Numerical solutions for image processing problems Part I

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Motivation

Karol Mikula's group of our department were working on European projects

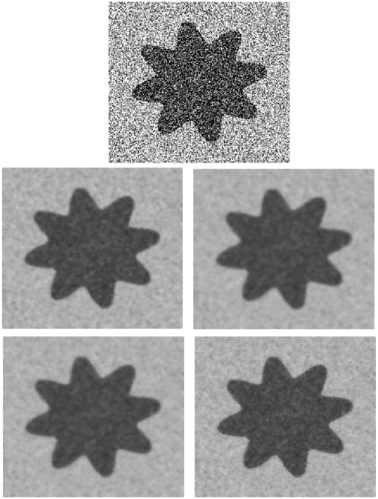
- cooperation with biologists (CNRS-Department of development biology, Institute Pasteur and Institute Curie, Paris), bioengineers (University Bologna), computer scientist (ecole Polytechnique Paris) and supercomupting centers ((IN2P3, Lyon, STUBA, Bratislava)-European projects Embryomics and BioEmergences
- an automated reconstruction of the vertebrate early embryogenesis in space and time
- extraction of the cell trajectories and the cell lineage tree
- reconstruction of the morphogenetics fields
- comparison of untreated and treated cell populations developement

Steps in computational embryogenesis reconstruction

Video

- data acquisition
- image filtering
- cell nuclei center detection
- cell nuclei segmentation
- whole embryo segmentation
- cell tracking and cell trajectories extraction

Smooth initial noisy image and preserve edges



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Denoising the image via nonlinear difusion equations

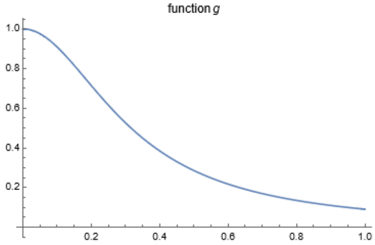
Modified Perona-Malik Equation

In the sense of Catté, Lions, Morel and Coll (1992) [CLMC]

$$\partial_t u - \nabla.(g(|\nabla G_\sigma * u|)\nabla u) = 0 \quad \text{in } Q_T \equiv I \times \Omega,$$

$$\partial_\nu u = 0 \quad \text{on } I \times \partial\Omega,$$

$$u(0, \cdot) = u_0 \quad \text{in } \Omega,$$



Data for Perona-Malik Equation

$\Omega \subset R^2$ - rectangular domain $I = [0, T]$ is a scaling interval
Let $g(s)$ -Lipschitz continuous decreasing function
$g(0) = 1, 0 < g(s) \to 0$ for $s \to \infty$,
$G_\sigma \in C^\infty(R^2)$ -smoothing kernel with compact support K_σ
$\int_{R^2} G_\sigma(x)dx = 1$ $G_\sigma(x) \to \delta_x$ for $\sigma \to 0$, δ_x- Dirac function at point x
$u_0 \in L_\infty(\Omega)$

Weak solution [CLMC]

$u \in L_2(I, W^{1,2}(\Omega))$ satisfying the identity

$$\int_0^T \int_\Omega u \frac{\partial \varphi}{\partial t}(t,x)\, dx\, dt + \int_\Omega u_0(x)\varphi(0,x)\, dx -$$

$$\int_0^T \int_\Omega g(|\nabla G_\sigma * u|)\nabla u(t,x) \nabla \varphi(t,x)\, dx\, dt = 0 \quad \forall \varphi \in \Psi.$$

$\Psi = \{\varphi \in C^{1,2}([0,T] \times \overline{\Omega}), \nabla\varphi.\vec{n} = 0$ on $(0,T) \times \partial\Omega\,, \varphi(T,.) = 0\}.$

Existence of unique weak solution - [CLMC]

Numerical approximation

Mikula and Ramarosy (2001) [MR]
scale discretization - uniform with constant scale step τ
space approximation - Finite volume method

Scale approximation

- uniform scale step $\tau = \frac{T}{N_T}$
- u^n an approximation of $u(t, x)$ at scale $t_n = n\tau$
- first time derivative is replaced by the backward difference $\frac{u^n - u^{n-1}}{\tau}$
- modified Perona-Malik equation can be rewritten into the form of semi-implicit scheme: $\frac{u^n - u^{n-1}}{\tau} = \nabla.(g(|\nabla G_\sigma * u^{n-1}|)\nabla u^n)$

Space approximation-finite volume method

- τ_h - uniform mesh of Ω with cells p of measure $m(p)$
- $N(p)$- set of all neighbours for cell p
- e_{pq} - the common interface of p and q -non-zero measure $m(e_{pq})$
- $\mathcal{E}$ -the set of all these edges for all volumes $p \in \tau_h$
- $x_p \in p$ - representative point for every p
- For every pair $p, q \in N(p)$, $\frac{x_q - x_p}{|x_q - x_p|}$ is equal to a unit normal vector n_{pq} to e_{pq} and oriented from p to q
- $d_{pq} := |x_p - x_q|$
- x_{pq} be the point of e_{pq} intersecting the segment $\overline{x_p x_q}$
- $T_{pq} := \frac{m(e_{pq})}{d_{pq}}$.
- Approximated solution is piecewise constant function in space and scale.

