Numerical scheme based on the Riemann solver problem applied to protoplanetary disks

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Outline

Formation of planets in a Keplerian disk

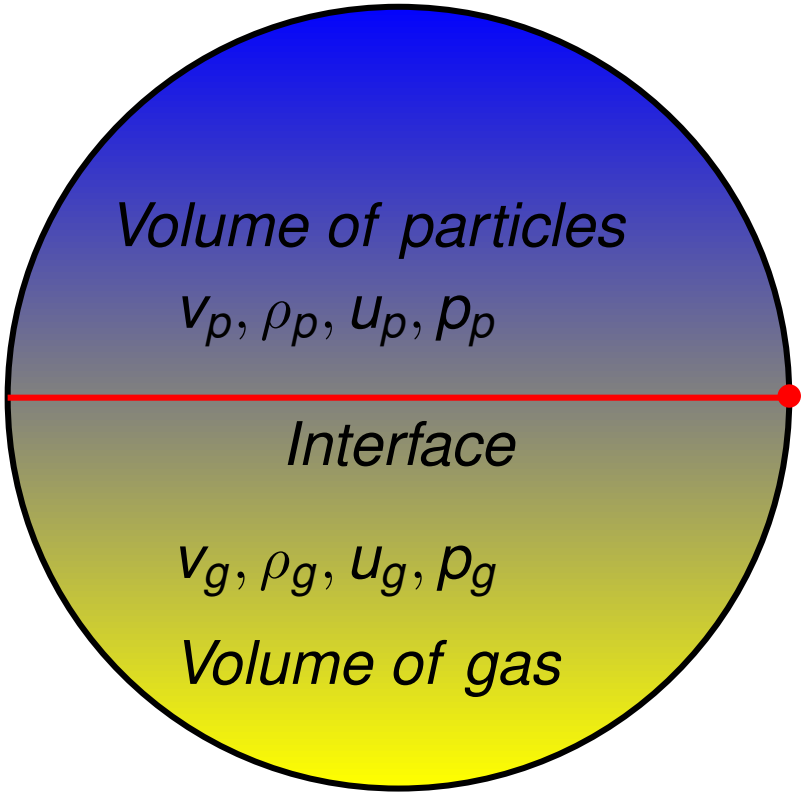


Figure: Modeling of two-phase flow.

Formation of planets in a Keplerian disk

- Assuming that the disk is at hydrostatic equilibrium along the z-axis. Denoting Φ the gravitational potential, the equilibrium hydrostatic equation involving the pressure p is given by:

$$\overrightarrow{grad(p)} = -\rho.\overrightarrow{grad(\Phi)}, \tag{1}$$

- Φ can be determined over the z-axis by the following equality:

$$\begin{aligned} \frac{1}{\rho}\frac{\partial p}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{GM}{\sqrt{r^2+z^2}}\right), \\ &= -\Omega^2 z. \end{aligned} \tag{2}$$

Euler equations in one/two-phase

- The conservation laws involving the cylindrical coordinates are written in a bi-dimensional space:

$$\frac{\partial w}{\partial t} + \frac{1}{r}\frac{\partial rF(w)}{\partial r} + \frac{1}{r}\frac{\partial G(w)}{\partial \theta} = Q(w). \tag{3}$$

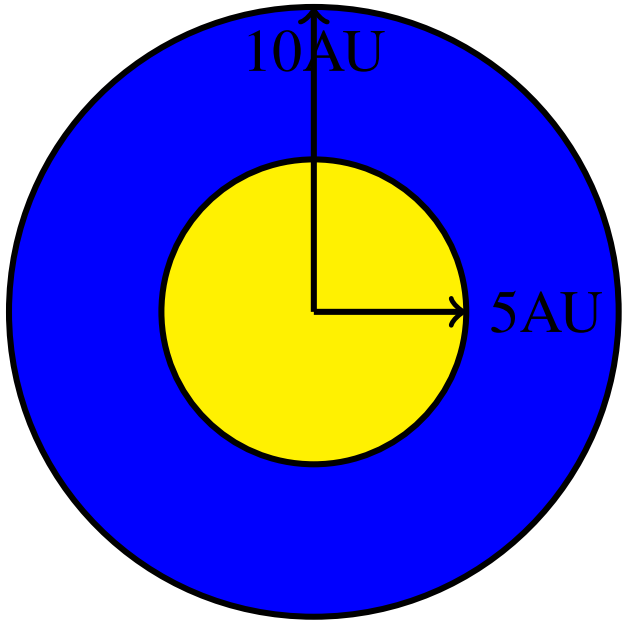
In which, F and G present respectively the fluxes in the radial and azimuthal directions, defined by:

