A DDFV Scheme For Incompressible Navier-Stokes Equations

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DDFV and applications,

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Unknowns

- ρ : the density
- $\mathbf{u}$: the velocity field
- p : the pressure

Equations

$$(NS) \quad \begin{cases} \partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \mathrm{div}\left(-2\eta\mathrm{D}\mathbf{u} + p\mathrm{Id}\right) = \mathbf{f}, \text{ in }]0,T[\times\Omega, \\ \mathrm{div}(\mathbf{u}) = 0, \text{ in }]0,T[\times\Omega, \\ \partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0, \text{ in }]0,T[\times\Omega. \end{cases}$$

with $\mathrm{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + {}^t\nabla\mathbf{u})$

- Ω a polygonal open bounded connected subset of $\mathbb{R}^2$, $T > 0$,
- $\mathbf{f} \in (L^2(\Omega))^2$,
- $\eta \in C^2(\Omega)$ with $0 < \underline{\mathrm{C}}_\eta \leq \eta(x) \leq \overline{\mathrm{C}}_\eta, \quad \forall x \in \Omega$.

Boundary conditions

$$\mathbf{u} = 0, \text{ on }]0,T[\times\partial\Omega.$$

Initial conditions

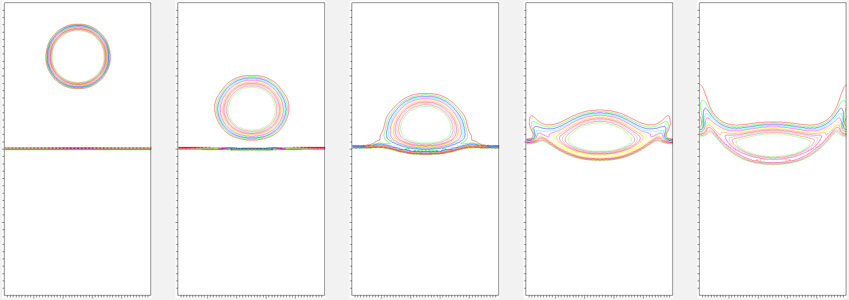
$$\mathbf{u}(0,.) = \mathbf{u}_{\mathrm{init}} \text{ in } \Omega,$$
$$\rho(0,.) = \rho_{\mathrm{init}} \geq 0 \text{ in } \Omega.$$

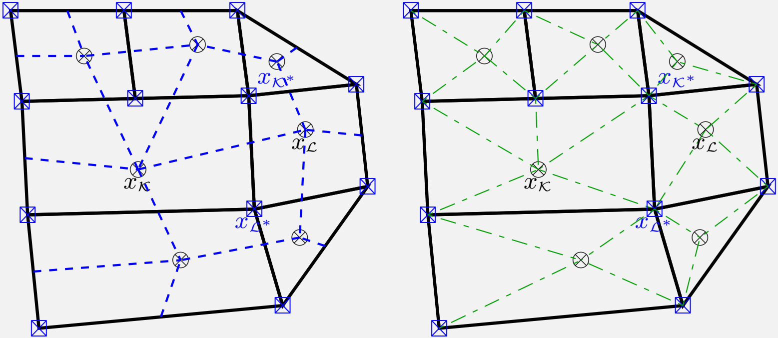
Closure to the model

$$\int_\Omega p(.,x)\mathrm{d}x = 0.$$

Experience

Numerical simulation





$\mathcal{K}$ Primal cells $\rightsquigarrow \mathbf{u}^{\mathfrak{M}} = (\mathbf{u}_{\mathcal{K}})_{\mathcal{K}\in\mathfrak{M}}$

$\mathcal{K}^*$ Dual cells $\rightsquigarrow \mathbf{u}^{\mathfrak{M}^*} = (\mathbf{u}_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*}$

$\rightsquigarrow \mathbf{u}_{\mathcal{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*})$,

$\mathcal{D}$ Diamond cells $\rightsquigarrow p_{\mathfrak{D}} = (p_{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}}$

$\rightsquigarrow \rho_{\mathfrak{D}} = (\rho_{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}}$

$\rightsquigarrow$ Discrete operators : $\nabla^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}$ and $\mathbf{div}^{\mathcal{T}}(\xi^{\mathfrak{D}})$.

