Convergence of a finite-volume scheme for a stochastic heat equation with a nonlinear multiplicative noise

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joint work with Caroline Bauzet, Flore Nabet and Aleksandra Zimmermann

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Discrete Duality Finite Volume Method and Applications, CIRM

The heat equation with multiplicative Lipschitz noise

We consider

$$\begin{aligned} u(t) - u(0) - \int_0^t \Delta u(s)\, ds &= \int_0^t g(u(s))\, dW_s && \text{in } \Omega \times (0,T) \times \Lambda \\ u(0,\cdot) &= u_0 && \text{in } \Omega \times \Lambda \\ \nabla u \cdot \mathbf{n} &= 0 && \text{on } \partial\Lambda \end{aligned} \qquad \text{(SHE)}$$

- $T > 0$, Λ open and bounded polygonal domain of $\mathbb{R}^2$
- $(\Omega, \mathcal{A}, \mathbb{P})$ probability space endowed with a right-continuous, complete filtration $(\mathcal{F}_t)_{t\geq 0}$
- $(W_t)_{t\geq 0}$ standard one-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$
- $g : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous, $g(0) = 0$
- $u_0 \in L^2(\Omega; H^1(\Lambda))$ $\mathcal{F}_0$-measurable

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Some useful stochastic definitions

- A filtration $(\mathcal{F}_t)_{t\geq 0}$ associated with $\mathcal{A}$ is a family of σ-fields $(\mathcal{F}_t)_{t\geq 0}$ satisfying $\mathcal{F}_t \subseteq \mathcal{A}$ for all $t \geq 0$ and $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leq t$.
- $(W_t)_{t\geq 0}$ Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$
  - $W_t : \Omega \to \mathbb{R}$ is a $\mathcal{F}_t$-measurable random variable for each $t \geq 0$ ($(\mathcal{F}_t)_{t\geq 0}$-adapted)
  - $W_0 = 0$
  - $W_t - W_s$ is independent of $\mathcal{F}_s$ for $s \leq t$
  - $\mathbb{E}[W_t - W_s] = 0$ and $\mathbb{E}[(W_t - W_s)^2] = t - s$ for all $s \leq t$

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- Properties of the Itô integral
  - Zero average: $\mathbb{E}\left[\int_0^T \phi(s)\, dW(s)\right] = 0$
  - Itô isometry: $\mathbb{E}\left[\left\|\int_0^T \phi(s)\, dW(s)\right\|_H^2\right] = \mathbb{E}\left[\int_0^T \|\phi(s)\|_H^2\, ds\right]$
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No Cauchy-Schwarz and $\left|\int_0^T \varphi(s)\, dW(s)\right| \not\leq \int_0^T |\varphi(s)|\, dW(s)$

Aim of the study

A *variational solution* to (SHE) is a $\mathcal{F}_t$-adapted stochastic process

$$u \in L^2(\Omega; C([0,T]; L^2(\Lambda))) \cap L^2(\Omega; L^2(0,T; H^1(\Lambda)))$$

such that for all $t \in [0,T]$, in $L^2(\Lambda)$, $\mathbb{P}$-a.s. in Ω

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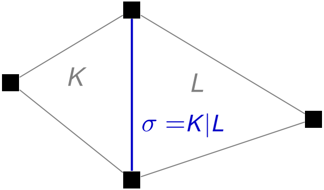
We propose a finite-volume scheme, semi-implicit in time and a Two-Point Flux Approximation (TPFA) in space and show its convergence to the *variational solution*.

The mesh on Λ

We consider an *admissible mesh* $\mathcal{T}$ on Λ consisting of open, polygonal and convex subsets, i.e. *control volumes* $K \in \mathcal{T}$ such that $\overline{\Lambda} = \bigcup_{K \in \mathcal{T}} \overline{K}$.



