A staggered scheme on unstructured meshes for the Euler equations

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- A staggered discretization for the Euler system in 2D

$$\begin{cases} \partial_t \rho + \operatorname{div}\left(\rho \mathbf{u}\right) = 0, \\ \partial_t \left(\rho \mathbf{u}\right) + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u}\right) + \nabla p = 0, \\ \partial_t \left(\rho E\right) + \operatorname{div}\left(\rho E \mathbf{u}\right) + \operatorname{div}\left(p \mathbf{u}\right) = 0. \end{cases}$$

$$E = \frac{\|\mathbf{u}\|^2}{2} + e, \quad \text{and} \quad p = (\gamma - 1)\rho e, \quad \gamma > 1.$$

- "Staggered" $\leftrightarrow$ the discrete unknowns are stored at different points
  - scalar unknowns (density ρ, internal energy e, pressure p) on centers
  - velocity $\mathbf{u}$ on vertices

Staggered grids are of interest for flows with incompressible features

$\rightsquigarrow$ avoid odd/even decoupling, checkerboard instabilities.

- Incompressible "multifluid" flows (a dispersed phase in a fluid phase) [Berthelin, Goudon, Minjeaud, 2016]

$$\begin{cases} \partial_t \alpha_p + \mathrm{div}\,(\alpha_p u_p) = 0, \\ \partial_t(\alpha_p u_p) + \mathrm{div}\,\big(\alpha_p u_p \otimes u_p + \pi(\alpha_p)\big) = D\alpha_p(u_f - u_p) - \dfrac{\alpha_p}{m_p}\nabla p + \alpha_p g, \end{cases}$$

$$\begin{cases} \partial_t \alpha_f + \mathrm{div}\,(\alpha_f u_f) = 0, \\ \partial_t(\alpha_f u_f) + \mathrm{div}\,(\alpha_f u_f \otimes u_f) - \mu\Delta u_f = \dfrac{m_p}{m_f} D\alpha_p(u_p - u_f) - \dfrac{\alpha_f}{m_f}\nabla p + \alpha_f g, \end{cases}$$

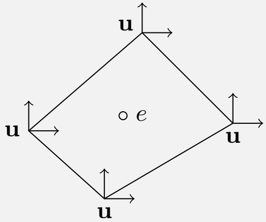
$$\alpha_f + \alpha_p = 1 \qquad \Longleftrightarrow \qquad \mathrm{div}\,\big(\alpha_p u_p + \alpha_f u_f\big) = 0 .$$

- Low Mach number flows [Goudon, Llobell, Minjeaud, 2020]

$$\begin{cases} \partial_t \rho^{(\varepsilon)} + \mathrm{div}\,\big(\rho^{(\varepsilon)}\mathbf{u}^{(\varepsilon)}\big) = 0, \\ \partial_t\big(\rho^{(\varepsilon)}\mathbf{u}^{(\varepsilon)}\big) + \mathrm{div}\,\big(\rho^{(\varepsilon)}\mathbf{u}^{(\varepsilon)} \otimes \mathbf{u}^{(\varepsilon)}\big) + \dfrac{1}{\varepsilon^2}\nabla p^{(\varepsilon)} = 0, \end{cases}$$

- Limit $\varepsilon \to 0 \quad \Longrightarrow \quad \mathrm{div}\,\mathbf{u}^{(0)} = 0$

In the definition of the total energy $E = \frac{\|\mathbf{u}\|^2}{2} + e$:
kinetic and internal energies do not share the same location



- discretization of a non-conservative version + corrective terms

[Herbin, Kheriji, Latché, 2013]

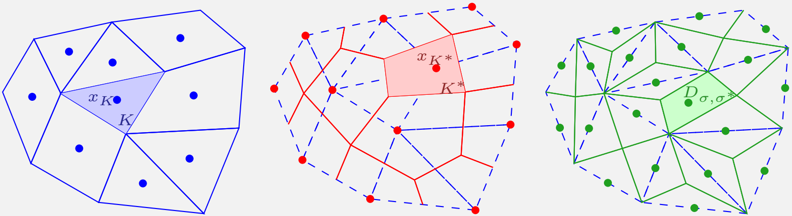
$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}\,(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \\ \partial_t(\rho e) + \mathrm{div}\,(\rho e \mathbf{u}) = -p \mathrm{div}\,\mathbf{u}. \end{cases} \qquad p = (\gamma - 1)\rho e, \quad \gamma > 1$$

- transfert of discrete fields and operators between the different grids

$\rightsquigarrow$ *ad hoc* averaged quantities and operators

[Goudon, Krell, Llobell, Minjeaud, 2021]

▶ The discrete unknowns are constant on different meshes

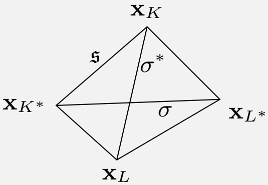


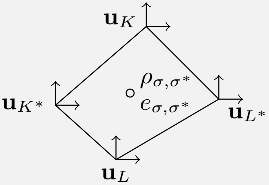
Primal Mesh | Dual mesh | Diamond mesh.

