The finite volume method for solving the oblique derivative geodetic boundary value problems

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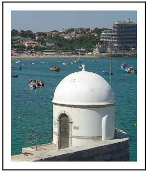
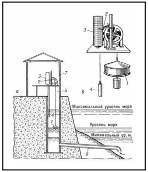
Outline of the talk

1. Motivation
2. Formulation of the oblique derivative BVP
3. Finite volume method for solving the oblique derivative BVP
   a) The central finite volume scheme for approximation of the oblique BC
      - Numerical experiments
   b) The upwind finite volume scheme for approximation of the oblique BC
      - Numerical experiments
4. Conclusion

Motivation

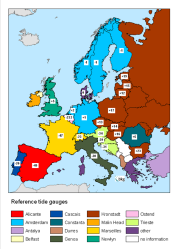
- Traditionally, the location of points on the Earth's surface was given separately by the horizontal position and vertical component - the height above the mean sea level.
- The height above the mean sea level refers to the local vertical datum that is defined by the selected tide gauge (mareograph).

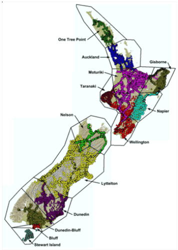



Motivation

- However, the sea water has a different temperature, structure, is subject to sea currents, earth's rotation, therefore the steady level of the seas and oceans has a different height with respect to the surface with the same gravity potential.
- So the tide gauges and thus local vertical datums are shifted.
- In Europe, we have more than 12 local vertical datums, and New Zealand, as an island country, has up to 13 local vertical datums.





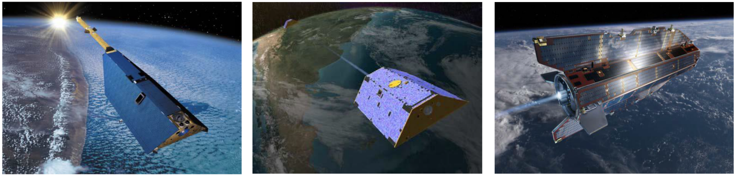
Motivation

- The case of the Laufenburg bridge or what can happen to a bridge when one side uses Mediterranean sea level and another the North sea

- Laufenburg is a town that straddles Germany and Switzerland. As two halves of a new bridge grew closer to one another in 2003, it became clear that one side was 54 cm higher than the other. Builders knew that there was a 27 cm difference between the LVDs - but they applied it with wrong sign. The German side had to be lowered before the bridge could be completed.

Motivation

- These practical reasons led to the idea of creating a global uniform vertical datum.
- This idea could begin to be fulfilled only with the arrival of space satellites (CHAMP, GRACE and GOCE).



- Thanks to their accurate monitoring of the Earth's gravity field, local vertical datums on different continents can be unified into one global vertical datum by finding the so-called geoid defined by W_0.

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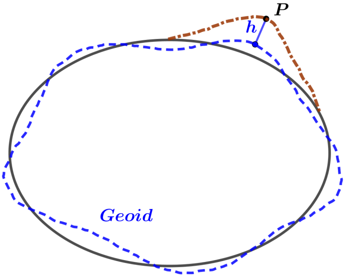
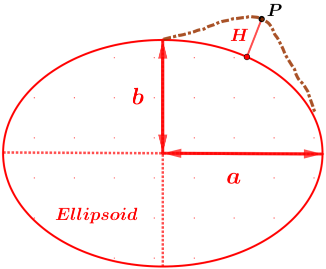
Formulation of the oblique derivative BVP

- Geoid: It is the equipotential surface that coincides with the mean sea surface and extends through the continents.
- Reference ellipsoid: A reference ellipsoid is a mathematically defined surface that approximates the geoid.





Formulation of the oblique derivative BVP

The gravity field - generated by the real Earth

- The gravity potential $W(\mathbf{x})$, $W(\mathbf{x}) = V_g(\mathbf{x}) + V_c(\mathbf{x})$
  - $V_g(\mathbf{x})$ is the gravitational potential (Newton formula)
  - $V_c(\mathbf{x})$ is the centrifugal potential (Earth spin velocity).