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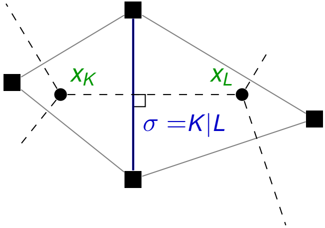

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- To each $K \in \mathcal{T}$ we associate a point $x_K \in K$, called *center*.
- For two neighbouring control volumes $K, L \in \mathcal{T}$ we have the orthogonality condition

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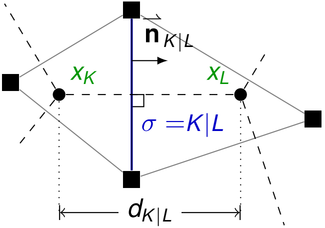

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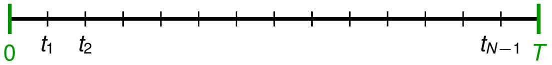
The discretization scheme

- For $N \in \mathbb{N}$, let $\Delta t := \frac{T}{N}$, and $t_n := n\Delta t$ for $n \in \{0, \dots, N\}$



- For $h > 0$, let $\mathcal{T}_h$ be an admissible mesh with size$(\mathcal{T}_h) = h$
- $(u_K^0)_{K \in \mathcal{T}_h}$ is a $\mathcal{F}_0$-measurable random vector associated to u_0

Proposition [Bauzet, Nabet, S., Zimmermann, '22]

For any $n \in \{0, \dots, N-1\}$ and any given $\mathcal{F}_{t_n}$-measurable random vector $(u_K^n)_{K \in \mathcal{T}_h}$ there exists a $\mathcal{F}_{t_{n+1}}$-measurable random vector $(u_K^{n+1})_{K \in \mathcal{T}}$ satisfying

$$\begin{aligned} &\frac{m_K}{\triangle t}(u_K^{n+1} - u_K^n) + \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}}(u_K^{n+1} - u_L^{n+1}) \\ &= \frac{m_K}{\triangle t} g(u_K^n)(W(t_{n+1}) - W(t_n)), \end{aligned} \tag{FV}$$

for all $K \in \mathcal{T}_h$, $\mathbb{P}$-a.s. in Ω.

Main result

For any $n \in \{0, \ldots, N-1\}$ let $(u_K^{n+1})_{K \in \mathcal{T}_h}$ be the solution of (FV) obtained by iteration starting with the random vector $(u_K^0)_{K \in \mathcal{T}_h}$, where for any $K \in \mathcal{T}$

$$u_K^0 := \frac{1}{m_K} \int_K u_0(x)\, dx.$$

Then the step functions

$$u_{h,N}^r(t,x) := u_K^{n+1}, \ t \in [t_n, t_{n+1}), x \in K \quad \textit{(not adapted)}$$
$$u_{h,N}^l(t,x) := u_K^{n}, \ t \in [t_n, t_{n+1}), x \in K \quad \textit{(adapted)}$$

converge in $L^p(\Omega; L^2(0,T; L^2(\Lambda)))$ for all $p \in [1,2)$ towards the unique variational solution of (SHE).

Bounds on the discrete solutions

$$\sum_{K\in\mathcal{T}_h} m_K(u_K^{n+1}-u_K^n)u_K^{n+1}+\triangle t\sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K}\frac{m_\sigma}{d_{K|L}}(u_K^{n+1}-u_L^{n+1})u_K^{n+1}$$
$$=\sum_{K\in\mathcal{T}_h} m_K g(u_K^n)(W(t_{n+1})-W(t_n))u_K^{n+1}$$

Bounds on the discrete solutions

$$= \sum_{\sigma \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} |u_K^{n+1} - u_L^{n+1}|^2$$

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$$\sum_{K\in\mathcal{T}_h} \frac{m_K}{2}\mathbb{E}\left[|u_K^{n+1}|^2 - |u_K^n|^2 + |u_K^{n+1} - u_K^n|^2\right]$$

$$+ \triangle t\mathbb{E}\left[\sum_{\sigma\in\mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}}|u_K^{n+1} - u_K^n|^2\right]$$

$$= \sum_{K\in\mathcal{T}_h} m_K\mathbb{E}\left[g(u_K^n)(W(t_{n+1}) - W(t_n))(u_K^{n+1} - u_K^n)\right]$$