$\frac{u^n - u^{n-1}}{\tau} = \nabla.(g(|\nabla G_\sigma * u^{n-1}|)\nabla u^n)$ Integrating the equation on finite volume p and by application of the divergence theorem we get the integral formulation

$$\frac{u_p^n - u_p^{n-1}}{\tau} m\left(p\right) = \sum_{q \in N(p)} g_{pq}^{\sigma, n-1} T_{pq} \left(u_q^n - u_p^n\right),$$

$u_p^0 = \frac{1}{m(p)} \int_p u_0(x) dx$
$g_{pq}^{\sigma,n-1} := g(|\nabla G_\sigma * \tilde{u}(t_{n-1}, x_{pq})|)$
where $\tilde{u}$ is a periodic extension of the discrete image computed in the $n-1$-th scale step.

Stability results

[MR]- Mikula and Ramarosy (2001)
Stability and convergence properties in $L_2(Q_T)$

$$\max_{0\leq l\leq N_{\max}} \sum_{p\in\tau_h} \left(u_p^l\right)^2 m(p) \leq C_1$$

$$\sum_{l=0}^{N_{\max}} k \sum_{(p,q)\in\mathcal{E}} \frac{\left(u_p^l - u_q^l\right)^2}{d_{pq}} m\left(e_{pq}\right) \leq C_2$$

and the constants C_1, C_2 do not depend on the h, τ.

Convergence [MR]

$\overline{u}_{h,\tau}$ -finite volume numerical solution This solution is piecewise constant on each finite volume and in each scale step.
There exists $u \in L^2(Q_T)$ (weak solution of modified Perona-Malik equation)

$$\overline{u}_{h,k} \to u \text{ in } L^2(Q_T)$$

as $h, \tau \to 0$.

Convergence results for explicit scheme Krivá (2003) [Kr]

Error estimates Krivá, H (2005) [KrH]

Let the relation between scale and space discretization fulfils

$$\tau = Ch,$$

Then for the error estimates for Perona-Malik weak solution and numerical solution obtained via finite volume method it holds

$$\sum_{n=0}^{N_{max}} \int_{I_n} \int_{\Omega} |u(t_{n+1},x) - \overline{u}_{h,k}(t_{n+1},x)|^2 \leq Ch$$

$$\sum_{n=0}^{m-1} \int_{I_n} \sum_{e_{pq} I} m(e_{pq}) d_{pq} \left(\frac{u_q^{n+1} - u_p^{n+1}}{d_{pq}} - \frac{1}{m(e_{pq})} \int_{e_{pq}} \nabla u \cdot \mathbf{n_{pq}} \right)^2 \leq Ch.$$

Numerical experiment Krivá (2011) [FVCA 6]

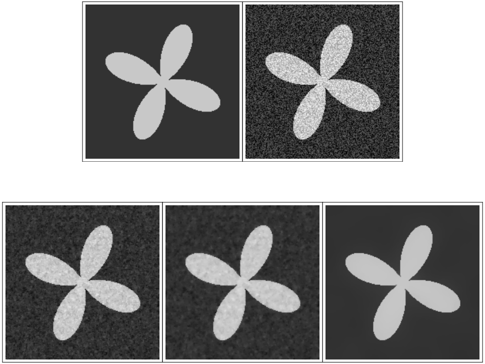


Figure: Image filtering by Perona-Malik model: the original image (left top) noisy image (right top) and the filtering results after 5, 10, 20 time steps

Nonlinear tensor diffusion model Weickert (1998) [W1,W2,W3]

$$\frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) = 0, \text{ in } Q_T \equiv I \times \Omega,$$

$$u(x,0) = u_0(x), \quad \text{in}\Omega,$$

$$(D\nabla u) \cdot \mathbf{n} = 0, \quad \text{on } I \times \partial\Omega.$$

Motivation

The diffusion tensor D steers the smoothing process such that the diffusion is:

- strong in preferred directions, e.g. along edges (in 2D images) or along 2D edge surfaces (in 3D images),
- low in the perpendicular direction.

One can achieve a better connectivity of coherent structures.