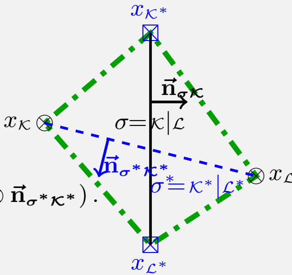
DISCRETE GRADIENT $\nabla^{\mathfrak{D}} : \left(\mathbb{R}^2\right)^{\mathcal{T}} \longrightarrow (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$

where $\begin{cases} \nabla^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}}(x_{\mathcal{L}} - x_{\mathcal{K}}) = \mathbf{u}_{\boldsymbol{\mathcal{L}}} - \mathbf{u}_{\boldsymbol{\mathcal{K}}}, \\ \nabla^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}}(x_{\mathcal{L}^*} - x_{\mathcal{K}^*}) = \mathbf{u}_{\boldsymbol{\mathcal{L}^*}} - \mathbf{u}_{\boldsymbol{\mathcal{K}^*}}. \end{cases}$

$$\nabla^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}} = \frac{1}{2m_{\mathcal{D}}} \left(m_\sigma (\mathbf{u}_{\boldsymbol{\mathcal{L}}} - \mathbf{u}_{\boldsymbol{\mathcal{K}}}) \otimes \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}} + m_{\sigma^*} (\mathbf{u}_{\boldsymbol{\mathcal{L}^*}} - \mathbf{u}_{\boldsymbol{\mathcal{K}^*}}) \otimes \vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}} \right).$$

$$\rightsquigarrow \mathrm{D}^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}} = \frac{1}{2} \left(\nabla^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}} + {}^t(\nabla^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}}) \right).$$

$$\rightsquigarrow \mathrm{div}^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}} = \mathrm{Tr} \nabla^{\mathcal{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}}.$$



Diamond

DISCRETE DIVERGENCE $\mathbf{div}^{\boldsymbol{\mathcal{T}}} : (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \longrightarrow \left(\mathbb{R}^2\right)^{\mathcal{T}}$

$$\mathcal{K} \in \mathfrak{M}, \quad \mathbf{div}^{\boldsymbol{\mathcal{K}}} \xi^{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}}} \sum_{\sigma \subset \partial\mathcal{K}} m_\sigma \xi^{\mathcal{D}} \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}.$$

DISCRETE DUALITY PROPERTY

$$[\![\mathbf{div}^{\boldsymbol{\mathcal{T}}} \xi_{\mathfrak{D}}, \mathbf{u}_{\boldsymbol{\mathcal{T}}}]\!]_{\mathcal{T}} = -(\xi_{\mathfrak{D}} : \nabla^{\mathfrak{D}} \mathbf{u}_{\boldsymbol{\mathcal{T}}})_{\mathfrak{D}}$$

On the continuous level

$$\int_{\mathcal{D}} \mathrm{div}(\rho(x)\mathbf{u}(x))\mathrm{d}x = \sum_{\mathfrak{s}\in\partial\mathcal{D}} \int_{\mathfrak{s}} \mathbf{u}(s)\cdot\vec{\mathbf{n}}\ \rho(s), \qquad \forall \mathcal{D}\in\mathfrak{D}.$$

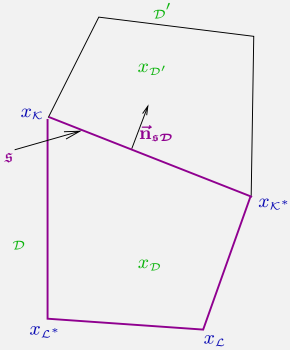
On the discrete level, with upwind fluxes

$$\mathrm{divc}^{\mathfrak{D}} : (\rho_{\mathfrak{D}}, \mathbf{u}_{\mathcal{T}}) \in \mathbb{R}^{\mathfrak{D}} \times \left(\mathbb{R}^2\right)^{\mathcal{T}} \mapsto \left(\mathrm{divc}^{\mathcal{D}}(\rho_{\mathfrak{D}}, \mathbf{u}_{\mathcal{T}})\right)_{\mathcal{D}\in\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$$

$$\mathrm{divc}^{\mathcal{D}}(\rho_{\mathfrak{D}}, \mathbf{u}_{\mathcal{T}}) = \frac{1}{m_{\mathcal{D}}} \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}'\in\partial\mathcal{D}} F_{\mathfrak{s},\mathcal{D}}$$

where

- $F_{\mathfrak{s},\mathcal{D}} = m_{\mathfrak{s}}\left((u_{\mathfrak{s},\mathcal{D}})^+ \rho_{\mathcal{D}} - (u_{\mathfrak{s},\mathcal{D}})^- \rho_{\mathcal{D}'}\right)$
- $u_{\mathfrak{s},\mathcal{D}} = \dfrac{\mathbf{u}_{\mathcal{K}} + \mathbf{u}_{\mathcal{K}^*}}{2}\cdot\vec{\mathbf{n}}_{\mathfrak{s}\mathcal{D}}$ for $\mathfrak{s} = [x_{\mathcal{K}}, x_{\mathcal{K}^*}] \in \partial\mathcal{D}$,
- $x^+ = \max(x,0)$ and $x^- = -\min(x,0)$
- $\mathrm{div}^{\mathcal{D}}\mathbf{u}_{\mathcal{T}} = \dfrac{1}{m_{\mathcal{D}}} \displaystyle\sum_{\mathfrak{s}\in\partial\mathcal{D}} m_{\mathfrak{s}} u_{\mathfrak{s},\mathcal{D}}$
- $\mathrm{div}^{\mathcal{D}}\mathbf{u}_{\mathcal{T}} = \mathrm{divc}^{\mathcal{D}}(\mathbf{1}_{\mathfrak{D}}, \mathbf{u}_{\mathcal{T}})$