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$$\Rightarrow\quad\sum_{K\in\mathcal{T}_h}\frac{m_K}{2}\mathbb{E}\left[|u_K^{n+1}|^2-|u_K^n|^2\right]+\triangle t\mathbb{E}\left[\sum_{\sigma\in\mathcal{E}_{\text{int}}}\frac{m_\sigma}{d_{K|L}}|u_K^{n+1}-u_L^{n+1}|^2\right]$$

$$\leq\sum_{K\in\mathcal{T}_h}\frac{m_K}{2}\triangle t\mathbb{E}\left[|g(u_K^n)|^2\right]$$

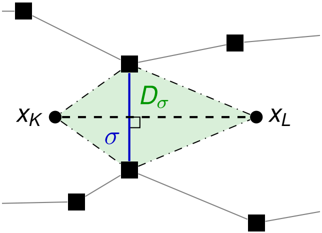
Bounds on the discrete solution

Using the Lipschitz continuity of g and a discrete Gronwall inequality yields

- The sequences $(u^r_{h,N})_{h,N}$ and $(u^l_{h,N})_{h,N}$ are bounded in $L^2(\Omega; L^2(0,T; L^2(\Lambda)))$
- The sequence of *discrete gradients* $(\nabla^h u^r_{h,N})_{h,N}$ is bounded in $L^2(\Omega; L^2(0,T; L^2(\Lambda)^2))$, where

$$\nabla^h u^r_{h,N} := \sum_{\sigma\in\mathcal{E}_{\text{int}}} \mathbb{1}_{D_\sigma} \frac{u_L^{n+1} - u_K^{n+1}}{d_{K|L}} \vec{\mathbf{n}}_{K|L}$$

piecewise constant on the *diamond cells D_σ*.



Weak convergence and improved regularity

There exists a function $u \in L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ such that, passing to a subsequence if necessary,

$$u^r_{h,N}, u^l_{h,N} \rightharpoonup u \text{ in } L^2(\Omega; L^2(0, T; L^2(\Lambda))) \text{ for } h \downarrow 0, N \to \infty$$

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Proposition [Eymard, Gallouët, '02]
$u \in L^2(\Omega; L^2(0,T; H^1(\Lambda)))$ and

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Because of the nonlinearity in the stochastic integral, weak convergence is not sufficient to pass to the limit.

Tightness

Lemma [Bauzet, Nabet, S., Zimmermann, '22]

For any $\alpha \in (0, \frac{1}{2})$, the sequence $(u^l_{h,N})_{h,N}$ is bounded in

$$L^2(\Omega; L^2(0, T; W^{\alpha,2}(\Lambda))) \cap L^2(\Omega; W^{\alpha,2}(0, T; L^2(\Lambda))).$$

Idea of Proof: Uniform estimates on the time and space translates of approximate solutions associated with (FV) are useful to find bounds on the Gagliardo seminorms for $(u^l_{h,N})_{h,N}$.

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Using

$$L^2(\Omega; L^2(0,T; W^{\alpha,2}(\Lambda))) \cap L^2(\Omega; W^{\alpha,2}(0,T; L^2(\Lambda))) \overset{\text{compact}}{\hookrightarrow} L^2(0,T; L^2(\Lambda))$$

the sequence of laws $(\mathbb{P} \circ (u^l_{h,N})^{-1})_{h,N}$ on $L^2(0,T; L^2(\Lambda))$ is tight.

Convergence in law

For $h > 0$, $N \in \mathbb{N}$ we consider the sequence of random vectors

$$Y_{h,N} = (u^l_{h,N}, u^r_{h,N} - u^l_{h,N}, W, u^0_h).$$

on $\mathcal{X} = (L^2(0,T;L^2(\Lambda)))^2 \times C([0,T]) \times L^2(\Lambda)$

Lemma [Bauzet, Nabet, S. Zimmermann, '22]