$$F(w) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (\rho e + p)u \end{pmatrix}, G(w) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (\rho e + p)v \end{pmatrix}, Q(w) = \begin{pmatrix} 0 \\ \frac{\rho v^2}{r} - \rho\frac{GM}{r^2} + \frac{p}{r} + F_r \\ -\frac{\rho uv}{r} + F_\theta \\ -\rho u\frac{GM}{r^2} \end{pmatrix}.$$

Computational domain

- The spatial distribution of cell is defined by the radial step Δr and the angular step $\Delta\theta$ that given by:

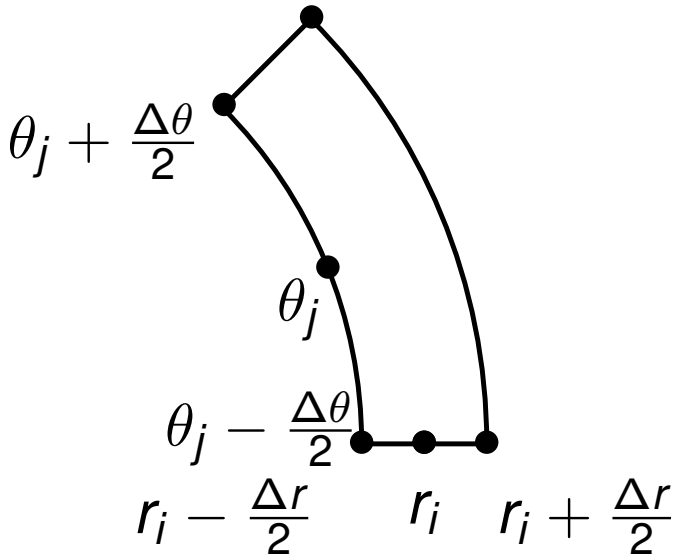
$$\begin{cases} \Delta r = \frac{r_{\text{out}} - r_{\text{in}}}{n_r}, \\ \Delta\theta = \frac{2\pi}{n_\theta}. \end{cases} \tag{4}$$



Cylindrical cell

• The time-space complexity of the cell (r_i, θ_j) is configurated as follows:

$$\begin{cases} r_i = r_{\text{in}} + (i - \frac{1}{2})\Delta r, \\ \theta_j = (j - \frac{1}{2})\Delta\theta. \end{cases} \tag{5}$$

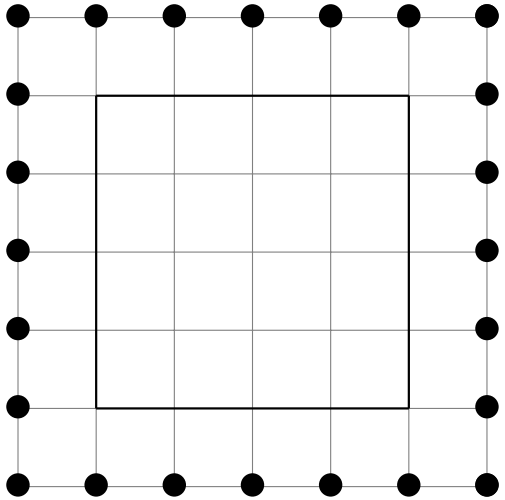


Ghost cells

- The ghost cells provide the well-posed boundary conditions for the conservative equations, that are defined for each cell $J_{i,j}$ by:

$$J_{i,j} = \left[r_{i-1/2}, r_{i+1/2}\right] \times \left[\theta_{j-1/2}, \theta_{j+1/2}\right]. \tag{6}$$

- The periodic boundary conditions are imposed in the azimuthal direction.