Structure tensor

Structure tensor

- $u_{\tilde{t}}(x,t) = (G_{\tilde{t}} * u(\cdot,t))(x), \qquad \tilde{t} > 0,$
- $J_\rho(\nabla u_{\tilde{t}}) = (G_\rho * (\nabla u_{\tilde{t}} \nabla u_{\tilde{t}}^T))(x),\ \rho > 0,$
- $J_\rho = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$

The structure tensor possesses the eigenvalues $\mu_1 \geq \mu_2$ and the orthogonal eigenvectors v and w.
The orientation of w is identical with the coherence

$$\kappa_1 = \alpha, \quad \alpha \in (0,1),\ \alpha \ll 1,$$

$$\kappa_2 = \begin{cases} \alpha, & \text{if } \mu_1 = \mu_2, \\ \alpha + (1-\alpha)\exp\left(\frac{-C}{(\mu_1-\mu_2)^2}\right), & C > 0, \text{ else} \end{cases}$$

Diffusion tensor

$$D = ABA^{-1} = \begin{pmatrix} \lambda & \beta \\ \beta & \nu \end{pmatrix},$$

where

$$A = \begin{pmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{pmatrix},$$

$$B = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

Numerical discretization Drblíková Mikula (2007)[DM]

Original equation

$$\frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u) = 0,$$

Discretization in time (k is uniform time step) - semi-implicit method finite volume method in space (W is arbitrary finite volume)

$$\frac{u_W^n - u_W^{n-1}}{k} m(W) - \sum_{\sigma \in \mathcal{E}_W \cap \mathcal{E}_{int}} \int_\sigma (D^{n-1} \nabla u^n) \cdot \mathbf{n}_{W,\sigma} ds = 0,$$

$$\frac{u_W^n - u_W^{n-1}}{k} - \frac{1}{m(W)} \sum_{\sigma \in \mathcal{E}_W \cap \mathcal{E}_{int}} \phi_\sigma^n (u_{h,k}^n) m(\sigma) = 0,$$

$\phi_\sigma^n(u_{h,k}^n) \approx \frac{1}{m(\sigma)} \int_\sigma (D^{n-1} \nabla u^n) \cdot \ \mathbf{n}_{W,\sigma} ds.$

Five-point scheme

	E_1	
E_2	W	E_3
	E_4	

Nine-point scheme

E_1	E_2	E_3
E_4	W	E_5
E_6	E_7	E_8

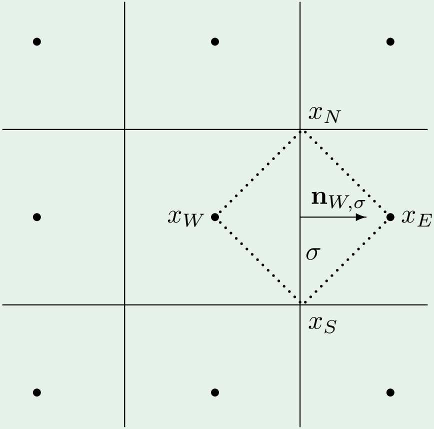
Perona-Malik equation

$$u_t - \nabla \cdot (g(|\nabla G_{\tilde{t}} * u|)\nabla u) = 0.$$

Nonlinear tensor diffusion equation

$$u_t - \nabla \cdot (D\nabla u) = 0.$$

Diamond-cell finite volume scheme



Semi-implicit finite volume scheme

$$\frac{u_W^n - u_W^{n-1}}{k} \frac{1}{m(W)} \sum_{\sigma \in \mathcal{E}_W \cap \mathcal{E}_{int}} \phi_\sigma^n(u_{h,k}^n) m(\sigma) = 0,$$

$$\phi_\sigma^n(u_{h,k}^n) = \bar{\lambda}_\sigma \frac{u_E^n - u_W^n}{h} + \bar{\beta}_\sigma \frac{u_N^n - u_S^n}{h}.$$

Convergence analysis Drblíková Mikula (2007)[DM]

Convergence analysis

- Lemma 1 (Uniform boundedness) There exists a positive constant C such that $\|u_{h,k}\|_{L^2(Q_T)} \leq C$.
- Lemma 2(Time translate estimate) For any $s \in (0,T)$ there exists a positive constant C such that
$$\int\limits_{\Omega\times(0,T-s)} \left(u_{h,k}\left(x,t+s\right) - u_{h,k}\left(x,t\right)\right)^2 dxdt \leq Cs.$$
- Lemma 3 (Space translate estimate) For any vector $\xi \in R^d$ there exists a positive constant C such that
$$\int\limits_{\Omega\times(0,T)} \left(u_{h,k}\left(x+\xi,t\right) - u_{h,k}\left(x,t\right)\right)^2 dxdt \leq C\left|\xi\right|.$$

Theorem

The sequence $u_{h,k}$ converges strongly in $L^2(Q_T)$ to the unique weak solution u as $h, k \to 0$.