There exists a not relabeled subsequence of $(Y_{h,N})_{h,N}$ converging in law, i.e. there exists a probability measure μ_∞ with martingale laws $\mu^1_\infty, \delta_0, \mathbb{P} \circ W^{-1}, \mathbb{P} \circ (u_0)^{-1}$ such that for all bounded and continuous $f : \mathcal{X} \to \mathbb{R}$

$$\mathbb{E}[f(Y_{h,N})] \to \int_{\mathcal{X}} f \, d\mu_\infty \quad N \to \infty, h \downarrow 0.$$

Convergence in law - Idea of proof

- $(u^l_{h,N})_{h,N}$ bounded in $L^2(\Omega; W^{\alpha,2}(0,T; L^2(\Lambda))) \cap L^2(\Omega; L^2(0,T; W^{\alpha,2}(\Lambda)))$ implies by Prokhorov's theorem that $(u^l_{h,N})_{h,N}$, up to a subsequence, converges in law to a probability measure μ^1_∞ on $L^2(0,T; L^2(\Lambda))$.

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- $u^r_{h,N} - u^l_{h,N} \to 0$ in $L^2(\Omega; L^2(0,T;L^2(\Lambda)))$ implies for all $f : L^2(0,T;L^2(\Lambda)) \to \mathbb{R}$ bounded and continuous

$$\int_{L^2(0,T;L^2(\Lambda))} f\, d(\mathbb{P} \circ (u^r_{h,N} - u^l_{h,N})^{-1}) = \mathbb{E}[f(u^r_{h,N} - u^l_{h,N})] \to \mathbb{E}[f(0)].$$

So $(u^l_{h,N} - u^r_{h,N})_{h,N}$ converges in law to δ_0.

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- $u^r_{h,N} - u^l_{h,N} \to 0$ in $L^2(\Omega; L^2(0,T;L^2(\Lambda)))$ implies for all $f : L^2(0,T;L^2(\Lambda)) \to \mathbb{R}$ bounded and continuous

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  So $(u^l_{h,N} - u^r_{h,N})_{h,N}$ converges in law to δ_0.
- W converges in law to $\mathbb{P} \circ W^{-1}$ obviously
- $u^0_h \to u_0$ in $L^2(\Lambda)$ $\mathbb{P}$-a.s. in Ω
  $\Rightarrow u^0_h$ converges in law towards $\mathbb{P} \circ (u_0)^{-1}$.

Stochastic compactness argument

Skorokhod's theorem implies:
On a new probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ there exist random variables

$$Y'_{h,N} = (v^l_{h,N}, z_{h,N}, W_{h,N}, v^0_{h,N}), u_\infty, W_\infty, v_0$$

satisfying for all $h > 0$, $N \in \mathbb{N}$

$$\mathbb{P}' \circ (Y'_{h,N})^{-1} = \mathbb{P} \circ (Y_{h,N})^{-1}, \qquad \mathbb{P}' \circ (u_\infty)^{-1} = \mu^1_\infty,$$
$$\mathbb{P}' \circ (W_\infty)^{-1} = \mathbb{P} \circ (W)^{-1}, \qquad \mathbb{P}' \circ (v_0)^{-1} = \mathbb{P} \circ (u_0)^{-1}$$

and

- $v^l_{h,N} \longrightarrow u_\infty$ in $L^2(0,T;L^2(\Lambda))$, $\mathbb{P}'$-a.s. in Ω'
- $z_{h,N} \longrightarrow 0$ in $L^2(0,T;L^2(\Lambda))$, $\mathbb{P}'$-a.s. in Ω'
- $W_{h,N} \longrightarrow W_\infty$ in $C([0,T])$, $\mathbb{P}'$-a.s. in Ω'
- $v^0_{h,N} \longrightarrow v_0$ in $L^2(\Lambda)$, $\mathbb{P}'$-a.s. in Ω'.