- Density ρ_{σ,σ^*} and internal energy e_{σ,σ^*} on the diamond cell D_{σ,σ^*}

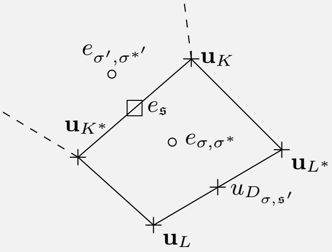
$$p_{\sigma,\sigma^*} = (\gamma - 1)\rho_{\sigma,\sigma^*} e_{\sigma,\sigma^*}.$$

- Velocity fields $(\mathbf{u}_K, \mathbf{u}_{K^*})$ on the primal cell K and on the dual cell K^*.





- We introduce also the following additional notation

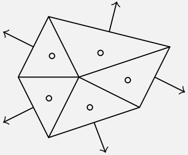


For an edge $\mathfrak{s} = [\mathbf{x}_K, \mathbf{x}_{K^*}]$ between two diamonds D_{σ,σ^*} and $D_{\sigma',\sigma^{*\prime}}$

$$e_{\mathfrak{s}} := \frac{e_{\sigma,\sigma^*} + e_{\sigma',\sigma^{*\prime}}}{2}, \quad \text{and} \quad u_{D_{\sigma,\mathfrak{s}}} := \frac{\mathbf{u}_K + \mathbf{u}_{K^*}}{2} \cdot \mathbf{n}_{D_{\sigma,\mathfrak{s}}}.$$

Discrete operators

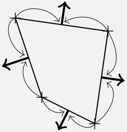
- discrete pressure gradient (on primal and dual cells)



$$(\boldsymbol{\nabla} p)_K = \frac{1}{|K|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K} |\sigma| p_{\sigma,\sigma^*} \mathbf{n}_{K,\sigma},$$

$$(\boldsymbol{\nabla} p)_{K^*} = \frac{1}{|K^*|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_{K^*}} |\sigma^*| p_{\sigma,\sigma^*} \mathbf{n}_{K^*,\sigma^*},$$

- discrete divergence operator (on diamond cells)



$$(\boldsymbol{\nabla} \cdot \mathbf{u})_{\sigma,\sigma^*} = \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s} \in \partial D_{\sigma,\sigma^*}} |\mathfrak{s}| u_{D_{\sigma,\mathfrak{s}}},$$

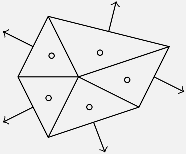
with $u_{D_{\sigma,\mathfrak{s}}} := \frac{\mathbf{u}_K + \mathbf{u}_{K^*}}{2} \cdot \mathbf{n}_{D_{\sigma,\mathfrak{s}}}$.

Discrete operators

- discrete pressure gradient (on primal and dual cells)

$$(\nabla p)_K = \frac{1}{|K|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K} |\sigma| p_{\sigma,\sigma^*} \mathbf{n}_{K,\sigma},$$

$$(\nabla p)_{K^*} = \frac{1}{|K^*|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_{K^*}} |\sigma^*| p_{\sigma,\sigma^*} \mathbf{n}_{K^*,\sigma^*},$$



- discrete divergence operator (on diamond cells)

$$(\nabla \cdot \mathbf{u})_{\sigma,\sigma^*} = \frac{1}{2|D_{\sigma,\sigma^*}|} \Big(|\sigma| \big(\mathbf{u}_L - \mathbf{u}_K\big) \cdot \mathbf{n}_{K,\sigma} + |\sigma^*| \big(\mathbf{u}_{L^*} - \mathbf{u}_{K^*}\big) \cdot \mathbf{n}_{K^*,\sigma^*} \Big),$$

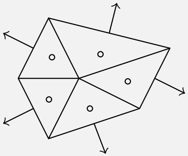
[Domelevo, Omnes 2005]

DISCRETE OPERATORS

- discrete pressure gradient (on primal and dual cells)

$$(\nabla p)_K = \frac{1}{|K|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K} |\sigma| p_{\sigma,\sigma^*} \mathbf{n}_{K,\sigma},$$

$$(\nabla p)_{K^*} = \frac{1}{|K^*|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_{K^*}} |\sigma^*| p_{\sigma,\sigma^*} \mathbf{n}_{K^*,\sigma^*},$$