SCHEME

$$\rightsquigarrow \frac{\rho_{\mathfrak{D}}^{n+1} - \rho_{\mathfrak{D}}^{n}}{\delta t} + \mathrm{divc}^{\mathfrak{D}}(\rho_{\mathfrak{D}}^{n}, \mathbf{u}_{\mathcal{T}}^{n}) = 0.$$

MAXIMAL PRINCIPLE

- $\rho_{\mathfrak{D}}^{n} \geq 0$
- $\delta t \leq \left(\|\mathbf{u}_{\mathcal{T}}\|_{\infty} \frac{1}{m_{\mathcal{D}}} \sum_{\mathfrak{s} \in \partial \mathcal{D}} m_{\mathfrak{s}} \right)^{-1}$

▶ $\rho_{\mathfrak{D}}^{n+1} \geq 0$

HOMOGENEOUS STATES ARE PRESERVED

- $\rho_{\mathfrak{D}}^{n} \equiv 1$
- $\mathrm{div}^{\mathcal{D}} \mathbf{u}_{\mathcal{T}}^{n} = 0$

▶ $\rho_{\mathfrak{D}}^{n+1} \equiv 1$

PROJECTION

- $\rho_{\mathcal{K}}^{n+1} = \frac{1}{m_{\mathcal{K}}} \sum_{\mathcal{D} \in \mathfrak{D}_{\mathcal{K}}} m(\mathcal{D} \cap \mathcal{K}) \rho_{\mathcal{D}}^{n+1}, \forall \mathcal{K} \in \mathfrak{M},$
- $\rho_{\mathcal{K}^*}^{n+1} = \frac{1}{m_{\mathcal{K}^*}} \sum_{\mathcal{D} \in \mathfrak{D}_{\mathcal{K}^*}} m(\mathcal{D} \cap \mathcal{K}^*) \rho_{\mathcal{D}}^{n+1}, \forall \mathcal{K}^* \in \mathfrak{M}^*.$

On the continuous level

$$\int_{\mathcal{K}} \mathrm{div}(\rho(x)\mathbf{u}(x) \otimes \mathbf{u}(x))\mathrm{d}x = \sum_{\sigma\in\partial\mathcal{K}} \int_{\sigma} (\rho(s)\mathbf{u}(s)\cdot\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}})\mathbf{u}(s), \qquad \forall \mathcal{K}\in\mathfrak{M}.$$

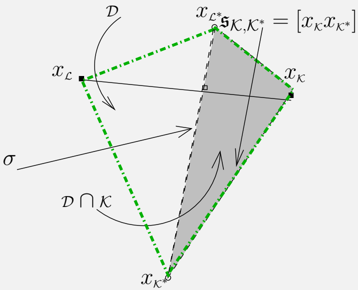
On the discrete level, with upwind fluxes

$$\mathbf{divc}^{\boldsymbol{\mathcal{K}}}(\rho^n_{\mathfrak{D}}, \mathbf{u}^n_{\mathcal{T}}, \mathbf{u}^{n+1}_{\mathcal{T}}) = \frac{1}{m_{\mathcal{K}}} \sum_{\sigma\in\partial\mathcal{K}} (F^n_{\mathcal{K},\sigma})^+ \mathbf{u}^{n+1}_{\boldsymbol{\mathcal{K}}} - (F^n_{\mathcal{K},\sigma})^- \mathbf{u}^{n+1}_{\boldsymbol{\mathcal{L}}}$$

How to define $F_{\mathcal{K},\sigma} \sim \int_{\sigma} \rho\mathbf{u}\cdot\vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}$?