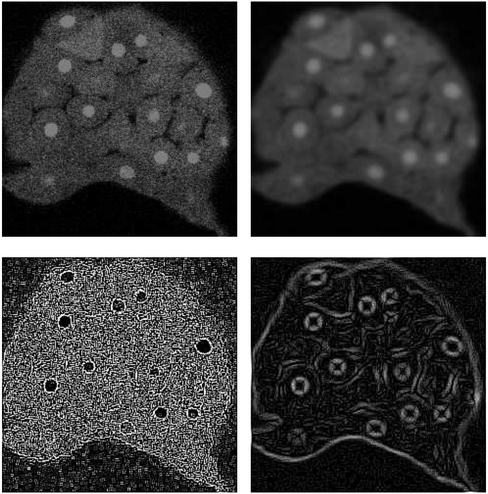
Drblíková, H., Mikula [DHM] (2009) Error estimate

Theorem

Let the exact solution fulfill the following regularity properties: $\nabla u \in L_\infty(Q_T)$, $u_{tt} \in L_2(Q_T)$, $u \in L_2(I, W^{2,2}(\Omega))$, $\nabla u_t \in L_2(I, L_\infty(\Omega))$. Let $e_W^n = u(x_W, t_n) - u_W^n$ and $e_{h,k}^m(x,t) = \sum_{W \in \mathcal{T}_h} e_W^n \chi\{x \in W\} \chi\{t_{n-1} < t \leq t_n\}$. Then, there exist a constant C, such that for sufficiently small h

$$\int_\Omega |e_{h,k}^m|^2 dx + \sum_{n=1}^{m} \int_\Omega |e_{h,k}^n - e_{h,k}^{n-1}|^2 dx + \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \sum_{\sigma \in \mathcal{E}_{int}} (e_E^n - e_W^n)^2 dt \leq C(h^2 + k)$$

for every $m = 1, ..., N_{\max}$.



Top: the original image and the filtered image after 10 time steps.
Bottom: the edge detections of these images.

Denoising the image using mean curvature flow problem

Level set formulation of the mean curvature flow problem Sethian Osher (1988), (1996) [S], [OS]

$$u_t - |\nabla u| \nabla . \left(\frac{\nabla u}{|\nabla u|} \right) = 0, \text{in } \Omega \times [0, T]$$

the initial condition

$$u(x, 0) = u_0(x), \text{ a.e. } x \in \Omega,$$

zero Dirichlet or Neumann boundary conditions

Regularized level set equation

Evans-Spruck regularization (1991)[ES]
given small parameter $\varepsilon > 0$

$$\frac{u_t}{\sqrt{\varepsilon^2 + |\nabla u|^2}} - \nabla . \left(\frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}} \right) = 0$$

$$|\nabla u|_\varepsilon = \sqrt{\varepsilon^2 + |\nabla u|^2}$$

Existence of "viscose solution" of the problem.

Explicit finite difference scheme on rectangular grids

S. Osher and J. A. Sethian,(1988), J. A. Sethian, (1999)
Equation in 2D- case:

$$u_t - \frac{u_{xx}(u_y^2 + \varepsilon^2) + u_{yy}(u_x^2 + \varepsilon^2) - 2u_x u_y u_{xy}}{u_x^2 + u_y^2 + \varepsilon^2} = 0$$

Discretization- squares with the side of the length h
Unknowns at the n-th time step - $u_{i,j}^n$.

$$u_x{}_{i,j}^n = \frac{u_{i+1,j}^n - u_{i-1,j}^n}{2h}, \quad u_y{}_{i,j}^n = \frac{u_{i,j+1}^n - u_{i,j-1}^n}{2h},$$

$$u_{xx}{}_{i,j}^n = \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2}, \quad u_{yy}{}_{i,j}^n = \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2},$$

$$u_{xy}{}_{i,j}^n = \frac{u_{i+1,j+1}^n + u_{i-1,j-1}^n - u_{i-1,j+1}^n - u_{i+1,j-1}^n}{4h^2},$$

Discretization in time semi-implicit scheme

Uniform discrete time step
$\tau = \frac{T}{N}$
the time derivative - the backward difference
The nonlinear terms of the equation - previous time step

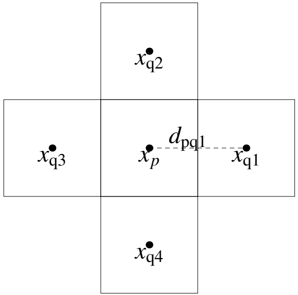
$$\frac{1}{|\nabla u^{n-1}|_\varepsilon} \frac{u^n - u^{n-1}}{\tau} = \nabla . \left(\frac{\nabla u^n}{|\nabla u^{n-1}|_\varepsilon} \right)$$

Space discretization

We can denote p as the finite volume with measure of $m(p)$, e^{pq} as the edge (face) between two finite volumes p and q and $N(p)$ as the set of all finite volume neighbors

- by application of the **divergence theorem we get the integral formulation**
$$\int_p \frac{1}{|\nabla u^{n-1}|_\varepsilon} \frac{u^n - u^{n-1}}{\tau} dx = \sum_{q \in N(p)} \int_{e^{pq}} \frac{1}{|\nabla u^{n-1}|_\varepsilon} \frac{\partial u^n}{\partial \nu} ds$$