- discrete divergence operator (on diamond cells)

$$(\nabla \cdot \mathbf{u})_{\sigma,\sigma^*} = \frac{1}{2|D_{\sigma,\sigma^*}|} \Big(|\sigma| \big(\mathbf{u}_L - \mathbf{u}_K\big) \cdot \mathbf{n}_{K,\sigma} + |\sigma^*| \big(\mathbf{u}_{L^*} - \mathbf{u}_{K^*}\big) \cdot \mathbf{n}_{K^*,\sigma^*} \Big),$$

[Domelevo, Omnes 2005]

LOCAL DUALITY RELATIONSHIP

There exists conservative fluxes $q_{D_\sigma,\mathfrak{s}}$ for all D_σ, for all $\mathfrak{s}$, $(\sigma = K|L)$,

$$|K \cap D_\sigma| \mathbf{u}_K \cdot (\nabla p)_K + |L \cap D_\sigma| \mathbf{u}_L \cdot (\nabla p)_L + p_{\sigma,\sigma^*} |\sigma| \big(\mathbf{u}_L - \mathbf{u}_K\big) \cdot \mathbf{n}_{K,\sigma} = \sum_{\mathfrak{s} \in \partial D_\sigma} |\mathfrak{s}| q_{D_\sigma,\mathfrak{s}}$$

[Goudon, Krell, Llobell, Minjeaud, 2020]

$$\partial_t(\rho) + \operatorname{div}(\rho \mathbf{u}) = 0$$

- The densities are updated as follows

$$\frac{\overline{\rho}_{\sigma,\sigma^*} - \rho_{\sigma,\sigma^*}}{\delta t} + \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}| \mathcal{F}_{D_{\sigma,\mathfrak{s}}} = 0,$$

where the mass fluxes $\mathcal{F}_{D_{\sigma,\mathfrak{s}}} = \mathcal{F}^+_{D_{\sigma,\mathfrak{s}}} + \mathcal{F}^-_{D_{\sigma,\mathfrak{s}}}$ $(\sim \rho\mathbf{u}\cdot\mathbf{n})$ with

$$\mathcal{F}^+_{D_{\sigma,\mathfrak{s}}} = \mathcal{F}^+\big(\rho_{\sigma,\sigma^*}, c(e_{\mathfrak{s}}), u_{D_{\sigma,\mathfrak{s}}}\big) \quad \text{and} \quad \mathcal{F}^-_{D_{\sigma,\mathfrak{s}}} = \mathcal{F}^-\big(\rho_{\sigma',\sigma^{*\prime}}, c(e_{\mathfrak{s}}), u_{D_{\sigma,\mathfrak{s}}}\big).$$

- Sound speed $c(e) = \sqrt{\gamma(\gamma-1)e}$.

- Flux splitting functions $\mathcal{F}^+$ and $\mathcal{F}^-$ inspired from the kinetic framework

$$\mathcal{F}^+(\rho, c, u) = \frac{\rho}{2c} \int_{\xi>0} \xi \mathbb{1}_{|\xi-u|\leqslant c}(\xi) d\xi$$

$$\partial_t(\rho) + \mathrm{div}\,(\rho\mathbf{u}) = 0$$

- The densities are updated as follows

$$\frac{\overline{\rho}_{\sigma,\sigma^*} - \rho_{\sigma,\sigma^*}}{\delta t} + \frac{1}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{F}_{D_{\sigma,\mathfrak{s}}} = 0,$$

where the mass fluxes $\mathcal{F}_{D_{\sigma,\mathfrak{s}}} = \mathcal{F}^+_{D_{\sigma,\mathfrak{s}}} + \mathcal{F}^-_{D_{\sigma,\mathfrak{s}}}$ $(\sim \rho\mathbf{u}\cdot\mathbf{n})$ with

$$\mathcal{F}^+_{D_{\sigma,\mathfrak{s}}} = \mathcal{F}^+\big(\rho_{\sigma,\sigma^*}, c(e_{\mathfrak{s}}), u_{D_{\sigma,\mathfrak{s}}}\big) \quad \text{and} \quad \mathcal{F}^-_{D_{\sigma,\mathfrak{s}}} = \mathcal{F}^-\big(\rho_{\sigma',\sigma^{*\prime}}, c(e_{\mathfrak{s}}), u_{D_{\sigma,\mathfrak{s}}}\big).$$

- Sound speed $c(e) = \sqrt{\gamma(\gamma-1)e}$.

