field} \\ \tilde{Z} &= \tilde{a_j}(z)\,\partial_{z_j} + \tilde{b_j}(z)\,\partial_{\bar{z}_j} && \text{almost analytic extension} \end{aligned}$$

Find *almost analytic BRT coordinates* $(w_1(z), w_2(z), w_3(z))$ on $\mathbb{C}^3$ such that

$$\tilde{Z} = \frac{1}{2}\left(\partial_{w_1} + i\partial_{w_2}\right) - \frac{i}{2}\left(\partial_{w_1}\varphi + i\partial_{w_2}\varphi\right)\partial_{w_3} \quad \text{for}$$

$$\varphi = \underbrace{\varphi_0(w_1, w_2)}_{\text{homogeneous, real coefficients}} + O\left(|w|^{r+1}\right).$$

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Step 2.
The local Bergman kernel $B_t(p, p')$ for the Hermitian metric $e^{-t\varphi(p_1,p_2)}dp$, $p = \mathrm{Re}\, w$ satisfies symbolic estimates/expansion in $t \to \infty$ by local index method.

Proof sketch

Step 3.

Construct almost analytic extensions for Szegő kernel $\Pi(x,x')$ on $X=\{\mathrm{Re}z=0\}$ and t-Fourier transform of $B_t(p,p')$ on $\mathbb{R}^3_p=\{\mathrm{Re}w=0\}$.
This part requires Christ estimates.

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Step 4.

Relate almost analytic extensions Π and t-Fourier transform of B_t.

Relation is pointwise Boutet de Monvel-Sjöstrand theorem.
This requires constructing a calculus of symbols $S^{m,k}_{\frac{1}{r},\mathrm{cl}}$ satisfying Christ estimates.

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Step 5.

For Fefferman's expansion; the relation between Bergman & Szegő kernels Π_U, Π via the Poisson operator still holds.
Also uses requires solution of $\bar{\partial}$-Neumann problem for weakly pseudoconvex domains $U \subset \mathbb{C}^2$ (Kohn '72).

Application: Canonical metrics

No new application to boundary extension problem (Bell-Ligocka '81)

Theorem (Cheng-Yau '79, Mok-Yau '83)

Any $U \subset \mathbb{C}^n$ pseudoconvex admits a Kahler-Einstein metric

$$g^{U,KE} = \partial\bar{\partial}u.$$

Here u solves MA equation

$$\begin{aligned} J(u) := \det \begin{pmatrix} u & u_{\bar{\beta}} \\ u_{\alpha} & u_{\alpha\bar{\beta}} \end{pmatrix} &= 1 \qquad \text{in } D \\ u &= 0 \qquad \text{on } \partial D \end{aligned}.$$

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Theorem (Lee-Melrose '81)

Let $U \subset \mathbb{C}^n$ strongly pseudoconvex. Then $u \sim \rho \sum_{j=0}^{\infty} c_j \left(\rho^{n+1} \ln \rho\right)^j$.

.

Question (Yau '82): For which $U \subset \mathbb{C}^n$ pseudoconvex is $g^{U,\mathrm{KE}} = g^{U,\mathrm{Bergman}}$?

Conjecturally:
U = homogeneous domain ($\mathrm{Aut}(U)$ acts transitively. VGPS classification '63).
U homogeneous + smooth boundary $\implies U = \mathbb{D}^n$

Application: Canonical metrics

Known in strongly pseudoconvex case

Theorem (Fu-Wong '97, Nemirovski-Shafikov '06 (n=2), Huang-Xiao '18 (n>2))

Let $U \subset \mathbb{C}^n$ strongly pseudoconvex with $g^{U,KE} = g^{U,Bergman}$. Then $U = \mathbb{D}^n$.

Application of the new asymptotics

Theorem (Xiao-S.)

Let $U \subset \mathbb{C}^2$ pseudoconvex and finite type with $g^{U,KE} = g^{U,Bergman}$. Then $U = \mathbb{D}^2$.

Proof sketch (of Fu-Wong & Nemirovski-Shafikov (n=2))

Step 1.
If $g^{U,\mathrm{KE}} = g^{U,\mathrm{Bergman}}$ then $u = \Pi_U(z)^{-1/n+1}$ satisfies

$$\det \begin{pmatrix} u & u_{\bar{\beta}} \\ u_{\alpha} & u_{\alpha\bar{\beta}} \end{pmatrix} = \frac{\pi^n}{n!} u^{n+2}$$

Step 2.
Comparing log singularity on both sides: $b = 0$ in Fefferman expansion.

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Step 4.
Uniformization theorems (Nemirovski-Shafikov) $\implies$ $U = \mathbb{D}^2$

Proof sketch (of Xiao-S)

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Along *critically tangent* (order r) path

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B_0=model kernel, v =tangent vector.

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No model can satisfy above equation. Different asymptotics as $|v| \to \infty$.

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Step 4.
Reduces to strongly pseudoconvex case.

Thank you.