Initial conditions

• The initial profile is previously introduced by Hayashi (1985). The density profile is calculated from the mass of planets.

$$\begin{cases} \rho_{init} = \rho_0 \left(\frac{r}{r_0} \right)^{-3/2}, \\ T_{init} = T_0 (\frac{r}{r_0})^{-1/2}, \\ p_{init} = p_0 (\frac{r}{r_0})^{-2}, \\ u_{init} = 0, \\ v_{init} = \sqrt{\frac{GM}{r_0}(r_0/r) - 2\frac{p}{\rho}}, \end{cases} \qquad \begin{cases} \rho_0 = 1.4.10^{-6} Kg.m^{-3}, \\ T_0 = 280K, \\ p_0 = 1.42Pa, \\ r_0 = 1AU. \end{cases} \tag{7}$$

Discretization

- The conservation laws defined by:

$$\partial_t w + \nabla . f(w) = Q(w). \tag{8}$$

- The integration of this equation over the control volume $\Omega_{i,j}$ delimited by the surface $S_{i,j}$ using Gauss's theorem gives:

$$\frac{\partial w}{\partial t}\Omega_{i,j} + \int_{\partial\Omega_{i,j}} f(w) d\Omega = \int_{\Omega_{i,j}} Q d\Omega. \tag{9}$$

- The discrete-time linear system is written in following form:

$$\frac{\partial w_{i,j}\Omega_{i,j}}{\partial t} + \sum_{l} \vec{F}.\vec{S} + \sum_{l'} \vec{G}.\vec{S} = Q_{i,j}\Omega_{i,j}. \tag{10}$$

Discretization

- The Taylor expansion in order 1 of pressure term is used to obtain:

$$r_{i+\frac{1}{2}} p^L_{i+\frac{1}{2}} = r_i p_i + \frac{\partial r p_i}{\partial r} \frac{\Delta r}{2} + \mathcal{O}(\Delta r^2), \tag{11}$$

$$p^R_{i+\frac{1}{2}} r_{i-\frac{1}{2}} = r_i p_i - \frac{\partial r p_i}{\partial r} \frac{\Delta r}{2} + \mathcal{O}(\Delta r^2). \tag{12}$$

- Denoting l the length of each facet, a general discretization is:

$$\begin{aligned} w^{n+1}_{i,j} = w^n_{i,j} + \Delta t \left(Q^n_{i,j} - \frac{l_{i+\frac{1}{2},j} F^n_{i+\frac{1}{2},j} - l_{i-\frac{1}{2},j} F^n_{i-\frac{1}{2},j}}{S^n_{i,j}} \right) \\ + \Delta t \left(\frac{G^n_{i,j+\frac{1}{2}} l'_{i,j+\frac{1}{2}} - G^n_{i,j-\frac{1}{2}} l'_{i,j-\frac{1}{2}}}{S^n_{i,j}} \right). \end{aligned} \tag{13}$$

Discretization

- The flux terms are evaluated on the interface $(r_{i+\frac{1}{2}}, \theta_j)$ that separates two cells $J_{i-1,j}$ and $J_{i,j}$ with the approximated Riemann solver $\mathcal{H}$

$$l_{i+\frac{1}{2},j} F^n_{i+\frac{1}{2},j} = r_{i+1/2} \Delta\theta \mathcal{H}(w^L_{i+\frac{1}{2},j}, w^R_{i+\frac{1}{2},j}). \tag{14}$$

- For instance, considering the Godunov method, then, the conservative vector w at this interface is given as:

$$w^L_{i+\frac{1}{2}} = \begin{pmatrix} \rho_i r_{i+\frac{1}{2}} \\ u_i \\ v_i r_{i+\frac{1}{2}} \\ p_i r_{i+\frac{1}{2}} \end{pmatrix}, w^R_{i+\frac{1}{2}} = \begin{pmatrix} \rho_{i+\frac{1}{2}} r_{i+\frac{1}{2}} \\ u_{i+1} \\ v_{i+1} r_{i+\frac{1}{2}} \\ p_{i+1} r_{i+\frac{1}{2}} \end{pmatrix}. \tag{15}$$

Discretization

- The conservative equation is resolved by a splitting extrapolation method. It allows to give the following equivalent system:

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{1}{r}\frac{\partial rF(w)}{\partial r} = Q(w), \\ \frac{\partial w}{\partial t} + \frac{1}{r}\frac{\partial G(w)}{\partial r} = Q(w). \end{cases} \tag{16}$$