Stochastic compactness argument

Skorokhod's theorem implies:
On a new probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ there exist random variables

$$Y'_{h,N} = (v^l_{h,N}, z_{h,N}, W_{h,N}, v^0_{h,N}), u_\infty, W_\infty, v_0$$

satisfying for all $h > 0$, $N \in \mathbb{N}$

$$\mathbb{P}' \circ (Y'_{h,N})^{-1} = \mathbb{P} \circ (Y_{h,N})^{-1}, \qquad \mathbb{P}' \circ (u_\infty)^{-1} = \mu^1_\infty,$$
$$\mathbb{P}' \circ (W_\infty)^{-1} = \mathbb{P} \circ (W)^{-1}, \qquad \mathbb{P}' \circ (v_0)^{-1} = \mathbb{P} \circ (u_0)^{-1}$$

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Thanks to equality in law, $v^l_{h,N}$ and $z_{h,N}$, respectively, have the same piecewise constant structure as $u^l_{h,N}$ and $u^r_{h,N} - u^l_{h,N}$, respectively.

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- $W_{h,N} \longrightarrow W_\infty$ in $L^2(\Omega'; C([0,T]))$
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Thanks to equality in law, $v^l_{h,N}$ and $z_{h,N}$, respectively, have the same piecewise constant structure as $u^l_{h,N}$ and $u^r_{h,N} - u^l_{h,N}$, respectively.

Consequences of Skorokhod's theorem

- For any $n \in \{0, \dots, N-1\}$, the random vector $(v_K^{n+1})_{K \in \mathcal{T}_h}$ is a solution of

$$m_K(v_K^{n+1} - v_K^n) + \triangle t \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (v_K^{n+1} - v_L^{n+1})$$
$$= m_K g(v_K^n)(W_{h,N}(t_{n+1}) - W_{h,N}(t_n))$$

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- There exists a filtration $(\mathfrak{F}_t^{h,N})_{t \geq 0}$ such that $v_{h,N}^l$ is adapted to $(\mathfrak{F}_t^{h,N})_{t \geq 0}$ and $W_{h,N} = (W_{h,N}(t))_{t \geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^{h,N})_{t \geq 0}$.

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- There exists a filtration $(\mathfrak{F}_t^\infty)_{t \geq 0}$ such that u_∞ has a predictable $d\mathbb{P}' \otimes dt$-representative and $W_\infty = (W_\infty(t))_{t \geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^\infty)_{t \geq 0}$.
  $u_\infty(t, \cdot)$ is only defined for a.e. $t \in [0, T]$
  $\rightsquigarrow$ $(\mathfrak{F}_t^\infty)_{t \geq 0}$ is generated with the use of v_0, W_∞ and $\int_0^\cdot u_\infty(s, \cdot)\, ds$

Consequences of Skorokhod's theorem

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- There exists a filtration $(\mathfrak{F}_t^{h,N})_{t \geq 0}$ such that $v_{h,N}^l$ is adapted to $(\mathfrak{F}_t^{h,N})_{t \geq 0}$ and $W_{h,N} = (W_{h,N}(t))_{t \geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^{h,N})_{t \geq 0}$.
- There exists a filtration $(\mathfrak{F}_t^{\infty})_{t \geq 0}$ such that u_∞ has a predictable $d\mathbb{P}' \otimes dt$-representative and $W_\infty = (W_\infty(t))_{t \geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^{\infty})_{t \geq 0}$.
- Lemma [Debussche, Glatt-Holtz, Temam, '11] implies

$$\int_0^\cdot g(v_{h,N}^l)\, dW_{h,N} \to \int_0^\cdot g(u_\infty)\, dW_\infty \text{ in } L^2(0,T;L^2(\Lambda)) \ \mathbb{P}'\text{-a.s. in } \Omega'.$$

Passage to the limit

For $t \in [t_n, t_{n+1}),\ n \in \{0, \dots, N-1\},\ K \in \mathcal{T}_h$

$$\frac{m_K}{\triangle t}\left[v_K^{n+1} - v_K^n - g(v_K^n)(W_{h,N}(t_{n+1}) - W(t_n))\right] + \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}}(v_K^{n+1} - v_L^{n+1}) = 0$$

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$$\frac{m_K}{\triangle t}\left[v_K^{n+1} - v_K^n - \int_{t_n}^{t_{n+1}} g(v_{h,N}^l(x_K))\, dW_{h,N}\right] + \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}}(v_K^{n+1} - v_L^{n+1}) = 0$$