  - scalar quantity - unable to measure
- The gravity (acceleration) $\vec{g}(\mathbf{x})$
  - vector quantity - direction (astronomy) and size (gravimetry)

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  - vector quantity - direction (astronomy) and size (gravimetry)

The normal field - generated by the normal body

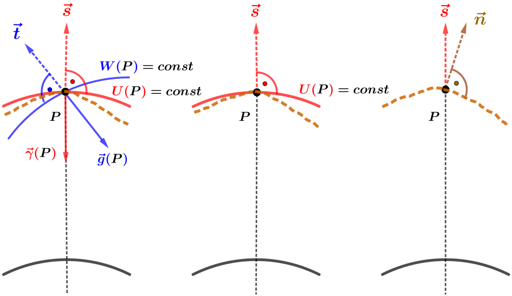
Defined by parameters (major semi-axis, flattening, geocentric gravitational constant, spin velocity) that are taken from the Earth

- The normal gravity potential $U(\mathbf{x})$
  $U(\mathbf{x}) = U_g(\mathbf{x}) + U_c(\mathbf{x})$
- The normal gravity $\vec{\gamma}(\mathbf{x})$

Formulation of the oblique derivative BVP

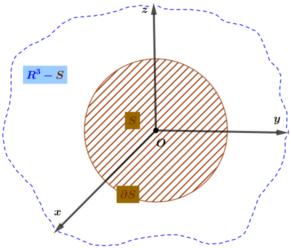
The disturbing field

- The disturbing potential $T(\mathbf{x})$, $T(\mathbf{x}) = W(\mathbf{x}) - U(\mathbf{x}) = V_g(\mathbf{x}) - U_g(\mathbf{x})$
- The gravity disturbance $\delta g(\mathbf{x})$,
  $\langle \nabla T(\mathbf{x}), \bar{s} \rangle = \langle \nabla W(\mathbf{x}), \bar{s} \rangle - \langle \nabla U(\mathbf{x}), \bar{s} \rangle \approx$
  $\langle \nabla W(\mathbf{x}), \bar{t} \rangle - \langle \nabla U(\mathbf{x}), \bar{s} \rangle = -g(\mathbf{x}) + \gamma(\mathbf{x}) = -\delta g(\mathbf{x})$, $\mathbf{x} \in R^3$.



Formulation of the oblique derivative BVP

- Let us consider the infinite computational domain $R^3 - S$, where S denotes the Earth and ∂S its boundary.



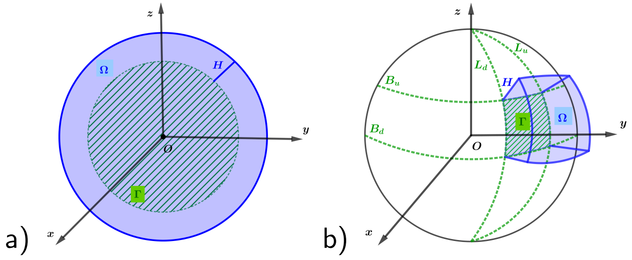
$$\Delta T(\mathbf{x}) = 0, \quad \mathbf{x} \in R^3 - S, \tag{1}$$

$$\nabla T(\mathbf{x}) \cdot \mathbf{s}(\mathbf{x}) = -\delta g(\mathbf{x}), \quad \mathbf{x} \in \partial S, \tag{2}$$

$$T(\mathbf{x}) \rightarrow 0, \quad \text{as} \quad |\mathbf{x}| \rightarrow \infty. \tag{3}$$

Formulation of the oblique derivative BVP

- For the FVM, we will create a finite domain Ω by adding a boundary/boundaries



$$
\begin{aligned}
\Delta T(\mathbf{x}) &= 0, \quad \mathbf{x} \in \Omega \subset R^3, &(4)\\
\nabla T(\mathbf{x}) \cdot \mathbf{s}(\mathbf{x}) &= -\delta g(\mathbf{x}), \quad \mathbf{x} \in \Gamma \subset \partial\Omega, &(5)\\
T(\mathbf{x}) &= T_{SAT}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega - \Gamma. &(6)
\end{aligned}
$$