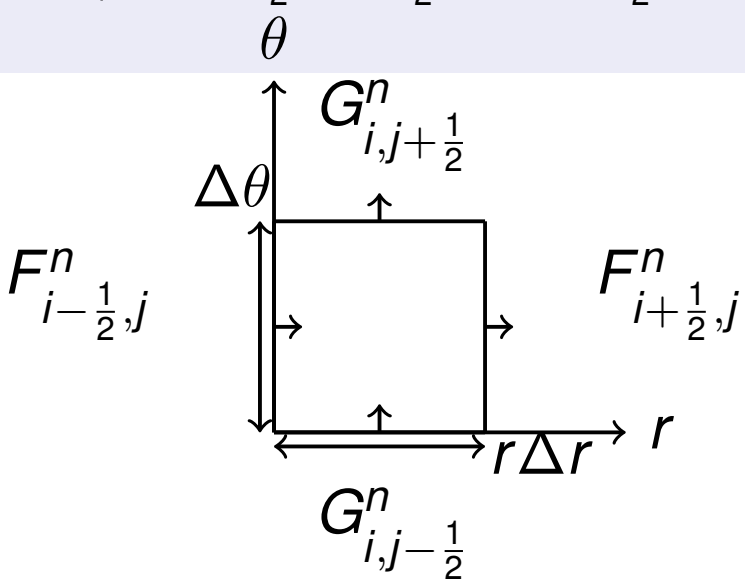
- The azimuthal flux G is approached by a cell-centred scheme wih keeping the same undimensional radial flux F. The Taylor-Young expansion in points $(r_i, \theta_j + \frac{\Delta\theta}{2})$ and $(r_i, \theta_j - \frac{\Delta\theta}{2})$ gives:

$$\frac{G_{i,j+\frac{1}{2}} - G_{i,j-\frac{1}{2}}}{\Delta\theta} = \frac{\partial G}{\partial \theta} + \mathcal{O}((\Delta\theta)^2). \tag{17}$$

Discretization

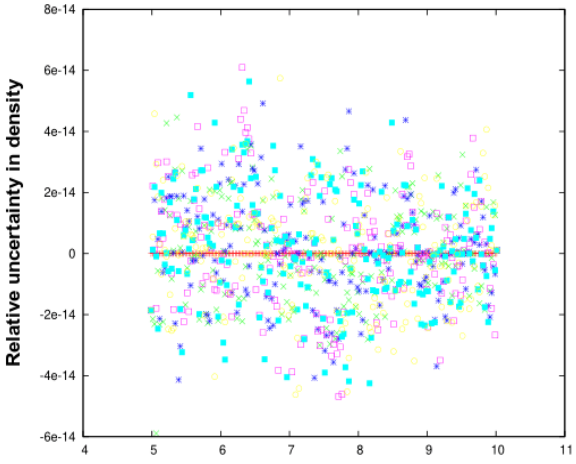
- The second-order discretization for the flux G is defined as follows:

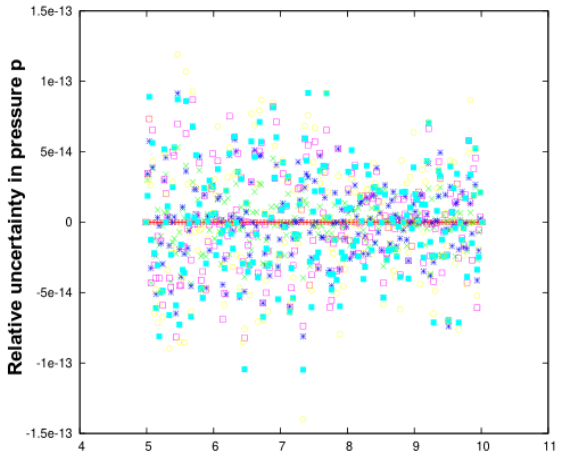
$$G_{i,j+\frac{1}{2}} = \begin{pmatrix} \rho_{i,j+\frac{1}{2}} v_{i,j+\frac{1}{2}} \\ \rho_{i,j+\frac{1}{2}} u_{i,j+\frac{1}{2}} v_{i,j+\frac{1}{2}} \\ \rho_{i,j+\frac{1}{2}} v^2_{i,j+\frac{1}{2}} + p_{i,j+\frac{1}{2}} \\ (\rho_{i,j+\frac{1}{2}} e_{i,j+\frac{1}{2}} + p_{i,j+\frac{1}{2}}) v_{i,j+\frac{1}{2}} \end{pmatrix}.$$

