Discrete Duality Finite Volume Schemes

Pascal Omnes – thanks to (in order of appearance on stage):
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Overview

A bit of history

Known and less known properties of DDFV methods with applications

A few open problems

Conclusions and perspectives

In 1994, I worked with F. Hermeline during my MSc internship on his 1993 article in the JCP ”Two coupled particle-finite volume methods using Delaunay-Voronoï meshes for the approximation of Vlasov-Poisson and Vlasov-Maxwell equations”.

There are ”Two” finite volume methods, because you can either integrate on the primal (Delaunay), or on the dual (Voronoï) mesh.

But you need orthogonality of the edges.

As of 1999, I was working at the CEA on the numerical simulation of isotope separation by gas centrifuges.

The model is the compressible Navier-Stokes equations (coupled to an energy equation) in cylindrical coordinates (r, z).

There is an equilibrium gas density $\rho_0(r) = \rho_C \exp\left(\frac{M}{2RT}\omega^2 r^2\right)$.

A building block of the linearization is a Stokes-like system:

$$
\begin{aligned}
-\nu \Delta U + \rho_0 \nabla p &= f \\
\nabla \cdot (\rho_0 U) &= 0
\end{aligned}
$$

Since this problem turns out to be ill conditioned when ωr is "large" and ν "small", I looked for help and started working with K. Domelevo in Toulouse.

$$-\nu \Delta U + \rho_0 \nabla p = f$$
$$\nabla \cdot (\rho_0 U) = 0$$

The discretization was a MAC scheme; the conservative form of $\nabla \cdot (\rho_0 U)$ does not cause any problem in a FV scheme. But an issue arises because of the non-conservative term $\rho_0 \nabla p$.

On a mesh with irregular Δr and because of the strong variations of ρ_0, there is no obvious choice for the discretization of ρ_0 (Mean value? Value at the center of the cell?).

I shared with K. Domelevo my observation that things were numerically better when the discretization of $\rho_0 \nabla \bullet$ was the (negative) adjoint of the discretization of $\nabla \cdot (\rho_0 \bullet)$.

Then Komla realized that the MAC discretization of the velocity Laplacian could itself be interpreted as the discrete divergence of a discrete gradient that were in duality.

Since my aim was to relax the constraints of Cartesian meshes involved in the MAC scheme and use unstructured, possibly non-conforming meshes for adaptive mesh refinement, we started looking for ways to generalize consistent discretizations of a gradient and a divergence on general meshes, such that they would respect a duality property, with some coercivity in the gradient, first for the Laplacian operator.

There are several ways that fail!

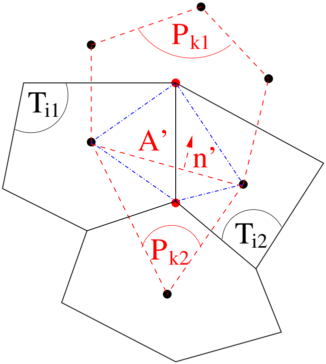
The most advanced scheme for the Laplace equation with numerical analysis was in the PhD thesis of Y. Coudière (see his 1999 article in M2AN "Convergence rate of a finite volume scheme for a two dimensional convection-diffusion problem")

And his "diamond-cell" gradient involving cell and vertex unknowns was a very good candidate for our purpose: consistent on any mesh, coercive. And I realized that the divergence operator that you obtain by duality would simply mean integrating both on the primal and on the dual mesh, which was familiar to me!

It turned out that in parallel, F. Hermeline had also developed the same ideas of double integration and reconstruction of a full gradient, but without the discrete duality point of view.

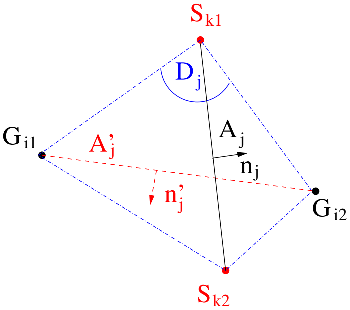
Hermeline JCP 2003, Domelevo Omnes M2AN 2005, see also the FVCA 5 Benchmark on diffusion problems (with possibly discontinuous and anisotropic diffusion tensors).

For the DDFV approximation of $-\nabla \cdot K\nabla\phi = f$, integrate on the primal cells T_i and on the dual cells P_k; associated unknowns ϕ_i^T and ϕ_k^P are used to construct 4-point gradients on the diamond cells D_j.