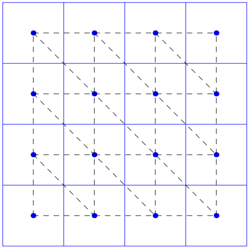
Space discretization in 2D H, Mikula, Sgallari (2003)



Ω -union of finite volumes p, x_p -representative point in each p
$N(p)$ the set of all nodes q connected to the node p by an edge
σ_{pq} cardinality $N(p) = N_p$, $|x_p - x_q| = d_{pq}$

Co volume scheme in 2D

Evaluate regularized gradients on each triangle :$|\nabla u_T^{n-1}|_\varepsilon$
$N(\sigma_{pq})$ the set of all triangles connected with the edge σ_{pq}
$c_{pq}^T = m(T \cap \sigma_{pq})$



Fully discrete semi implicit co-volume scheme

$$b_p^{n-1}\ m(p)\ u_p^n + \tau \sum_{q \in N(p)} a_{pq}^{n-1} \frac{(u_p^n - u_q^n) m(\sigma_{pq})}{d_{pq}} = b_p^{n-1}\ m(p)\ u_p^{n-1}$$

$$a_{pq}^{n-1} = \sum_{T \in N(\sigma_{pq})} \frac{c_{pq}^T}{|\nabla u_T^{n-1}|_\varepsilon}$$

$$b_p^{n-1} := \frac{1}{|\nabla u_p^{n-1}|_\varepsilon},$$

$$|\nabla u_p^{n-1}|_\varepsilon = \sum_{T;\ p \cap T \neq \emptyset} \frac{m(p \cap T)}{m(p)} |\nabla u_T^{n-1}|_\varepsilon$$

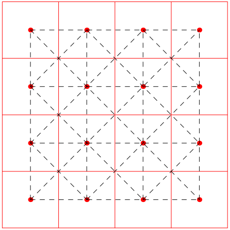
Results H. Mikula, Sgallari (2003) [HMS]

Theorem There exists limit u_h^n of a subsequence $u_{h,\varepsilon}^n$, the solutions of proposed numerical scheme, for $\varepsilon \to 0$. Moreover for this solution the following estimates hold:

$$||u_h^n||_{L_\infty(\Omega)} \leq ||u_h^0||_{L_\infty(\Omega)}$$
$$||\nabla u_h^n||_{L_1(\Omega)} \leq ||\nabla u_h^0||_{L_1(\Omega)}, \quad 1 \leq n \leq N$$

Modified co volume scheme in 2D

Co volume grids
Evaluate gradients on each triangle
$m(p) = h^2, \; m(\sigma_{pq}) = h, \; d_{pq} = h$



Fully discrete semi implicit modified co-volume scheme

$$a_{pq}^{n-1} = \frac{1}{|\nabla u_{pq}^{n-1}|_\varepsilon} := \frac{1}{2}\left(\frac{1}{|\nabla u_{T_{pq}^1}^{n-1}|_\varepsilon} + \frac{1}{|\nabla u_{T_{pq}^2}^{n-1}|_\varepsilon}\right),$$

$$b_p^{n-1} := \frac{1}{|\nabla u_p^{n-1}|_\varepsilon} = \frac{1}{N_p}\sum_{q\in N(p)} \frac{1}{|\nabla u_{pq}^{n-1}|_\varepsilon},$$

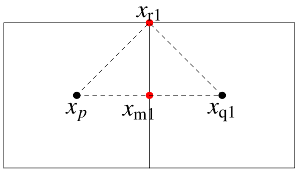
where $T_{pq}^1, T_{pq}^2 \in N(\sigma_{pq})$.

Fully discrete semi implicit modified co-volume scheme

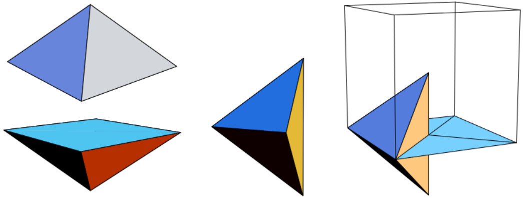
For example for triangle with points x_p, x_{q_1}, x_{r_1} we have

$$|\nabla u^{n-1}_{T^1_{pq_1}}|_\varepsilon = \sqrt{\frac{(u_{q_1} - u_p)^2}{h^2} + \frac{(2(u_{r_1} - u_{m_1}))^2}{h^2} + \varepsilon^2}.$$

$$b^{n-1}_p \; m(p) \; u^n_p + \tau \sum_{q \in N(p)} a^{n-1}_{pq} \frac{(u^n_p - u^n_q) m(\sigma_{pq})}{d_{pq}} = b^{n-1}_p \; m(p) \; u^{n-1}_p.$$



Co volume scheme in 3D S.Corsaro, K.Mikula, A.Sarti, F.Sgallari [CMSS] 2006



3D implementation - every cubic voxel is splitted into 6 pyramids. The neighbouring pyramids of neighbouring voxels are joined together to form octahedron (**diamond cell** for the face) which can be itself used to evaluate gradients of solution on the face or it can be further split into 4 tetrahedras, elements of 3D triangulation on which we can evaluate nonlinearities depending on gradients.

