DDFV scheme for the regularised Heston model

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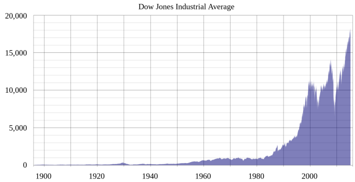
Financial derivatives

Derivative is a contract that derives its value from the performance of an underlying entity.
If this entity is a financial product (stock, interest rate, index) we speak about financial derivatives.

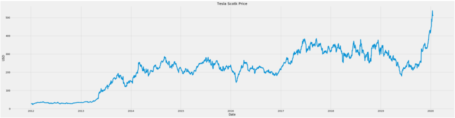
Financial derivatives are used for two main purposes to speculate and to hedge investments.

Price of the financial derivative is a function of the time and the underlying asset price, which is considered to be a stochastic variable: $V = V(S, t)$.

Stock price modelling



(a) Dow Jones index.



(b) Tesla stock prices development.

The stock price is modelled by the stochastic differential equation:

$$dS = \mu(S,t)dt + \sigma(S,t)dw$$

Black-Scholes model [1973]

Introduced by Black and Scholes and by Merton in the same year.

The stock price follows geometric Brownian motion with constant drift and volatility: $dS = \mu S dt + \sigma S dw$.

Using the Itô's lemma, no arbitrage opportunities assumption and principle of hedging we obtain the linear Black-Scholes partial differential equation in the usual form:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{1}$$

Heston model [1993]

Volatility of the underlying v is taken to be a stochastic variable modelled by the equation

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dz_t$$

and the price of underlying is driven by the equation:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dw_t$$

Therefore is V function of time and two stochastic variables - S and v.

Repeating the approach given by Heston we get the Heston partial differential equation in the form:

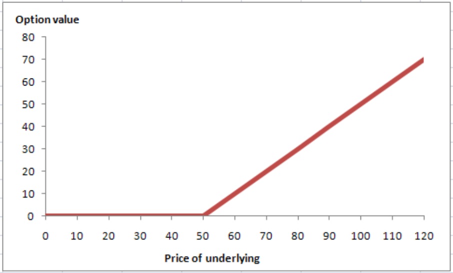
$$\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S v \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2} v \sigma^2 \frac{\partial^2 V}{\partial v^2} + rS \frac{\partial V}{\partial S} + [\kappa(\theta - v) - \lambda v] \frac{\partial V}{\partial v} - rV = 0. \quad (2)$$

Model parameters explanation

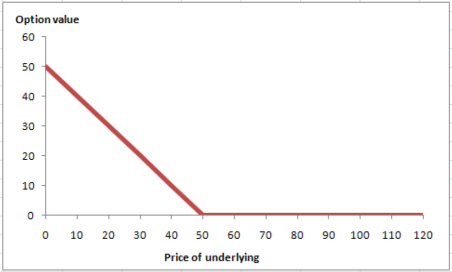
- $\rho \in < -1, 1 >$ is the correlation parameter between underlying asset price and the volatility of the financial derivative;
- $\sigma > 0$ is the volatility variance, which is taken to be a stochastic variable as stated above;
- $\theta > 0$ is the long term variance, around which the financial derivative volatility oscillate;
- $\kappa > 0$ is the reversion speed of the underlying asset volatility return to the long term variance;
- $r > 0$ is the interest rate;
- $\lambda > 0$ is the market price of risk, which models the risk impact.

Terminal condition for European options

$$V(S, v, T) = max(0, S - E) \vee V(S, v, T) = max(0, E - S),$$



(a) Call option.



(b) Put option.

Figure: Pay off diagrams for European type of options.

Boundary conditions for European call option

$$V(0, v, t) = 0,$$
$$\frac{\partial V}{\partial S}(S \to \infty, v, t) = 1,$$
$$V(S, v \to \infty, t) = S,$$
$$\frac{\partial V}{\partial t}(S, 0, t) + r\frac{\partial V}{\partial S}(S, 0, t) + \kappa\theta\frac{\partial V}{\partial v}(S, 0, t) - rV(S, 0, t) = 0.$$

Last condition plays no role in reality as it is consequence of the so-called Fichera condition, here considered in the form:

$$-\begin{pmatrix} r - \frac{1}{2}\rho\sigma \\ \kappa\theta - \frac{1}{2}\sigma^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \kappa\theta - \frac{1}{2}\sigma^2 \geq 0.$$

Compact version of the Heston model

Using the substitutions

$$x = ln\left(\frac{S}{E}\right), y = v, \tau = T - t, v(x, y, \tau) = \frac{V(S, v, t)}{E}$$