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Let $A \in \mathcal{A}'$, $\xi \in C^\infty(\mathbb{R})$, $\xi(T) = 0$, $\varphi \in C^\infty(\mathbb{R}^2)$, $\nabla\varphi \cdot \vec{n} = 0$ on $\partial\Lambda$,
$\varphi_h(x) := \varphi(x_K)$ for $x \in K$.
For $t \in [t_n, t_{n+1}),\ n \in \{0, \dots, N-1\},\ K \in \mathcal{T}_h$

$$\mathbb{1}_A(\omega)\xi(t)\frac{m_K}{\triangle t}\left[v_K^{n+1} - v_K^n - \int_{t_n}^{t_{n+1}} g(v_{h,N}^l(x_K))\, dW_{h,N}\right]\varphi(x_K)$$
$$+\, \mathbb{1}_A(\omega)\xi(t) \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}}(v_K^{n+1} - v_L^{n+1})\varphi(x_K) = 0$$

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$$+\,\mathbb{E}'\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\mathbb{1}_A(\omega)\xi(t)\sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}_{\text{int}}\cap\mathcal{E}_K}\frac{m_\sigma}{d_{K|L}}(v_K^{n+1} - v_L^{n+1})\varphi(x_K)\, dt = 0$$

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$$+ \mathbb{E}' \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{1}_A(\omega)\xi(t) \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (v_K^{n+1} - v_L^{n+1})\varphi(x_K)\, dt = 0$$

$\widehat{v}_{h,N}$, $\widehat{M}_{h,N}$, respectively, piecewise affine in time and piecewise constant in space functions associated to $v_{h,N}^l$, $\int_0^\cdot g(v_{h,N}^l)\, dW_{h,N}$, respectively.

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$$-\mathbb{E}'\int_0^T \int_\Lambda \mathbb{1}_A(\omega)\xi'(t)[\widehat{v}_{h,N} - \widehat{M}_{h,N}](t)\varphi_h(x)\,dx\,dt + \mathbb{E}'\int_\Lambda v_h^0 \xi(0)\varphi_h(x)\,dx$$
$$+\,\mathbb{E}'\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \mathbb{1}_A(\omega)\xi(t)\sum_{\sigma\in\mathcal{E}_{\text{int}}}\frac{m_\sigma}{d_{K|L}}(v_K^{n+1} - v_L^{n+1})(\varphi(x_K) - \varphi(x_L))\,dt = 0$$

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Passage to the limit results

$$\int_0^T \int_\Lambda \xi'(t)\left[u_\infty(t) - \int_0^t g(u_\infty)\,dW_\infty\right]\varphi(x)\,dx\,dt + \int_\Lambda v_0\xi(0)\varphi(x)\,dx$$
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Since $\{\varphi \in C^\infty(\mathbb{R}^2) : \nabla\varphi \cdot \mathbf{n} = 0 \text{ on } \partial\Lambda\}$ is dense in $H^1(\Lambda)$ the equation also holds for all $\varphi \in H^1(\Lambda)$.

Regularity

- $u_\infty \in L^2(\Omega'; L^2(0,T; H^1(\Lambda)))$ implies $\Delta u_\infty \in L^2(\Omega'; L^2(0,T; H^1(\Lambda)^*))$ and

$$-\int_0^T \int_\Lambda \nabla u_\infty \nabla \varphi(x) \xi(t)\, dx\, dt = \int_0^T \langle \Delta u_\infty, \varphi \rangle \xi(t)\, dt$$

$\mathbb{P}'$-a.s. in Ω' for all $\varphi \in H^1(\Lambda), \xi \in C^\infty(\mathbb{R}), \xi(T) = 0$.