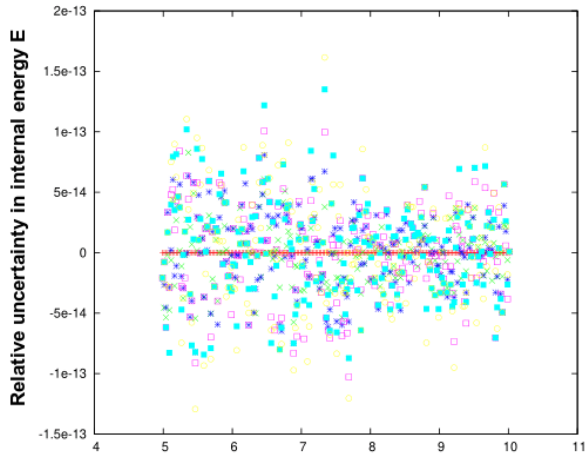


Relative error

- The estimated relative error is stated in some physical variable. This error shows perturbation analysis in density, pressure and internal energy as shown:

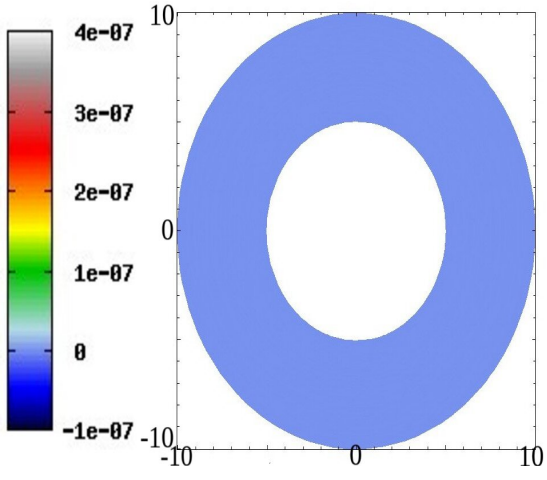


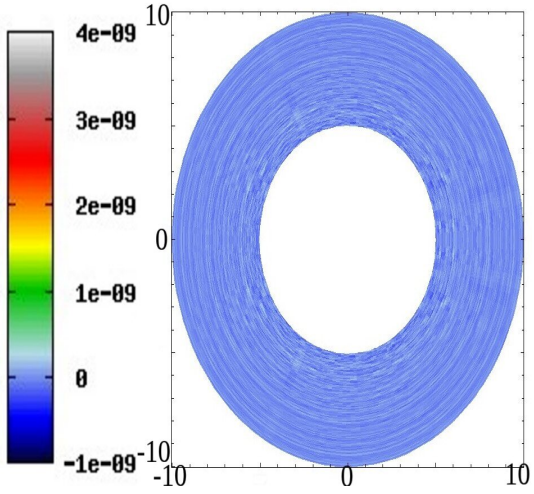


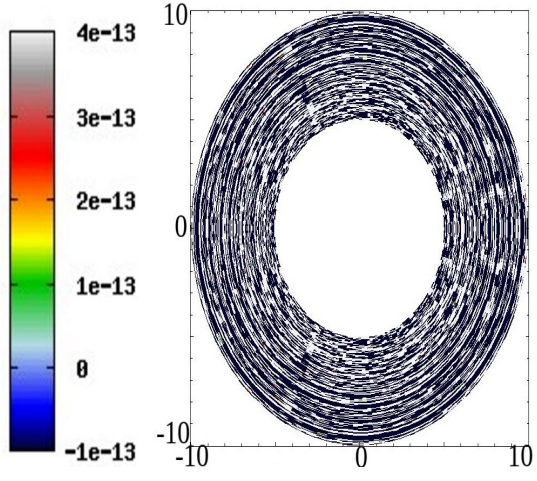


Regime steady-state

• The code accuracy is tested with computing the stationary solutions that remain uniform over time with good precision.