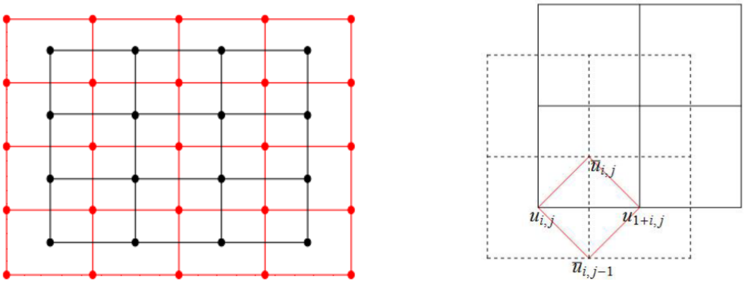
we can rewrite the Heston PDE in the compact form:

$$\frac{\partial v}{\partial \tau} + \vec{A} \cdot \nabla v = \nabla \cdot (\mathbf{B}\nabla \mathrm{v}) - \mathrm{rv}, \qquad (3)$$

where

$$B = \frac{1}{2}y\begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{pmatrix}, \vec{A} = -\begin{pmatrix} r - \frac{1}{2}y - \frac{1}{2}\rho\sigma \\ \kappa(\theta - y) - \lambda y - \frac{1}{2}\sigma^2 \end{pmatrix}. \qquad (4)$$

DDFV principle



(a) Primal (red) and dual (black) mesh. (b) DDFV diamond cell approximation.

Figure: Basic principle of the DDFV approach explanation.

FVM and DDFV in financial derivatives pricing problems

The uniqueness of the numerical solution and stability of three different FVM scheme were proven by Mikula and Kútik in 2015.

Regarding the DDFV approach for Heston model analysis the $L_\infty([t_1; t_2]; L_2(\Omega))$ estimate for the numerical solution were obtained by Handlovičová in 2017.

But convergence of the schemes remains an open problem.

Studied model formulation

To simplify the problem we are studying we will find the function $v = v(x, y, \tau)$ in the form $v(x, y, \tau) = u(x, y, \tau) + w(x, y, \tau)$, where

$$w(x, y, \tau) = max\{0, e^{x} - e^{-r\tau}\}. \tag{5}$$

and

$$\frac{\partial u}{\partial \tau} + \vec{A} \cdot \nabla u = \underline{\epsilon \triangle u} + \nabla \cdot (\mathbf{B} \nabla \mathrm{u}) - \mathrm{ru} + \mathrm{f}(\mathrm{x}, \mathrm{y}, \tau), \ (\mathrm{x}, \mathrm{y}, \tau) \in \Omega \times [\mathrm{t}_1, \mathrm{t}_2], \tag{6}$$

where

$$f(x, y, \tau) = \begin{cases} 0 \text{ for } x < -r\tau, \\ \epsilon e^{x} \text{ for } x \geq -r\tau, \end{cases}$$

$\Omega = (X_a, X_b) \times (0, Y)$ and u fulfils homogeneous initial and boundary conditions.

Weak solution of the problem

We say that u is a weak solution of (6) if, for all $I =< t_1, t_2 >$, $0 < t_1 < t_2 < \infty$,

1. $u \in L^2(I; V(\Omega))$, where $V(\Omega) := \{u \in H^1(\Omega) : u|_{\Gamma_D} = 0\}$.
2. $u(\cdot, 0) = 0$.
3. Following holds

$$\int_I \int_\Omega -u(x,y,\tau)\frac{\partial \psi}{\partial \tau}(x,y,\tau) + \vec{A} \cdot \nabla u(x,y,\tau)\psi(x,y,\tau) + \\ + \epsilon \nabla u(x,y,\tau)\nabla\psi(x,y,\tau) + \\ + \mathbf{B}\nabla u(x,y,\tau)\nabla\psi(x,y,\tau) + ru(x,y,\tau)\psi(x,y,\tau)dxdyd\tau = \\ = \int_I \int_\Omega f(x,y,\tau)\psi(x,y,\tau)dxdyd\tau, \\ \forall \psi \in \mathcal{A} := \{\varphi \in C^1(I; C^1(\Omega)) : \varphi(t_2, \cdot) = 0 \wedge \varphi|_{\Gamma_D} = 0\}. \tag{7}$$

Fully implicit finite volumes formulation

We create fully implicit formulation of the problem based on finite volumes principle:

$$\frac{u^n - u^{n-1}}{k} m(V_{ij}) - \epsilon \sum_{|p|+|q|=1} \int_{\sigma_{ij}^{pq}} \nabla u^n \vec{n}_{ij}^{pq} ds - \sum_{|p|+|q|=1} \int_{\sigma_{ij}^{pq}} \mathbf{B} \nabla u^n \vec{n}_{ij}^{pq} ds + \\ + \sum_{|p|+|q|=1} \int_{\sigma_{ij}^{pq}} \vec{A} u^n \vec{n}_{ij}^{pq} ds - u_{ij}^n \int_{V_{ij}} (\nabla \cdot \vec{A}) dx + r u_{ij}^n m(V_{ij}) = f_{ij}^n m(V_{ij}). \tag{8}$$