Results H., Mikula (2008) [HM]

Theorem: There exists unique solution u_h^n of the numerical scheme for any value of the regularization parameter ε and for any time step $n = 1, \ldots, N$. Moreover approximation scheme has **stability** and **consistency** property.

Eymard, H., Mikula: Regularized mean curvature flow

$$u_t - g(|\nabla u|)\,\mathrm{div}\left(\frac{\nabla u}{f(|\nabla u|)}\right) = r,$$

with the initial condition

$$u(x,0) = u_0(x),\ \text{a.e. } x \in \Omega,$$

and the boundary condition

$$u(x,t) = 0,\ \text{a.e. } (x,t) \in \partial\Omega \times \mathbb{R}_+,$$

For regularized level set equation $f(x) = g(x) = \min(\sqrt{x^2 + a^2}, b)$

Hypotheses (H)

1. Ω is a finite connected open subset of $\mathbb{R}^d$, $d \in \mathbb{N}^\star$, with boundary $\partial\Omega$ defined by a finite union of subsets of hyperplanes of $\mathbb{R}^d$,
2. $u_0 \in H_0^1(\Omega)$,
3. $r \in L^2(\Omega \times]0, T[)$ for all $T > 0$,
4. $g \in C^0(\mathbb{R}_+; [a, b])$, with $0 < a < b$,
5. $f \in C^0(\mathbb{R}_+; [a, b])$ is a Lipschitz continuous (non-strictly) increasing function, and $x \mapsto x/f(x)$ is strictly increasing on $\mathbb{R}_+$.

Weak solution

Under hypotheses (H), we say that u is a weak solution if, for all $T > 0$,

1. $u \in L^2(0,T;H_0^1(\Omega))$ and $u_t \in L^2(\Omega\times]0,T[)$ (hence $u \in C^0(0,T;L^2(\Omega))$).
2. $u(\cdot,0) = u_0$

and

$$\int_0^T \int_\Omega \left(\frac{u_t(x,t)v(x,t)}{g(|\nabla u(x,t)|)} + \frac{\nabla u(x,t)\cdot\nabla v(x,t)}{f(|\nabla u(x,t)|)} \right) \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \frac{r(x,t)v(x,t)}{g(|\nabla u(x,t)|)} \mathrm{d}x\mathrm{d}t,$$
$$\forall v \in L^2(0,T;H_0^1(\Omega)).$$

Eymard et all finite volume method in 2D

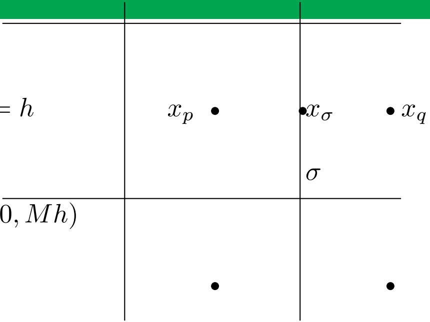
$h > 0,\ x_q - x_p = h$

$T = N_T \cdot \tau$

$\Omega = (0, Nh) \times (0, Mh)$

$(x_p, n\tau) \ldots . u_p^n$

$(x_\sigma, n\tau) \ldots . u_\sigma^n$



$\mathcal{M} = \{$all finite volumes $p,$ with representative point $x_p\}$,
$\mathcal{E} = \{$ all edges σ with representative point $x_\sigma\}$,
$\mathcal{E}_p$ is the subset of all $\sigma \in \mathcal{E}$ such that $\sigma \subset \partial p$, for all $p \in \mathcal{M}$,
$\mathcal{N}_p$ is the subset of all $q \in \mathcal{M}$ neighboring to p, for all $\sigma \in \mathcal{E}$,
$\mathcal{M}_\sigma$ is the subset of $p \in \mathcal{M}$ such that $\sigma \in \mathcal{E}_p$.

Discrete H norm Eymard, Gallouet, Herbin

$H_{\mathcal{D}} \subset \mathbb{R}^{\mathcal{M}} \times \mathbb{R}^{\mathcal{E}}$ such that $u_\sigma = 0$ for all $\sigma \in \mathcal{E}_{\text{ext}}$.

$$N_p(u)^2 = \frac{1}{|p|} \sum_{\sigma \in \mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} (u_\sigma - u_p)^2, \ \forall p \in \mathcal{M}, \ \forall u \in H_{\mathcal{D}}. \tag{1}$$

$$\|u\|_{1,\mathcal{D}}^2 = \sum_{p \in \mathcal{M}} |p| N_p(u)^2 \tag{2}$$

defines a norm on $H_{\mathcal{D}}$.