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- Using Fubini's theorem

$$\int_0^T \int_\Lambda \xi'(t) \left[u_\infty(t) - \int_0^t g(u_\infty)\, dW_\infty \right] \varphi(x)\, dx\, dt + \int_\Lambda v_0 \xi(0) \varphi(x)\, dx$$
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$$\begin{aligned} &\langle -\int_0^T \xi'(t) \left[u_\infty(t) - \int_0^t g(u_\infty)\, dW_\infty - v_0 \right] dt, \varphi \rangle \\ &= \langle \int_0^T \Delta u_\infty \xi(t)\, dt, \varphi \rangle \end{aligned}$$

- So $u_\infty - \int_0^\infty g(u_\infty)\, dW_\infty - v_0 \in W^{1,2}(0,T; H^1(\Lambda)^*)$ $\mathbb{P}'$-a.s. in Ω' and

$$\frac{d}{dt}\left(u_\infty(t) - \int_0^t g(u_\infty)\, dW_\infty - v_0 \right) = \Delta u_\infty \text{ in } L^2(\Omega'; L^2(0,T; H^1(\Lambda)^*))$$

Regularity

- $u_\infty - \int_0^\cdot g(u_\infty)\, dW_\infty \in L^2(\Omega'; L^2(0,T; H^1(\Lambda)))$ implies $u_\infty \in L^2(\Omega'; C([0,T]; L^2(\Lambda)))$ and

$$\left(u_\infty(t) - \int_0^t g(u_\infty)\, dW_\infty - v_0, \varphi\right)\xi(t) - (u_\infty(0) - v_0, \varphi)\xi(0)$$
$$= \int_0^t \xi(s)\langle \Delta u_\infty(s), \varphi\rangle\, ds + \int_0^t \xi'(s)\left(u_\infty(s) - \int_0^s g(u_\infty)\, dW_\infty - v_0, \varphi\right) ds$$

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for all $\varphi \in H^1(\Lambda), \xi \in C^\infty(\mathbb{R}), \xi(T) = 0$ $\mathbb{P}'$-a.s. in Ω'.

- Special choices of $t \in [0,T]$ and ξ imply $u_\infty(0) = v_0$ $\mathbb{P}'$-a.s. in Ω' and

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  in $L^2(\Lambda)$ $\mathbb{P}'$-a.s. in Ω'.

Strong convergence of finite-volume approximations

Proposition [Bauzet, Nabet, S., Zimmermann, '22]

$(\Omega', \mathbb{P}', \mathcal{A}', (\mathfrak{F}_t^\infty)_{t\geq 0}, W_\infty, u_\infty, v_0)$ is a martingale solution to (SHE).

- Solutions are pathwise unique.
- We may construct two martingale solutions with respect to the same stochastic basis and with the same initial value.
- According to [Gyöngy, Krylov, '96]: Up to a not relabeled subsequence, $(u^l_{h,N})_{h,N}$ and $(u^r_{h,N})_{h,N}$ converge in $L^2(0,T;L^2(\Lambda))$ $\mathbb{P}$-a.s. in Ω.

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- According to [Gyöngy, Krylov, '96]: Up to a not relabeled subsequence, $(u^l_{h,N})_{h,N}$ and $(u^r_{h,N})_{h,N}$ converge in $L^2(0,T;L^2(\Lambda))$ $\mathbb{P}$-a.s. in Ω.
- Thanks to the uniform bounds in $L^2(\Omega;L^2(0,T;L^2(\Lambda)))$, by Vitali's theorem the convergence also holds in $L^p(\Omega;L^2(0,T;L^2(\Lambda)))$ for $p \in [1,2)$.

Strong convergence of finite-volume approximations

Proposition [Bauzet, Nabet, S., Zimmermann, '22]

$(\Omega', \mathbb{P}', \mathcal{A}', (\mathfrak{F}_t^\infty)_{t\geq 0}, W_\infty, u_\infty, v_0)$ is a martingale solution to (SHE).

- Solutions are pathwise unique.
- We may construct two martingale solutions with respect to the same stochastic basis and with the same initial value.
- According to [Gyöngy, Krylov, '96]: Up to a not relabeled subsequence, $(u^l_{h,N})_{h,N}$ and $(u^r_{h,N})_{h,N}$ converge in $L^2(0,T;L^2(\Lambda))$ $\mathbb{P}$-a.s. in Ω.
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- The joint limit u is the unique variational solution to (SHE).

Thank you for your attention.