DDFV fully implicit numerical scheme

For unknown values u_{ij} (primal mesh) we get

$$\begin{aligned}
&\frac{u_{ij}^{n} - u_{ij}^{n-1}}{k} h_x h_y - \epsilon\left(h_y[u_x^{ij,n} - u_x^{i-1j,n}] + h_x[\bar{u}_y^{ij,n} - \bar{u}_y^{ij-1,n}]\right) - \\
&\quad - h_y[b_{ij,10}^{11} u_x^{ij,n} + b_{ij,10}^{12} u_y^{ij,n}] - h_x[b_{ij,01}^{21} \bar{u}_x^{ij,n} + b_{ij,01}^{22} \bar{u}_y^{ij,n}] + \\
&\quad + h_y[b_{ij,-10}^{11} u_x^{i-1j,n} + b_{ij,-10}^{12} u_y^{i-1j,n}] + h_x[b_{ij,0-1}^{21} \bar{u}_x^{ij-1,n} + b_{ij,0-1}^{22} \bar{u}_y^{ij-1,n}] + \\
&\quad + h_y a_{ij,10}^{1} \frac{u_{i+1j}^{n} - u_{ij}^{n}}{2} + h_x a_{ij,01}^{2} \frac{u_{ij+1}^{n} - u_{ij}^{n}}{2} - h_y a_{ij,-10}^{1} \frac{u_{i-1j}^{n} - u_{ij}^{n}}{2} - \\
&\quad - h_x a_{ij,0-1}^{2} \frac{u_{ij-1}^{n} - u_{ij}^{n}}{2} + r u_{ij}^{n} h_x h_y = f_{ij}^{n} h_x h_y
\end{aligned} \tag{9}$$

and analogously for unknown values $\bar{u}_{ij}$ (dual mesh).

Stability estimate

Let it hold

$$\sigma \geq |\rho|, \quad 1 \geq |\rho|\sigma,$$

$$Y(\lambda + \kappa) \geq \kappa\theta - \frac{1}{2}\sigma^2.$$

In addition let ϵ be fixed regularisation parameter. Then for the numerical solution of the scheme the following stability estimates hold:

$$||u_{k,h}||_{L_\infty(I;L_2(\Omega))} \leq C, \;\; ||\nabla u_{k,h}||^2_{L_2(I;L_2(\Omega))} \leq C(\epsilon), \tag{10}$$

where $C(\epsilon)$ is generic constant depending only on the data of the problem and the regularisation parameter ϵ, not on parameters k (time step), h_x and h_y (space steps).

Convergence of the scheme

Convergence of the fully implicit DDFV scheme to the weak solution of the regularised Heston model theorem:

Let Ω be the rectangular domain and $I = [t_1, t_2]$ be the time interval, $0 < t_1 < t_2 < \infty$. Let $u_{k,h}$ be the solution of the fully implicit DDFV scheme for the regularised Heston model. Let (k_m, h_m) be the sequence of the space-time discretisations such that $k_m \to 0$ and $h_m \to 0$ for $m \to \infty$. Then the function $\tilde{u} \in L^2(I; H^1(\Omega))$ such that $u_{k_m,h_m} \rightharpoonup \tilde{u}$ in $L^2(I; H^1(\Omega))$ for $m \to \infty$ is the weak solution of the problem (6).

Regularisation parameter importance experiment

$$\rho = -0.5,\ \sigma = 0.5,\ r = 0.1,\ \kappa = 5,\ \theta = 0.07,\ \lambda = 0.$$

N_x	N_y	N_{ts}	$L_2 D$	$L_2 R, \epsilon = 10^{-2}$	$L_2 R, \epsilon = 10^{-4}$	$L_2 R, \epsilon = 10^{-6}$
20	10	1	0.00815061	0.00827543	0.00815181	0.00815063
40	20	4	0.00607821	0.00613972	0.00607879	0.00607822
80	40	16	0.00548663	0.00553226	0.00548706	0.00548664
160	80	64	0.00529716	0.00533972	0.00529756	0.00529716

Table: Classic and regularised DDFV scheme errors comparison.

The experiment conclusion is that errors of both models are decreasing with the increasing number of the time and space steps. In addition one can see that $L_2 R(\epsilon) \to L_2 D$ as $\epsilon \to 0$ for all listed meshes and for ϵ sufficiently small are the results for the regularised model almost the same as for the non-regularised case.

Conclusion and future research plans

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2. necessary theoretical features of the scheme, stability estimate and convergence, were proven;

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