Coudière, Vila, Villedieu M2AN 99

$$|D_j| \; \langle \nabla \phi \rangle_{|D_j} = \int_{D_j} \nabla \phi = \int_{\partial D_j} \phi \, n$$



Using $\int_{[SG]} \phi \approx \ell_{SG} \, \frac{[\phi(S)+\phi(G)]}{2}$ and geometrical properties in the diamond, one obtains the discrete gradient ∇_h over D_j:

$$(\nabla_h \phi)_j := \frac{1}{2\,|D_j|} \left\{ \left[\phi^P_{k_2} - \phi^P_{k_1}\right] |A'_j| \, n'_j + \left[\phi^T_{i_2} - \phi^T_{i_1}\right] |A_j| \, n_j \right\}$$

and $\nabla_h \phi \cdot \overrightarrow{G_{i1} G_{i2}} = \phi^T_{i_2} - \phi^T_{i_1} \quad \nabla_h \phi \cdot \overrightarrow{S_{k1} S_{k2}} = \phi^P_{k_2} - \phi^P_{k_1}$

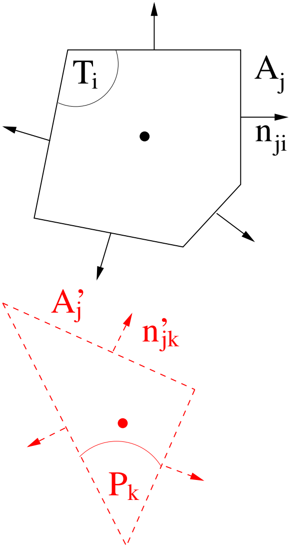
The divergence operators and the DD property

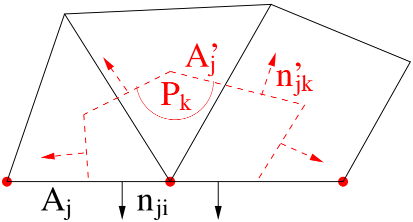
Definition of the discrete divergence of a vector field u on the primal mesh:

$$(\nabla_h^T \cdot u)_i := \frac{1}{|T_i|} \sum_{j \in \mathcal{V}(i)} |A_j| u_j \cdot n_{ji}$$

Definition of the discrete divergence on the inner elements of the dual mesh:

$$(\nabla_h^P \cdot u)_k := \frac{1}{|P_k|} \sum_{j \in \mathcal{E}(k)} |A'_j| u_j \cdot n'_{jk}$$





Definition of the discrete divergence on the boundary elements of the dual mesh:

$$(\nabla_h^P \cdot u)_k := \frac{1}{|P_k|}\left(\sum_{j\in\mathcal{E}(k)} |A'_j| u_j \cdot n'_{jk} + \sum_{j\in\mathcal{E}(k)\cap\partial\Omega} \frac{1}{2}|A_j| u_j \cdot n_j\right)$$

The discrete Green formula holds (Komla found the name "DDFV").

$$-(u, \nabla_h \phi)_D + (u\cdot n, \phi)_{\partial\Omega} = \frac{1}{2}\left[(\nabla_h^T \cdot u, \phi^T)_T + (\nabla_h^P \cdot u, \phi^P\right.$$

In the same way, we may construct the curl of a scalar field

$$(\nabla_h^D \times \phi)_j := \frac{1}{2\,|D_j|}\left\{\left[\phi_{k_2}^P - \phi_{k_1}^P\right]|A'_j|\,t'_j + \left[\phi_{i_2}^T - \phi_{i_1}^T\right]|A_j|\,t_j\right\}$$

And, with the DD property, curls of a vector field

$$(\nabla_h^T \times u)_i := \frac{1}{|T_i|}\sum_{j\in\mathcal{V}(i)} |A_j| u_j \cdot t_{ji}$$

$$(\nabla_h^P \times u)_k := \frac{1}{|P_k|}\sum_{j\in\mathcal{E}(k)} |A'_j| u_j \cdot t'_{jk}$$

$$(\nabla_h^P \times u)_k := \frac{1}{|P_k|}\left(\sum_{j\in\mathcal{E}(k)} |A'_j| u_j \cdot t'_{jk} + \sum_{j\in\mathcal{E}(k)\cap\partial\Omega} \frac{1}{2}|A_j| u_j \cdot t_j\right)$$