- Flux splitting functions $\mathcal{F}^+$ and $\mathcal{F}^-$ inspired from the kinetic framework

$$\mathcal{F}^+(\rho,c,u) = \begin{cases} 0 & \text{if } u \leqslant -c, \\ \dfrac{\rho}{4c}(u+c)^2 & \text{if } |u| \leqslant c, \\ \rho u & \text{if } u \geqslant c, \end{cases} \quad \text{and } \mathcal{F}^-(\rho,c,u) = -\mathcal{F}^+(\rho,c,-u).$$

▶ Transfer of the mass balance equation on primal and dual cells [Ansanay, Babik, Latché, Vola, '11][Goudon, Krell, Llobell, Minjeaud, '20]

- Average density on a primal cell K

$$\rho_K = \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K} \frac{|D_{\sigma,\sigma^*} \cap K|}{|K|} \rho_{\sigma,\sigma^*},$$

- Average mass fluxes $\mathcal{F}_{K,\sigma}$ outgoing from a primal cell K

$$\mathcal{F}_{K,\sigma} = \mathcal{F}^+_{K,\sigma} + \mathcal{F}^-_{K,\sigma},$$

with

$$\mathcal{F}^\pm_{K,\sigma} = \frac{|D_{\sigma,\sigma^*} \cap K|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset L}} \frac{|\mathfrak{s}|}{|\sigma|} \mathcal{F}^\pm_{D_{\sigma},\mathfrak{s}} - \frac{|D_{\sigma,\sigma^*} \cap L|}{|D_{\sigma,\sigma^*}|} \sum_{\substack{\mathfrak{s} \in \partial D_{\sigma,\sigma^*} \\ \mathfrak{s} \subset K}} \frac{|\mathfrak{s}|}{|\sigma|} \mathcal{F}^\mp_{D_{\sigma},\mathfrak{s}}.$$

[Goudon, Krell, 2014]

- The fluxes $\mathcal{F}_{K,\sigma}$ are conservative.
- The average densites ρ_K satisfy the following conservative equations

$$\frac{\overline{\rho}_K - \rho_K}{\delta t} + \frac{1}{|K|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K} |\sigma| \mathcal{F}_{K,\sigma} = 0.$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0$$

- The velocities are updated as follows

$$\frac{\overline{\rho}_K \overline{\mathbf{u}}_K - \rho_K \mathbf{u}_K}{\delta t} + \frac{1}{|K|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_K} |\sigma| \mathcal{G}_{K,\sigma} + (\boldsymbol{\nabla} p)_K = 0,$$

$$\frac{\overline{\rho}_{K^*} \overline{\mathbf{u}}_{K^*} - \rho_{K^*} \mathbf{u}_{K^*}}{\delta t} + \frac{1}{|K^*|} \sum_{D_{\sigma,\sigma^*} \in \mathfrak{D}_{K^*}} |\sigma^*| \mathcal{G}_{K^*,\sigma^*} + (\boldsymbol{\nabla} p)_{K^*} = 0,$$

where the momentum fluxes $\mathcal{G}_{K,\sigma}$ and $\mathcal{G}_{K^*,\sigma^*}$ $(\sim \rho(\mathbf{u} \cdot \mathbf{n})\mathbf{u})$ are defined by

$$\mathcal{G}_{K,\sigma} = \mathcal{F}^+_{K,\sigma} \mathbf{u}_K + \mathcal{F}^-_{K,\sigma} \mathbf{u}_L \qquad \text{and} \qquad \mathcal{G}_{K^*,\sigma^*} = \mathcal{F}^+_{K^*,\sigma^*} \mathbf{u}_{K^*} + \mathcal{F}^-_{K^*,\sigma^*} \mathbf{u}_{L^*}.$$

$$\left.\begin{aligned}&\partial_t(\rho) + \mathrm{div}\,(\rho\mathbf{u}) = 0\\ &\partial_t(\rho\mathbf{u}) + \mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = 0\end{aligned}\right\} \Longrightarrow \partial_t\left(\rho\frac{\|\mathbf{u}\|^2}{2}\right) + \mathrm{div}\left(\rho\mathbf{u}\frac{\|\mathbf{u}\|^2}{2}\right) + \nabla p\cdot\mathbf{u} = 0$$

- Multiplying the momemtum balance eq. on primal mesh by $\mathbf{u}_K$, we find

$$\frac{\overline{\rho}_K\frac{\|\overline{\mathbf{u}}_K\|^2}{2} - \rho_K\frac{\|\mathbf{u}_K\|^2}{2}}{\delta t} + \frac{1}{|K|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K}|\sigma|\mathcal{K}_{K,\sigma} + (\boldsymbol{\nabla}p)_K\cdot\overline{\mathbf{u}}_K = -\mathbb{R}_K,$$

where $\mathcal{K}_{K,\sigma} = \frac{1}{2}\left(\mathcal{F}^+_{K,\sigma}\|\mathbf{u}_K\|^2 + \mathcal{F}^-_{K,\sigma}\|\mathbf{u}_L\|^2\right)$ and

$$\mathbb{R}_K = \frac{\overline{\rho}_K}{2\delta t}\|\overline{\mathbf{u}}_K - \mathbf{u}_K\|^2 + \frac{1}{|K|}\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_K}|\sigma|\mathcal{F}^-_{K,\sigma}\left(\frac{\|\overline{\mathbf{u}}_K - \mathbf{u}_K\|^2}{2} - \frac{\|\overline{\mathbf{u}}_K - \mathbf{u}_L\|^2}{2}\right).$$