Relation on interior edges

$$\frac{u_\sigma^{n+1} - u_p^{n+1}}{f(N_p(u^n))\ d_{p\sigma}} + \frac{u_\sigma^{n+1} - u_q^{n+1}}{f(N_q(u^n))\ d_{q\sigma}} = 0, \tag{3}$$

$$\forall \sigma \in \mathcal{E}_{\text{int}} \text{ with } \mathcal{M}_\sigma = \{p, q\}, \ \forall n \in \mathbb{N}, \tag{4}$$

Semi-implicit scheme

Integrating regularized level set equation in every finite volume p and using divergence theorem

$$\frac{|p|(u_p^{n+1} - u_p^n)}{\tau\ g(N_p(u^n))} - \frac{1}{f(N_p(u^n))} \sum_{\sigma \in \mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} (u_\sigma^{n+1} - u_p^{n+1}) = \frac{r_p^{n+1}}{\tau\ g(N_p(u^n))}, \quad \forall p \in \mathcal{M},\ \forall n \in \mathbb{N},$$

$$u_p^0 = \frac{1}{|p|} \int_p u_0(x) \mathrm{d}x,\ \forall p \in \mathcal{M}, \tag{5}$$

$$r_p^{n+1} = \int_{n\tau}^{(n+1)\tau} \int_p r(x,t) \mathrm{d}x \mathrm{d}t,\ \forall p \in \mathcal{M},\ \forall n \in \mathbb{N}, \tag{6}$$

Discrete solutions and terms for fully implicit scheme

$$u_{\mathcal{D},\tau}(x,t) = u_p^{n+1},$$

$$z_{\mathcal{D},\tau}(x,t) = \frac{u_p^{n+1} - u_p^n}{\tau},$$

$$N_{\mathcal{D},\tau}(x,t) = N_p(u^{n+1}),$$

$$G_{\mathcal{D},\tau}(x,t) = d\frac{u_\sigma^{n+1} - u_p^{n+1}}{d_{p\sigma}}\mathbf{n}_{p\sigma},$$

$$H_{\mathcal{D},\tau}(x,t) = d\frac{u_\sigma^{n+1} - u_p^{n+1}}{d_{p\sigma} f(N_p(u^{n+1}))}\mathbf{n}_{p\sigma},$$

$$w_{\mathcal{D},\tau}(x,t) = -\frac{u_p^{n+1} - u_p^n}{\tau\ g(N_p(u^{n+1}))} + \frac{r_p^{n+1}}{|p| \quad \tau\ g(N_p(u^{n)1}))},$$

Convergence result

Let Hypotheses (H) be fulfilled. Let $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$ be a sequence of space-time discretizations of $\Omega\times]0,T[$, such that $h_{\mathcal{D}_m}$ and $\tau_m > 0$ tends to 0 as $m \longrightarrow \infty$. Let, for all $m \in \mathbb{N}$, $u_{\mathcal{D}_m,\tau_m}$ be such that semi implicit or fully implicit scheme hold.
Then there exists a subsequence of $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$, again denoted $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$, and there exists a function
$\bar{u} \in L^\infty(0,T; H_0^1(\Omega))$, weak solution, such that
$u_{\mathcal{D}_m,\tau_m}$ tends to $\bar{u}$ in $L^\infty(0,T; L^2(\Omega))$,
$N_{\mathcal{D}_m,\tau_m}$ tends to $|\nabla\bar{u}|$ in $L^2(\Omega\times]0,T[)$.

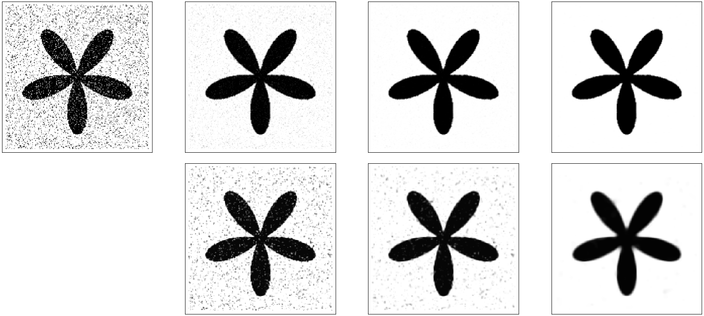
Sketch of the proof

- L^∞ stability of $u_{\mathcal{D}_m,\tau_m}$, existence and uniqueness of the discrete solution
- $L^2(\Omega \times (0,T))$ estimate on discrete u_t and estimate $L^\infty(0,T;H_{\mathcal{D}})$of $u_{\mathcal{D}_m,\tau_m}$
- convergence results:
  $u_{\mathcal{D}_m,\tau_m} \longrightarrow \bar{u}$ in $L^2(\Omega \times (0,T))$. $\bar{u} \in L^2(0,T;H_0^1(\Omega))$
- $G_{\mathcal{D}_m,\tau_m} \in L^\infty(0,T;L^2(\Omega))$ $G_{\mathcal{D}_m,\tau_m} \rightharpoonup \nabla\bar{u}$ weakly in $L^2(\Omega \times (0,T))^d$.