We want $F^n_{\mathcal{K},\sigma}$ to verify

$$m_{\mathcal{K}} \frac{\rho^{n+1}_{\mathcal{K}} - \rho^n_{\mathcal{K}}}{\delta t} + \sum_{\sigma\in\partial\mathcal{K}} F^n_{\mathcal{K},\sigma} = 0.$$



For this we define

$$F^n_{\mathcal{K},\sigma} = -\frac{m(\mathcal{D}\cap\mathcal{L})}{m_{\mathcal{D}}} \sum_{\mathfrak{s}\in\partial\mathcal{D},\mathfrak{s}\subset\mathcal{K}} F^n_{\mathfrak{s},\mathcal{D}} + \frac{m(\mathcal{D}\cap\mathcal{K})}{m_{\mathcal{D}}} \sum_{\mathfrak{s}\in\partial\mathcal{D},\mathfrak{s}\subset\mathcal{L}} F^n_{\mathfrak{s},\mathcal{D}}$$

(Goudon, Krell, Llobell & Minjeaud, '21)

DDFV scheme

Find $\mathbf{u}_{\mathcal{T}}^{n+1} \in \mathbb{E}_0$ and $p_{\mathfrak{D}}^{n+1} \in \mathbb{R}^{\mathfrak{D}}$ such that,

$$\frac{\rho_{\mathcal{T}}^{n+1}\mathbf{u}_{\mathcal{T}}^{n+1} - \rho_{\mathcal{T}}^{n}\mathbf{u}_{\mathcal{T}}^{n}}{\delta t} + \mathbf{divc}^{\mathcal{T}}(\rho_{\mathfrak{D}}^{n}, \mathbf{u}_{\mathcal{T}}^{n}, \mathbf{u}_{\mathcal{T}}^{n+1}) + \mathbf{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1} + p_{\mathfrak{D}}^{n+1}\mathrm{Id}) = \mathbf{f}_{\mathcal{T}}^{n+1},$$

$$\mathrm{div}^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1} = 0, \qquad \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} p_{\mathcal{D}}^{n+1} = 0,$$

Energy stability

$$\frac{1}{2\delta t}\|\sqrt{\rho_{\mathcal{T}}^{n+1}}\mathbf{u}_{\mathcal{T}}^{n+1}\|_{\mathcal{T}}^2 - \frac{1}{2\delta t}\|\sqrt{\rho_{\mathcal{T}}^{n}}\mathbf{u}_{\mathcal{T}}^{n}\|_{\mathcal{T}}^2 + \frac{1}{2\delta t}\|\sqrt{\rho_{\mathcal{T}}^{n}}(\mathbf{u}_{\mathcal{T}}^{n+1} - \mathbf{u}_{\mathcal{T}}^{n})\|_{\mathcal{T}}^2 + \underline{C}_{\eta}\|\nabla^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1}\|_2^2 \leq [\![\mathbf{f}_{\mathcal{T}}^{n+1}, \mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}}.$$

- Existence and uniqueness

(Goudon & Krell, '14)

Exact solution

$$\mathbf{u} = \begin{pmatrix} -\cos(2\pi x)\sin(2\pi y)e^{-2t\eta} \\ \sin(2\pi x)\cos(2\pi y)e^{-2t\eta} \end{pmatrix}$$

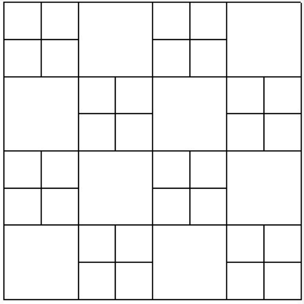
$$p = -\frac{1}{4}(\cos(4\pi x) + \cos(4\pi y))e^{-4t\eta}$$