- A similar result holds on dual cells K^*

$$\partial_t(\rho e) + \text{div}\,(\rho e \mathbf{u}) + p\text{div}\,\mathbf{u} = 0$$

▶ The internal energies are updated as follows

$$\frac{\overline{\rho}_{\sigma,\sigma^*}\overline{e}_{\sigma,\sigma^*} - \rho_{\sigma,\sigma^*}e_{\sigma,\sigma^*}}{\delta t} + \frac{1}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{E}_{D_{\sigma,\mathfrak{s}}} + p_{\sigma,\sigma^*}\left(\boldsymbol{\nabla}\cdot\overline{\mathbf{u}}\right)_{\sigma,\sigma^*} = \mathbb{R}_{\sigma,\sigma^*},$$

where the numerical fluxes $\mathcal{E}_{D_{\sigma,\mathfrak{s}}}$ $(\sim \rho e \mathbf{u}\cdot\mathbf{n})$ are defined by

$$\mathcal{E}_{D_{\sigma,\mathfrak{s}}} = e_{\sigma,\sigma^*}\mathcal{F}^+_{D_{\sigma,\mathfrak{s}}} + e_{\sigma',\sigma^{*\prime}}\mathcal{F}^-_{D_{\sigma,\mathfrak{s}}},$$

and

$$\mathbb{R}_{\sigma,\sigma^*} = \frac{|D_{\sigma,\sigma^*}\cap K|\mathbb{R}_K + |D_{\sigma,\sigma^*}\cap L|\mathbb{R}_L}{2|D_{\sigma,\sigma^*}|} + \frac{|D_{\sigma,\sigma^*}\cap K^*|\mathbb{R}_{K^*} + |D_{\sigma,\sigma^*}\cap L^*|\mathbb{R}_{L^*}}{2|D_{\sigma,\sigma^*}|}.$$

▶ Positivity of the density

Under the following CFL-like conditions

$$\frac{\delta t}{|D_{\sigma,\sigma^*}|} \sum_{\mathfrak{s} \in \partial D_{\sigma,\sigma^*}} |\mathfrak{s}| [\lambda_+(e_{\mathfrak{s}}, u_{D_{\sigma,\mathfrak{s}}})]^+ \leqslant 1, \qquad \big(\lambda_+(e,u) = u + c(e)\big),$$

the non negativity of the density ρ_{σ,σ^*} is preserved:

$$\rho_{\sigma,\sigma^*} \geqslant 0 \quad \Longrightarrow \quad \overline{\rho}_{\sigma,\sigma^*} \geqslant 0.$$

▶ Positivity of the internal energy

Under more restrictive CFL-like conditions, if $\rho_{\sigma,\sigma^*} \geqslant 0$ we have

- $\mathbb{R}_K \geqslant 0$, and $\mathbb{R}_{K^*} \geqslant 0$,
- the positivity of the internal energy is preserved

$$e_{\sigma,\sigma^*} \geqslant 0 \quad \Longrightarrow \quad \overline{e}_{\sigma,\sigma^*} \geqslant 0.$$

- An average kinetic energy $E^{\mathrm{kin}}_{\sigma,\sigma^*}$ on diamond cell

$$E^{\mathrm{kin}}_{\sigma,\sigma^*} = \frac{|D_{\sigma,\sigma^*} \cap K| \rho_K \|\mathbf{u}_K\|^2 + |D_{\sigma,\sigma^*} \cap L| \rho_L \|\mathbf{u}_L\|^2}{4|D_{\sigma,\sigma^*}|\rho_{\sigma,\sigma^*}} + \frac{|D_{\sigma,\sigma^*} \cap K^*| \rho_{K^*} \|\mathbf{u}_{K^*}\|^2 + |D_{\sigma,\sigma^*} \cap L^*| \rho_{L^*} \|\mathbf{u}_{L^*}\|^2}{4|D_{\sigma,\sigma^*}|\rho_{\sigma,\sigma^*}}.$$

- Total energy E_{σ,σ^*} on diamond cell

$$E_{\sigma,\sigma^*} = e_{\sigma,\sigma^*} + E^{\mathrm{kin}}_{\sigma,\sigma^*}.$$

- The total energy E_{σ,σ^*} satisfies the following conservative equation

$$\frac{\overline{\rho}_{\sigma,\sigma^*}\overline{E}_{\sigma,\sigma^*} - \rho_{\sigma,\sigma^*} E_{\sigma,\sigma^*}}{\delta t} + \frac{1}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}| \mathcal{T}_{D_{\sigma,\mathfrak{s}}} + \frac{1}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}| q_{D_{\sigma,\mathfrak{s}}} = 0,$$

where

- $\mathcal{T}_{D_{\sigma,\mathfrak{s}}}$ is a conservative flux through the interfaces of diamond cells,
- $\frac{1}{|D_{\sigma,\sigma^*}|}\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}| q_{D_{\sigma,\mathfrak{s}}}$ is a conservative discrete version of div $(p\overline{\mathbf{u}})$.