Sketch of the proof -continuation

- $H_{\mathcal{D}_m,\tau_m} \rightharpoonup \bar{H}$ and $\widetilde{H}_{\mathcal{D}_m,\tau_m} \rightharpoonup \bar{H}$ weakly in $L^2(\Omega \times (0,T))^d$
- $w_{\mathcal{D}_m,\tau_m} \rightharpoonup \bar{w}$
- $z_{\mathcal{D}_m,\tau_m} \rightharpoonup \bar{u}_t$ weakly in $L^2(\Omega \times (0,T))$.
- $$\lim_{m\to\infty} \int_0^T \int_\Omega \frac{N_{\mathcal{D}_m,\tau_m}(x,t)^2}{f(N_{\mathcal{D}_m,\tau_m}(x,t))} \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \bar{H}(x,t) \cdot \nabla \bar{u}(x,t) \mathrm{d}x\mathrm{d}t.$$
- $N_{\mathcal{D}_m,\tau_m} \to |\nabla \bar{u}|$ in $L^2(\Omega \times (0,T))$
- passing to the limit in equation

Example 1 noisy filtering 20 % salt-and-pepper noise

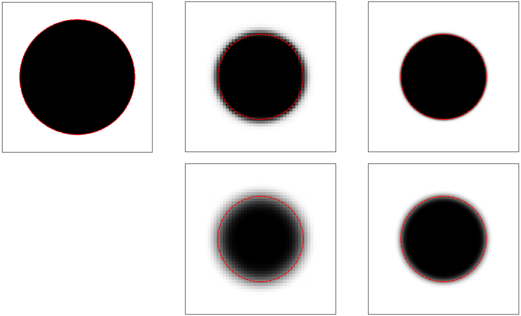


Dimensions of the image are $N_1 = N_2 = 200$.

Iinitial image (top left) filtering by FV after 1, 2, 3 time steps (top), and by the explicit FD after 1, 4, 30 time steps (bottom).

Example 2- moving circle

The exact viscosity solution - the characteristic function of $R_t = \{(x,y) \in \mathbb{R}^2, x^2 + y^2 + 2t \leq 1\}$ (the inside of the circle with centre $(0,0)$ and radius $r(t) = \sqrt{1-2t}$),$[0,T] = [0, 0.25]$.



Initial condition (top left), fully implicit FV with $n = 50,\ n = 250$ (top), explicit FD with $n = 50,\ n = 250$ (bottom) at time 0.25 (n-number of representation points along one side)

Geodesic mean curvature flow equation

Geodesic mean curvature flow equation (Caselles, Kimmel, Sapiro and Chen, Vemuri, Wang)

$$u_t = |\nabla u| \nabla . \left(g(|\nabla G_\sigma * u|) \frac{\nabla u}{|\nabla u|} \right)$$

$$u(0,x) = I^0(x),$$

Neumann boundary conditions

Numerical scheme combines approximation using in Perona-Malik equation (term with function G) and numerical approximation for level set equation.

Results: semi- implicit schemes:

Kačur, Mikula 1995

Weickert 1995

using co-volume scheme H. Mikula Sgallari 2003, 2006

Image segmentation

Subjective surface method -Sarti, Malladi, Sethian (2000)
ε-regularization of the geodesic mean curvature flow equation

$$u_t = \sqrt{\varepsilon^2 + |\nabla u|^2} \nabla . \left(g \frac{\nabla u}{\sqrt{\varepsilon^2 + |\nabla u|^2}} \right),$$

$$g = g(|\nabla G_\sigma * I^0|)$$

Numerical analysis for semi imlpicit scheme and space aproximation by finite volume method- H., Tibenský (2018) [HT]

Image segmentation II

Generalized version with different weigths to advective and diffusive parts
- K.Mikula., N.Peyriéras, M.Remešiková, A.Sarti (2008, FVCA5) and C.Zanella et al.(2010, IEEE TIP)

$$u_t = \mu_1 \; g|\nabla u|\nabla . \left(\frac{\nabla u}{|\nabla u|} \right) + \mu_2 \; \nabla g . \nabla u$$

efficient 3D implementations using semi-implicit scheme in curvature part and up-wind schemes in advective part - M.Remešiková, R.Čunderlik, K.Mikula

Video

K. Mikula, R.Čunderlik, O.Drbliková, M.Remešiková, M.Smišek, R.Špir
(Bratislava)

P.Bourgine (Paris), N.Peyrieras (Gif-sur-Yvette), A.Sarti (Bologna)

Thank you for your attention!

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