$$\rho = 1$$

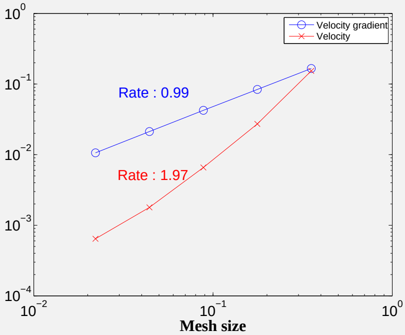
$$\eta = 1$$

$T = 1$ and $\delta t = 5.10^{-3}$

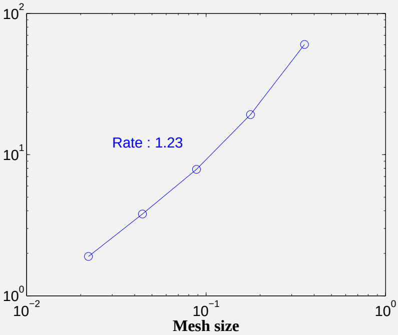
Primal mesh



Velocity gradient & Velocity



Pressure



- $\rho_{\mathfrak{D}}^{n} \equiv 1$

Exact solution

$$\mathbf{u} = \begin{pmatrix} -y\cos(t) \\ x\cos(t) \end{pmatrix}$$

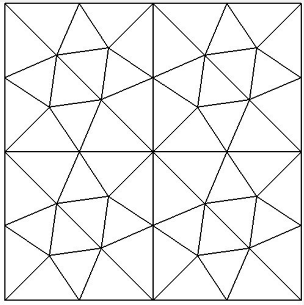
$$p = \sin(x)\sin(y)\sin(t)$$

$$\rho(r,\theta,t) = 2 + r\cos(\theta - \sin(t))$$

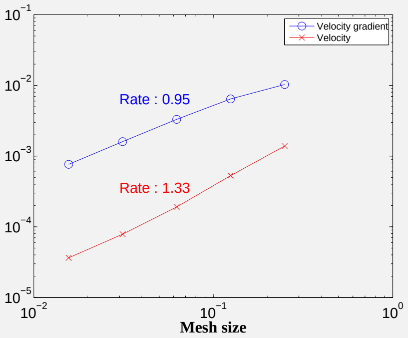
$$\eta = 1$$

$T = 3.10^{-2}$ and $\delta t = 7,5.10^{-5}$

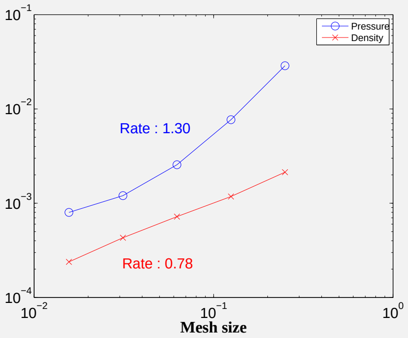
Primal mesh



Velocity gradient & Velocity



Pressure & Density



Rayleigh-Taylor instability

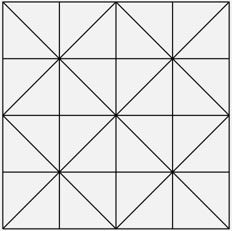
Parameter

- $\Omega =]0, 0.5[\times]-2, 2[$.
- $\rho_{\text{init}}(x, y) = 2 + \tanh\left(\frac{y + 0.1\cos(2\pi x)}{0.01}\right)$,
- $\mathbf{u}_{\text{Init}} \equiv 0$,
- $\eta = \frac{1}{1000}$,
- $\mathbf{f} = (0, -\rho)$

Primal mesh

$\mathbf{u} = 0$

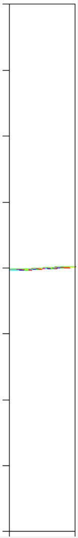
$\mathbf{u} \cdot \vec{\mathbf{n}} = 0$ $\quad$ $2\eta \mathrm{D}\mathbf{u}\vec{\mathbf{n}} \cdot \vec{\boldsymbol{\tau}} = 0$



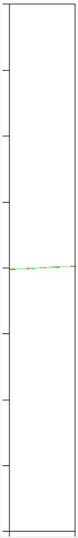
$\mathbf{u} \cdot \vec{\mathbf{n}} = 0$ $\quad$ $2\eta \mathrm{D}\mathbf{u}\vec{\mathbf{n}} \cdot \vec{\boldsymbol{\tau}} = 0$

$\mathbf{u} = 0$

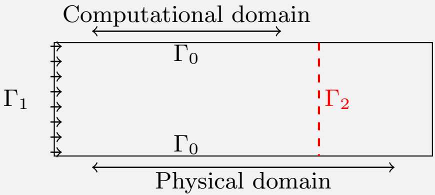
Density ratio equal to 7.



Density ratio equal to 19.



Outline



$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathrm{div}(\sigma(\mathbf{u},\mathrm{p})) = 0; & \text{in} \quad \Omega \times [0,T], \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega \times [0,T], \\ \mathbf{u} = \mathbf{g}_1 & \text{on} \quad \Gamma_1 \times (0,T), \\ \mathbf{u} = 0 & \text{on} \quad \Gamma_0 \times (0,T), \\ \sigma(\mathbf{u},\mathrm{p})\vec{\mathbf{n}} + \frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}-\mathbf{u}_{\mathrm{ref}}) = \sigma_{\mathrm{ref}}\vec{\mathbf{n}} & \text{on} \quad \Gamma_2 \times (0,T) \\ \mathbf{u}(0) = \mathbf{u}_{init} & \text{in} \quad \Omega \end{cases}$$

with $\mathbf{u}_{\mathrm{ref}}, \sigma_{\mathrm{ref}}$: reference flow, $\sigma(\mathbf{u},\mathrm{p}) = \frac{2}{\mathrm{Re}}\mathrm{D}\mathbf{u} - \mathrm{pId}$.