- This result is useful to prove the consistency *à la* Lax Wendroff

[Herbin, Latché, Minjeaud, Therme, 2021]

- It is based on
  - local discrete duality relationship $\rightsquigarrow q_{D_{\sigma,\mathfrak{s}}}$
  - transfert of operators between the grids $\rightsquigarrow \mathcal{T}_{D_{\sigma,\mathfrak{s}}}$
- Both are deduced from the following result

Let us assume that fluxes $F_{K,\sigma}$, for all K, for all σ are given.

There exist fluxes $F_{D_\sigma,\mathfrak{s}}$ for all D_σ, for all $\mathfrak{s}$ such that

$$F_{D_\sigma,\mathfrak{s}} = -F_{D_{\sigma'},\mathfrak{s}} \text{ if } \mathfrak{s} = D_\sigma | D_{\sigma'} \text{ (conservativity)},$$

and, for all D_σ, $\sigma = K|L$,

$$|D_\sigma|\mathrm{div}^{D_\sigma} = \Big(|K \cap D_\sigma|\mathrm{div}^K + |L \cap D_\sigma|\mathrm{div}^L\Big) - \Big(F_{K,\sigma} + F_{L,\sigma}\Big),$$

where

$$\mathrm{div}^{D_\sigma} = \frac{1}{|D_\sigma|}\sum_{\mathfrak{s}\in\partial D_\sigma} F_{D_\sigma,\mathfrak{s}} \quad \text{and} \quad \mathrm{div}^K = \frac{1}{|K|}\sum_{\sigma\in\partial K} F_{K,\sigma}.$$

[Goudon, Krell, Llobell, Minjeaud, 2021]

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and, for all D_σ, $\sigma = K|L$,

$$|D_\sigma| \mathrm{div}^{D_\sigma} = \Big(|K \cap D_\sigma| \mathrm{div}^K + |L \cap D_\sigma| \mathrm{div}^L \Big) - \Big(F_{K,\sigma} + F_{L,\sigma} \Big),$$

- First choice : $F_{K,\sigma} = \mathbf{u}_K |\sigma| p_{\sigma,\sigma^*} \mathbf{n}_{K,\sigma}$.
  - non-conservative quantities : $F_{L,\sigma} = -\mathbf{u}_L |\sigma| p_{\sigma,\sigma^*} \mathbf{n}_{K,\sigma}$.
  - $\mathrm{div}^K = \mathbf{u}_K \cdot (\boldsymbol{\nabla} p)_K$
  - The result above leads to the discrete Green formula

$$|K \cap D_\sigma| \mathbf{u}_K \cdot (\boldsymbol{\nabla} p)_K + |L \cap D_\sigma| \mathbf{u}_L \cdot (\boldsymbol{\nabla} p)_L + p_{\sigma,\sigma^*} |\sigma| \big(\mathbf{u}_L - \mathbf{u}_K \big) \cdot \mathbf{n}_{K,\sigma} = \sum_{\mathfrak{s} \in \partial D_\sigma} |\mathfrak{s}| q_{D_\sigma,\mathfrak{s}}$$

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and, for all D_σ, $\sigma = K|L$,

$$|D_\sigma|\mathrm{div}^{D_\sigma} = \Big(|K \cap D_\sigma|\mathrm{div}^K + |L \cap D_\sigma|\mathrm{div}^L\Big) - \Big(F_{K,\sigma} + F_{L,\sigma}\Big),$$

- Second choice : conservative fluxes $F_{K,\sigma}$.
  - $F_{K,\sigma} + F_{L,\sigma} = 0$
  - The result above leads to the transfert of conservative operators

$$|D_\sigma|\mathrm{div}^{D_\sigma} = \Big(|K \cap D_\sigma|\mathrm{div}^K + |L \cap D_\sigma|\mathrm{div}^L\Big).$$

  - Transfert of kinetic energy fluxes

$$\sum_{\mathfrak{s}\in\partial D_{\sigma,\sigma^*}} |\mathfrak{s}|\mathcal{K}_{D_\sigma,\mathfrak{s}} = \frac{|D_{\sigma,\sigma^*} \cap K|}{|K|} \sum_{\sigma\in\partial K} |\sigma|\mathcal{K}_{K,\sigma} + \frac{|D_{\sigma,\sigma^*} \cap L|}{|L|} \sum_{\sigma\in\partial L} |\sigma|\mathcal{K}_{L,\sigma}$$

Let us assume that fluxes $F_{K,\sigma}$, for all K, for all σ are given.