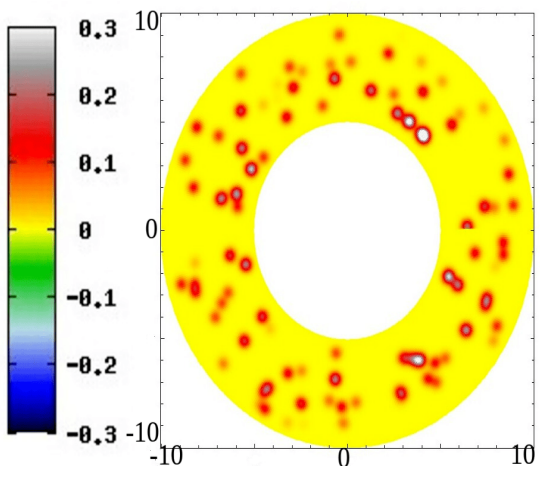


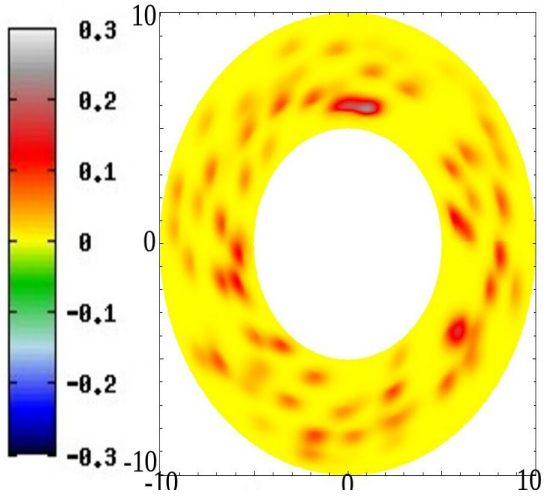


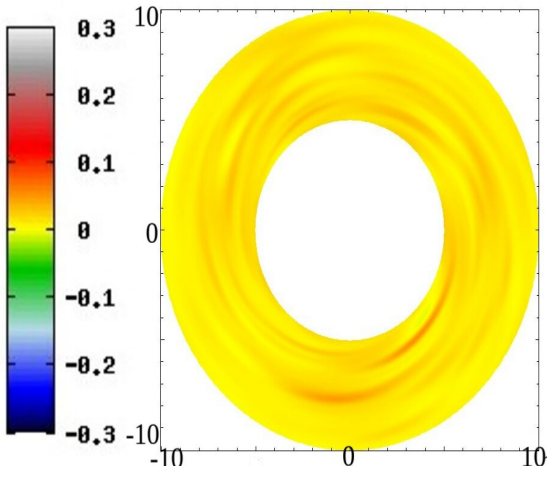


Stability and long-term evolution

- The perturbation of the vortices is studied in the disk with neglecting the turbulence effects,







Three-dimensional finite volume method

- The system of Euler equations is extended from bidimensional space to three-dimensional:

$$\frac{\partial w}{\partial t} + \frac{1}{r}\frac{\partial rF(w)}{\partial r} + \frac{1}{r}\frac{\partial G(w)}{\partial \theta} + \frac{1}{r}\frac{\partial H(w)}{\partial z} = Q(w). \qquad (18)$$

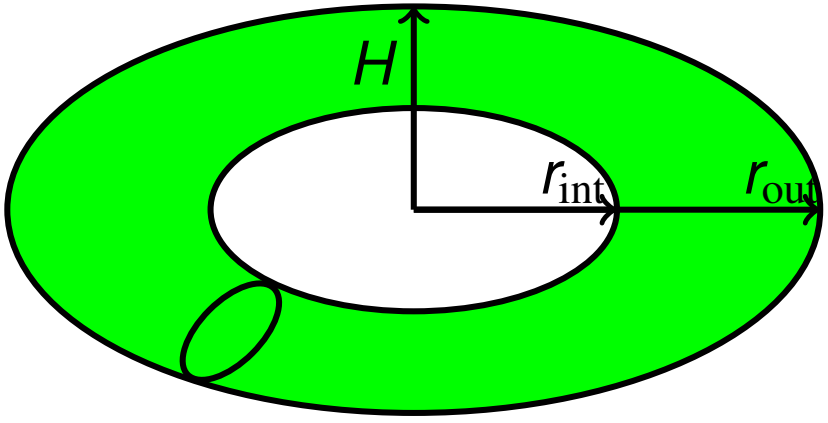


Figure: Computational domain in 3D.

Conclusions

- Develop in a second-order well-balanced scheme for the compressible Euler equations with keeping the gravitational source term.

- Numerical tests will be performed by comparing the accuracy and efficiency of the new multi-slopes methods with the first order method.