(Bruneau & Fabrie '94)

Let $\Psi \in V = \{\psi \in (H^1(\Omega))^2, \psi|_{\Gamma_1} = 0, \psi|_{\Gamma_0} = 0, \mathrm{div}(\psi) = 0\}$.

$$\int_\Omega \partial_t \mathbf{u} \cdot \Psi + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \Psi - \int_\Omega \mathrm{div}(\sigma(\mathbf{u}, \mathrm{p})) \cdot \Psi = 0$$

Let $\Psi \in V = \{\psi \in (H^1(\Omega))^2, \psi|_{\Gamma_1} = 0, \psi|_{\Gamma_0} = 0, \mathrm{div}(\psi) = 0\}$.

$$\int_\Omega \partial_t \mathbf{u} \cdot \Psi + \frac{1}{2}\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \Psi - \frac{1}{2}\int_\Omega (\mathbf{u} \cdot \nabla)\Psi \cdot \mathbf{u} - \int_\Omega \mathrm{div}(\sigma(\mathbf{u}, \mathrm{p})) \cdot \Psi = -\frac{1}{2}\int_{\Gamma_2} (\mathbf{u} \cdot \vec{\mathbf{n}})\mathbf{u} \cdot \Psi$$

Let $\Psi \in V = \{\psi \in (H^1(\Omega))^2, \psi|_{\Gamma_1} = 0, \psi|_{\Gamma_0} = 0, \mathrm{div}(\psi) = 0\}$.

$$\int_\Omega \partial_t \mathbf{u} \cdot \Psi + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u} + \frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}\mathbf{u} : \mathrm{D}\Psi - \int_{\Gamma_2} (\sigma(\mathbf{u},\mathrm{p})\vec{\mathbf{n}})\cdot\Psi = -\frac{1}{2}\int_{\Gamma_2} (\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\cdot\Psi$$

Let $\Psi \in V = \{\psi \in (H^1(\Omega))^2, \psi|_{\Gamma_1} = 0, \psi|_{\Gamma_0} = 0, \mathrm{div}(\psi) = 0\}$.

$$\int_\Omega \partial_t \mathbf{u} \cdot \Psi + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u}$$
$$+\frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}\mathbf{u} : \mathrm{D}\Psi - \int_{\Gamma_2} (\sigma(\mathbf{u},\mathrm{p})\vec{\mathbf{n}})\cdot\Psi = -\frac{1}{2}\int_{\Gamma_2} (\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\cdot\Psi$$

with

$$\boxed{\sigma(\mathbf{u},\mathrm{p})\vec{\mathbf{n}} = -\frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}-\mathbf{u}_{\mathrm{ref}}) + \sigma_{\mathrm{ref}}\vec{\mathbf{n}}}$$

Let $\Psi \in V = \{\psi \in (H^1(\Omega))^2, \psi|_{\Gamma_1} = 0, \psi|_{\Gamma_0} = 0, \mathrm{div}(\psi) = 0\}$.

$$\int_\Omega \partial_t \mathbf{u} \cdot \Psi + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u} + \frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}\mathbf{u} : \mathrm{D}\Psi - \int_{\Gamma_2} (\sigma(\mathbf{u},\mathrm{p})\vec{\mathbf{n}})\cdot\Psi = -\frac{1}{2}\int_{\Gamma_2} (\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u}\cdot\Psi$$

with

$$\boxed{\sigma(\mathbf{u},\mathrm{p})\vec{\mathbf{n}} = -\frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}-\mathbf{u}_{\mathrm{ref}}) + \sigma_{\mathrm{ref}}\vec{\mathbf{n}}}$$

The variational formulation writes :

$$\int_\Omega \partial_t \mathbf{u} \cdot \Psi + \frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}(\mathbf{u}) : \mathrm{D}(\Psi) + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u}$$
$$= -\frac{1}{2}\int_{\Gamma_2} (\mathbf{u}\cdot\vec{\mathbf{n}})^+(\mathbf{u}\cdot\Psi) + \frac{1}{2}\int_{\Gamma_2} (\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}_{ref}\cdot\Psi) + \int_{\Gamma_2} (\sigma_{ref}\vec{\mathbf{n}})\cdot\Psi$$

because $x + x^- = x^+$.

$$\int_\Omega \partial_t \mathbf{u}\cdot\Psi + \frac{2}{\mathrm{Re}}\int_\Omega \mathrm{D}(\mathbf{u}) : \mathrm{D}(\Psi) + \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Psi - \frac{1}{2}\int_\Omega (\mathbf{u}\cdot\nabla)\Psi\cdot\mathbf{u}$$
$$= -\frac{1}{2}\int_{\Gamma_2} (\mathbf{u}\cdot\vec{\mathbf{n}})^+(\mathbf{u}\cdot\Psi) + \frac{1}{2}\int_{\Gamma_2}(\mathbf{u}\cdot\vec{\mathbf{n}})^-(\mathbf{u}_{ref}\cdot\Psi) + \int_{\Gamma_2}(\sigma_{ref}\vec{\mathbf{n}})\cdot\Psi$$