There exist fluxes $F_{D_\sigma,\mathfrak{s}}$ for all D_σ, for all $\mathfrak{s}$ such that

$$F_{D_\sigma,\mathfrak{s}} = -F_{D_{\sigma'},\mathfrak{s}} \quad \text{if} \quad \mathfrak{s} = D_\sigma | D_{\sigma'} \ \text{(conservativity)},$$

and, for all D_σ, $\sigma = K|L$,

$$|D_\sigma| \mathrm{div}^{D_\sigma} = \Big(|K \cap D_\sigma| \mathrm{div}^K + |L \cap D_\sigma| \mathrm{div}^L \Big) - \Big(F_{K,\sigma} + F_{L,\sigma} \Big),$$

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  - $F_{K,\sigma} + F_{L,\sigma} = 0$
  - The result above leads to the transfert of conservative operators

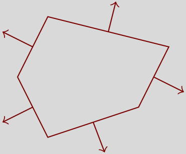
$$|D_\sigma| \mathrm{div}^{D_\sigma} = \Big(|K \cap D_\sigma| \mathrm{div}^K + |L \cap D_\sigma| \mathrm{div}^L \Big).$$

  - Transfert of kinetic energy fluxes

$$\mathcal{T}_{D_\sigma,\mathfrak{s}} = \frac{\mathcal{K}_{D_\sigma,\mathfrak{s}} + \mathcal{K}^*_{D_\sigma,\mathfrak{s}}}{2} + \mathcal{E}_{D_\sigma,\mathfrak{s}}$$

Let K a polygon. Let us assume that fluxes $F_{K,\sigma}$ are given.
There exists a function $\boldsymbol{\omega}_K \in H_{\mathrm{div}}(\mathring{K})$ such that

$$\operatorname{div} \boldsymbol{\omega}_K = \frac{1}{|K|} \sum_{\sigma \in \partial K} F_{K,\sigma} \quad \text{a.e. on } K$$



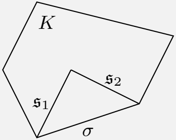
and

$$\int_\sigma \boldsymbol{\omega}_K \cdot \mathbf{n}_{K,\sigma} = F_{K,\sigma}, \quad \forall \sigma \in \partial K.$$

[Ansanay, Babik, Latché, Vola, '11] [Goudon, Krell, Llobell, Minjeaud, '21]

- For $\sigma \in \partial K$ and $\mathfrak{s} \subset K$, we define

$$F_{D_\sigma,\mathfrak{s}} = \int_{\mathfrak{s}} \boldsymbol{\omega}_K \cdot \mathbf{n}_{D_\sigma,\mathfrak{s}}.$$



- The Green formula on $|D_\sigma \cap K|$ gives

$$\int_{D_\sigma \cap K} \operatorname{div}(\boldsymbol{\omega}_K) = \int_{\mathfrak{s}_1} \boldsymbol{\omega}_K \cdot \mathbf{n}_{D_\sigma,\mathfrak{s}_1} + \int_{\mathfrak{s}_2} \boldsymbol{\omega}_K \cdot \mathbf{n}_{D_\sigma,\mathfrak{s}_2} + \int_\sigma \boldsymbol{\omega}_K \cdot \mathbf{n}_{K,\sigma}$$

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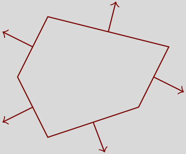

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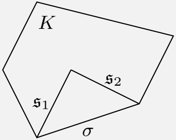

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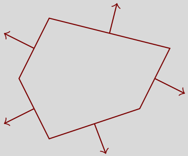

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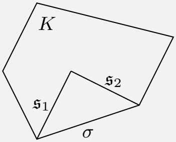

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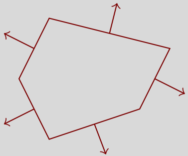

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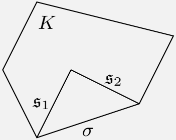

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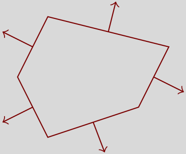

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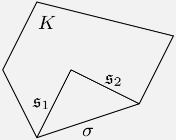

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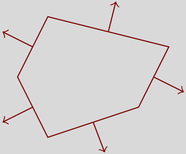

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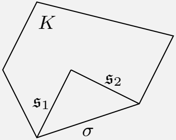