Delcourte, Domelevo, Omnes, "A discrete duality finite volume approach to Hodge decomposition and div-curl problems on almost arbitrary two-dimensional meshes", SINUM 2007.
For all $\phi = (\phi_i^T, \phi_k^P)$,

$$(\nabla_h^T \cdot (\nabla_h^D \times \phi))_i = 0 \;\; ; \;\; (\nabla_h^T \times (\nabla_h^D\, \phi))_i = 0 \;\; \forall i$$

$$(\nabla_h^P \cdot (\nabla_h^D \times \phi))_k = 0 \;\; ; \;\; (\nabla_h^P \times (\nabla_h^D\, \phi))_k = 0 \;\; \forall k \notin \Gamma$$

Helmholtz-Hodge orthogonal decomposition (in a simply connected domain): for all $u \in (\mathbb{R}^J)^2$, there exists ϕ and ψ with $\psi = 0$ on Γ and ϕ with 0 mean-value in Ω such that for all diamond cells:

$$u_j = (\nabla_h\, \phi)_j + (\nabla_h \times \psi)_j$$

Euler's formula plays a role here. Also works on non-simply connected domains.

Delcourte, Domelevo, Omnes, SINUM 2007

$\nabla \cdot u = f$, $\nabla \times u = g$ in Ω and $u \cdot n = h$ on Γ.

Unknowns $u = (u_j)$

$$\begin{aligned}
(\nabla_h^T \cdot u)_i &= f_i^T \ \forall i \\
(\nabla_h^P \cdot u)_k &= f_k^P \ \forall k \\
(\nabla_h^T \times u)_i &= g_i^T \ \forall i \\
(\nabla_h^P \times u)_k &= g_k^P \ \forall k \notin \Gamma
\end{aligned}$$

$$u_j \cdot n_j = h_j \ \forall j \in \Gamma$$

Existence and uniqueness: Euler's formula and Hodge decomposition

Also works on non-simply connected domains

Generalisation of Nicolaides' covolume method.

Sarah Delcourte's PhD thesis 2007 and Delcourte and Omnes "A discrete duality finite volume discretization of the vorticity-velocity-pressure Stokes problem on almost arbitrary two-dimensional grids", NMPDE 2015.

$-\Delta u + \nabla p = f \;,\; \nabla \cdot u = 0$ in Ω; $u \cdot n$ and p prescribed on Γ

Unknowns: $u = (u_j)$, $\omega = \nabla \times u = (\omega_i^T, \omega_k^P)$ and $p = (p_i^T, p_k^P)$

Using $-\Delta u = \nabla \times \nabla \times u - \nabla \, \nabla \cdot u$, first solve

$$(\nabla_h^D \times \omega)_j + (\nabla_h^D \, p)_j = f_j^D \;\; \forall j$$

(Hodge decomposition) and then solve the div-curl problem

$$(\nabla_h^T \cdot u)_i = 0 \;\; \forall i \quad ; \quad (\nabla_h^P \cdot u)_k = 0 \;\; \forall k$$

$$(\nabla_h^T \times u)_i = \omega_i^T \;\; \forall i \quad ; \quad (\nabla_h^P \times u)_k = \omega_k^P \;\; \forall k \notin \Gamma$$

$$u_j \cdot n_j = 0 \;\; \forall j \in \partial\Omega$$

Hermeline, Layouni, Omnes, "A finite volume method for the approximation of Maxwell's equations in two space dimensions on arbitrary meshes", JCP 2008.
Unknowns $E = (E_j)$ (vector field) and $B = (B_i^T, B_k^P)$ (scalar)

$$\begin{aligned} \frac{E_j^{n+1} - E_j^n}{\Delta t} - c^2(\nabla_h^D \times B^{n+1/2})_j &= -\frac{1}{\varepsilon_0} J_j \ \ \forall j \\ \frac{B_i^{n+1/2} - B_i^{n-1/2}}{\Delta t} + (\nabla_h^T \times E^n)_i &= 0 \ \ \forall i \\ \frac{B_k^{n+1/2} - B_k^{n-1/2}}{\Delta t} + (\nabla_h^T \times E^n)_k &= 0 \ \ \forall k \end{aligned}$$

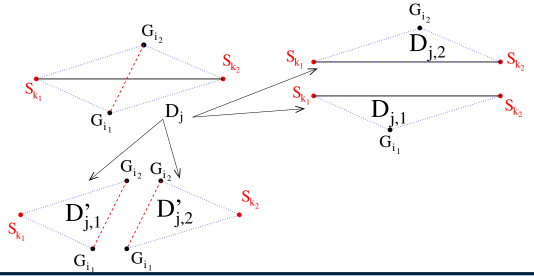
$$E_j^n \cdot t_j = 0 \ \ \forall j \in \Gamma$$