DDFV VARIATIONAL FORMULATION

$$\left[\!\!\left[\frac{\mathbf{u}_{\mathcal{T}}^{n+1}-\mathbf{u}_{\mathcal{T}}^{n}}{\delta t},\Psi_{\mathcal{T}}\right]\!\!\right]_{\mathcal{T}} + \frac{2}{\mathrm{Re}}(\mathrm{D}^{\mathfrak{D}}\mathbf{u}_{\mathcal{T}}^{n+1},\mathrm{D}^{\mathfrak{D}}\Psi_{\mathcal{T}})_{\mathfrak{D}} + \frac{1}{2}[\![\mathbf{divc}^{\mathcal{T}}(\mathbf{u}_{\mathcal{T}}^{n},\mathbf{u}_{\mathcal{T}}^{n+1}),\Psi_{\mathcal{T}}]\!]_{\mathcal{T}}$$
$$-\frac{1}{2}[\![\mathbf{divc}^{\mathcal{T}}(\mathbf{u}_{\mathcal{T}}^{n},\Psi_{\mathcal{T}}),\mathbf{u}_{\mathcal{T}}^{n+1}]\!]_{\mathcal{T}} = -\frac{1}{2}\sum_{\mathcal{D}\in\mathcal{D}_{ext}\cap\Gamma_2}(F_{\mathcal{K},\sigma}^{n})^{+}\gamma^{\sigma}(\mathbf{u}_{\mathcal{T}}^{n+1})\cdot\gamma^{\sigma}(\Psi_{\mathcal{T}})$$
$$+\frac{1}{2}\sum_{\mathcal{D}\in\mathcal{D}_{ext}\cap\Gamma_2}(F_{\mathcal{K},\sigma}^{n})^{-}\gamma^{\sigma}(\mathbf{u}_{\mathcal{T}}^{\mathrm{ref}})\cdot\gamma^{\sigma}(\Psi_{\mathcal{T}}) + \sum_{\mathcal{D}\in\mathcal{D}_{ext}\cap\Gamma_2} m_\sigma(\sigma_{\mathcal{D}}^{\mathrm{ref}}\vec{\mathbf{n}})\cdot\gamma^{\sigma}(\Psi_{\mathcal{T}}).$$

- Existence and uniqueness

(Goudon, Krell & Lissoni, '19)

♦ Let $(\mathbf{u}^{\mathfrak{T},[0,T]}, \mathrm{p}^{\mathfrak{D},[0,T]}) \in \left((\mathbb{R}^2)^{\mathfrak{T}}\right)^{N+1} \times (\mathbb{R}^{\mathfrak{D}})^{N+1}$ be the solution of the DDFV scheme, where $\mathbf{u}^{\mathfrak{T},[0,T]} = \mathbf{v}^{\mathfrak{T},[0,T]} + \mathbf{u}^{\mathrm{ref}}_{\mathcal{T}}$.

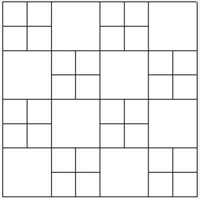
- For $N > 1$, $\exists C > 0$, depending on $\Omega, \mathbf{u}^{\mathrm{ref}}, \mathbf{u}_{init}, \mathrm{Re}$ such that :

$$\sum_{j=0}^{N-1} \|\mathbf{v}_{\mathcal{T}}^{j+1} - \mathbf{v}_{\mathcal{T}}^{j}\|_2^2 \leq C, \quad \|\mathbf{v}_{\mathcal{T}}^{N}\|_2^2 \leq C,$$

$$\sum_{j=0}^{N-1} \delta t \frac{1}{\mathrm{Re}} \|\mathrm{D}^{\mathfrak{D}} \mathbf{v}_{\mathcal{T}}^{j+1}\|_2^2 \leq C, \quad \delta t \frac{1}{\mathrm{Re}} \|\mathrm{D}^{\mathfrak{D}} \mathbf{v}_{\mathcal{T}}^{N}\|_2^2 \leq C,$$

$$\sum_{j=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}_{ext}} (F_{\mathcal{K},\sigma}(\mathbf{v}_{\mathcal{T}}^{j} + \mathbf{u}_{\mathcal{T}}^{\mathrm{ref}}))^{+} (\gamma^{\sigma}(\mathbf{v}_{\mathcal{T}}^{j+1}))^2 \leq C.$$