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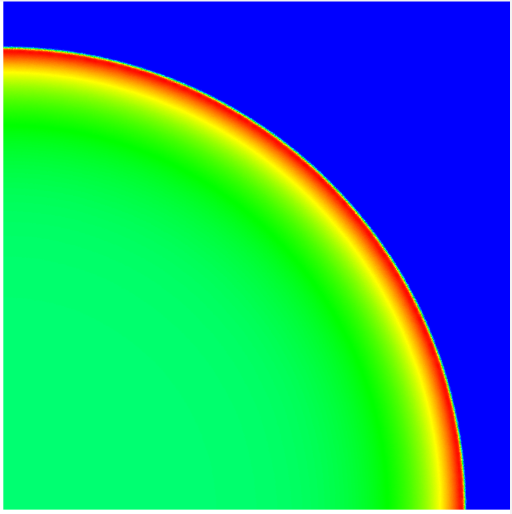
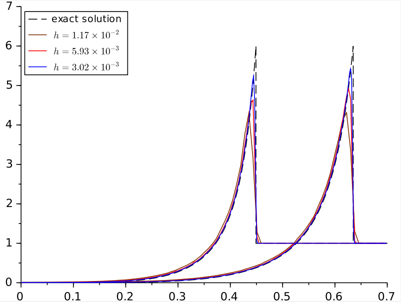
$$|D_\sigma \cap K|\mathrm{div}^K = F_{D_\sigma,\mathfrak{s}_1} + F_{D_\sigma,\mathfrak{s}_2} + F_{K,\sigma}$$

- The sum with the same equality for L gives

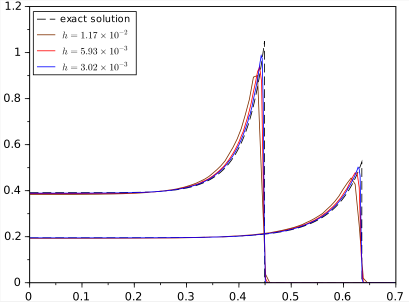
$$|D_\sigma \cap K|\mathrm{div}^K + |D_\sigma \cap L|\mathrm{div}^L = |D_\sigma|\mathrm{div}^{D_\sigma} + F_{K,\sigma} + F_{L,\sigma}$$



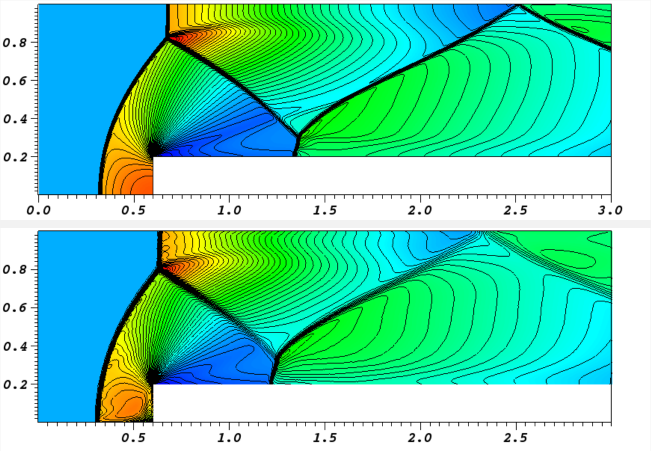
pressure



density



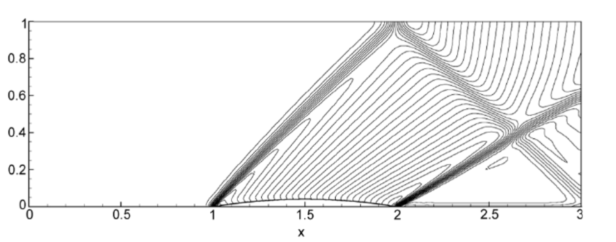
pressure



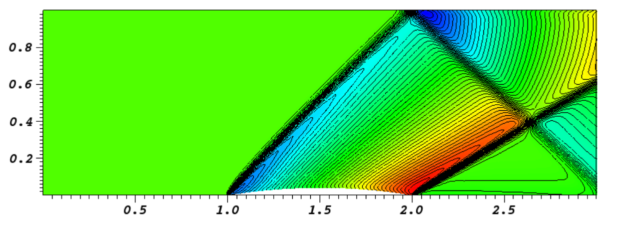
MAC grid

Triangle mesh

Density



[Tsui, Wu, 2007]



Triangle mesh

Iso-Mac

Conclusion

- An explicit staggered scheme for the Euler system
- Preserving the positivity of ρ and e (under CFL conditions)
- A local conservative equation for an averaged total energy
  - Transfert of conservative operators between general grids
  - Local duality relationship

Perspective

- Second order extension
- Time discretization for low Mach number flows
- Well-balanced discretization of source terms
(*e.g.* shallow-water with topography)