The scheme conserves a discrete energy and is stable under a (quite complicated) CFL type condition; no parasitic modes

Let ϕ_e be the exact solution of the Laplace equation (supposed to be in H^2 or H^3).
Let $\forall i, (\Pi\phi_e)_i^T = \phi_e(G_i)$ and $\forall k, (\Pi\phi_e)_k^P = \phi_e(S_k)$
Thanks to $(\nabla_h^D \phi, \nabla_h^D \psi)_\Omega = (\delta\phi_e, \nabla_h^D \psi)_\Omega$,

$$||\nabla_h^D \phi - \delta\phi_e||_{0,\Omega} \leq \; ||\nabla_h^D (\Pi\phi_e) - \delta\phi_e||_{0,\Omega}$$

$$||\nabla_h^D (\phi - \Pi\phi_e)||_{0,\Omega} \leq \; ||\nabla_h^D (\Pi\phi_e) - \delta\phi_e||_{0,\Omega}$$



The following error estimate holds:

$$||\delta\phi_e - \nabla_h^D(\Pi\phi_e)||_{0,D_j} \leq C\frac{h}{\sin\theta_j}|\phi_e|_{2,D_j}$$

with

$$\theta_j = \min(\text{angle max } D_{j,\alpha}, \text{angle max } D'_{j,\alpha})$$

If D_j is a parallelogram, then

$$||\delta\phi_e - \nabla_h^D(\Pi\phi_e)||_{0,D_j} \leq Ch^2|\phi_e|_{3,D_j}$$

Under the hypothesis that all θ_j are uniformly bounded with h:
In the general case: convergence with order 1.
Cases in which "almost all" D_j are parallelograms, convergence with order $1.5 - \varepsilon, \forall\varepsilon > 0$.

• LE and Omnes, "Discrete Poincaré inequalities for arbitrary meshes in the discrete duality finite volume context", ETNA 2013.
Under the condition that the angles in the diagonals of the diamond cells do not degenerate to 0, there exists a constant C such that

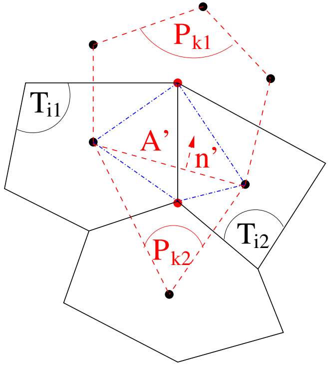
$$||\phi^T||_{L^2(\Omega)} + ||\phi^P||_{L^2(\Omega)} \leq C||\nabla_h^D \phi||_{L^2(\Omega)}$$

• Omnes Penel and Rosenbaum, "A posteriori error estimation for the discrete duality finite volume discretization of the Laplace equation", SINUM 2009.

• LE and Omnes, "An a posteriori error estimation for the discrete duality finite volume discretization of the Stokes equations", M2AN 2015.

From the FVCA 5 Benchmark on diffusion problems, it seems that on some families of meshes, the gradients super-converge.

The Hessians that can be computed from the gradients on the primal and dual meshes seem to converge with order $\mathcal{O}(h)$ on triangular meshes with barycentric dual mesh.



$-\Delta u + \nabla p = f \ , \ \ \nabla \cdot u = 0$ in Ω and $u = 0$ over Γ
Unknowns: $u = (u_j)$ and $p = (p_i^T, p_k^P)$

Using again $-\Delta u = \nabla \times \nabla u - \nabla \, \nabla \cdot u$,

$$(\nabla_h^D \times \nabla_h^{T,P} \times u - \nabla_h^D (\nabla_h^{T,P} \cdot u))_j + (\nabla_h^D \, p)_j = f_j^D \ \ \forall j \notin \partial\Omega$$

$$(\nabla_h^T \cdot u)_i = 0 \ \ \forall i \quad ; \quad (\nabla_h^P \cdot u)_k = 0 \ \ \forall k$$

$$u_j = 0 \ \ \forall j \in \partial\Omega$$

Existence and uniqueness if the cells of the primal mesh have only one edge on Γ.

- Uniformity of the inf-sup condition? (Numerical tests: OK),
- Rate of convergence seems to be lower than expected on arbitrary families of meshes (primal triangles with barycentric dual mesh are OK).

Conclusions and perspectives

- Construction of discrete differential operators on arbitrary meshes
- They verify properties similar to those verified by the continuous operators
- Discretization of various (linear) PDEs
- Numerical analysis could be performed

- Perspectives are yours!