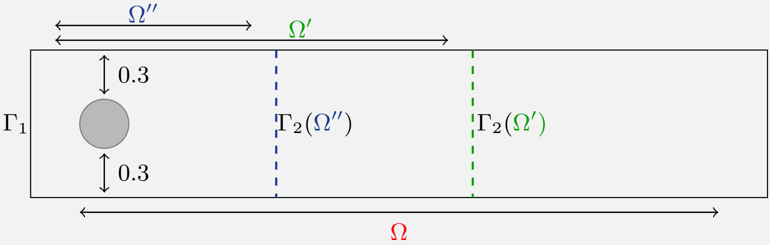
Exact solution and non-conformal mesh :



- $\mathbf{u}(t,x,y) = \begin{pmatrix} -2\pi\cos(\pi x)\sin(2\pi y)\exp(-5\eta t\pi^2) \\ \pi\sin(\pi x)\cos(2\pi y)\exp(-5\eta t\pi^2) \end{pmatrix}$,
- $\mathrm{p}(t,x,y) = -\frac{\pi^2}{4}(4\cos(2\pi x) + \cos(4\pi y))\exp(-10t\eta\pi^2)$

The final time is $\mathrm{T} = 0.03$ and we set $\delta t = 3 \times 10^{-5}$.

NbCell	Ervel	Ratio	Ergradvel	Ratio	Erpre	Ratio
64	1.424E-01	-	1.612E-01	-	6.127E+00	-
208	4.095E-02	1.80	7.316E-02	1.14	1.725E+00	1.83
736	1.019E-02	2.00	3.489E-02	1.07	5.836E-01	1.56
2752	2.559E-03	1.99	1.710E-02	1.03	1.947E-01	1.58
10624	6.493E-04	**1.98**	8.474E-03	**1.01**	6.189E-02	**1.65**



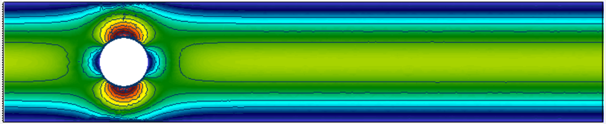
$\Omega = [0,5] \times [0,1] \rightarrow$ 12118 cells, $\Omega' = [0,3] \times [0,1] \rightarrow$ 8636 cells
$\Omega'' = [0,1.5] \times [0,1] \rightarrow$ 6534 cells

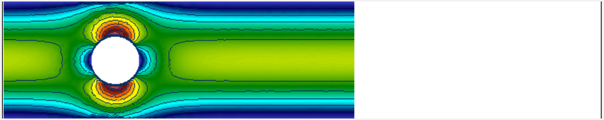
$$\mathbf{g}_1(x,y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \quad \text{on } \Gamma_1$$

$$\mathbf{u}_{\text{ref}}(x,y) = \begin{pmatrix} 6y(1-y) \\ 0 \end{pmatrix} \quad \text{on } \Gamma_2$$

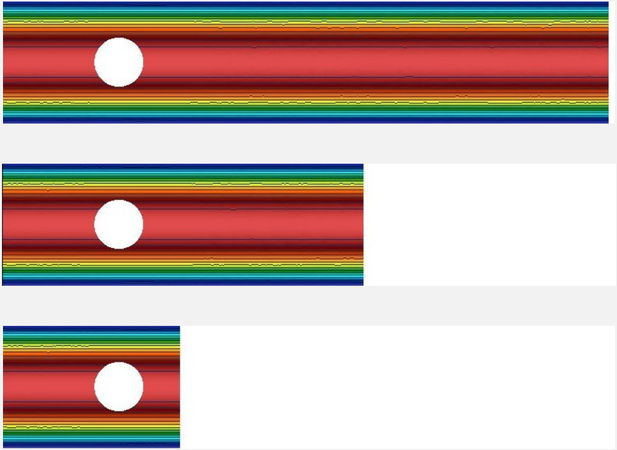
$$\sigma_{\text{ref}}(\mathbf{u},\text{p}) \cdot \vec{\mathbf{n}} = \begin{pmatrix} 0 \\ 6\eta(1-2y) \end{pmatrix} \quad \text{on } \Gamma_2$$

with $\eta = 4 \times 10^{-3}$, $\delta t = 0,035$.





$\mathrm{Re} = 100$



Re = 1000

- More numerical tests.
- The proof of the convergence of the scheme.
- Mixture flows with a complex constraint

$$\mathrm{div}(\mathbf{u}) = \mathrm{div}\left(\frac{\bar{\phi}}{\eta}(1-\rho)\nabla p - \frac{\nabla \rho}{\eta}\right), \text{ in }]0,T[\times\Omega.$$

- Improved computation of $\mathbf{u}_{\mathrm{ref}}$ and σ_{ref}

Thank you for your attention