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FVM for solving the oblique derivative BVP

- We multiply the Laplace equation (4) by -1 and integrate over the finite volume $V_{i,j,k}$

$$-\int_{V_{i,j,k}} \Delta T \, \mathrm{d}V = -\int_{\partial V_{i,j,k}} \nabla T \cdot \mathbf{n} \, \mathrm{d}S, \tag{7}$$

and it gives us a weak formulation of the equation (4)

$$-\int_{\partial V_{i,j,k}} \frac{\partial T}{\partial \mathbf{n}} \mathrm{d}S = 0, \tag{8}$$

where $\mathbf{n}$ is the unit normal vector to the boundary $V_{i,j,k}$.

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- We denote by $\mathbf{x}_{i,j,k}$ a representative point of $V_{i,j,k}$ and N_1 the set of all triples (p, q, r), $|p| + |q| + |r| = 1$, then the finite volumes $V_{i+p,j+q,k+r}, \ (i, j, k) \in N_1$ share a common edge $e^{p,q,r}_{i,j,k}$ with $V_{i,j,k}$.

FVM for solving the oblique derivative BVP

- Using this discretization we can then write

$$-\sum_{(p,q,r)\in N_1} \int_{e_{i,j,k}^{p,q,r}} \frac{\partial T}{\partial \mathbf{n}_{i,j,k}^{p,q,r}} \mathrm{d}S = 0, \tag{9}$$

where $\mathbf{n}_{i,j,k}^{p,q,r}$ is a unit vector in the normal direction oriented from the finite volume $V_{i,j,k}$ to $V_{i+p,j+q,k+r}$.

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- We approximate the derivative in the normal direction

$$\frac{\partial T}{\partial \mathbf{n}_{i,j,k}^{p,q,r}} \approx \frac{T_{i+p,j+q,k+r} - T_{i,j,k}}{d_{i,j,k}^{p,q,r}}, \tag{10}$$

where the unknown values $T_{i,j,k}$ are considered at the points $\mathbf{x}_{i,j,k}$ and $d_{i,j,k}^{p,q,r}$ denotes the distance between $\mathbf{x}_{i,j,k}$ and $\mathbf{x}_{i+p,j+q,k+r}$.

FVM for solving the oblique derivative BVP

- If we assume that the derivative in the normal direction is constant on the boundary $e_{i,j,k}^{p,q,r}$, we get

$$\sum_{(p,q,r)\in N_1} \frac{m\left(e_{i,j,k}^{p,q,r}\right)}{d_{i,j,k}^{p,q,r}} \left(T_{i,j,k} - T_{i+p,j+q,k+r}\right) = 0. \tag{11}$$

where $m\left(e_{i,j,k}^{p,q,r}\right)$ is the size of the area $e_{i,j,k}^{p,q,r}$.

- Finally, we take into account the boundary conditions.

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The central finite volume scheme

- We start by decomposing the gradient T in the equation (5) into one normal and two tangential components to $\Gamma \subset \partial\Omega \subset R^3$.

$$\nabla T = (\nabla T \cdot \mathbf{n})\mathbf{n} + (\nabla T \cdot \mathbf{t_1})\mathbf{t_1} + (\nabla T \cdot \mathbf{t_2})\mathbf{t_2} = \frac{\partial T}{\partial \mathbf{n}}\mathbf{n} + \frac{\partial T}{\partial \mathbf{t_1}}\mathbf{t_1} + \frac{\partial T}{\partial \mathbf{t_2}}\mathbf{t_2}.$$

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- We substitute it into BC (5)

$$\nabla T \cdot \mathbf{s} = \left(\frac{\partial T}{\partial \mathbf{n}}\mathbf{n} + \frac{\partial T}{\partial \mathbf{t_1}}\mathbf{t_1} + \frac{\partial T}{\partial \mathbf{t_2}}\mathbf{t_2}\right) \cdot \mathbf{s} = \frac{\partial T}{\partial \mathbf{n}}\mathbf{n} \cdot \mathbf{s} + \frac{\partial T}{\partial \mathbf{t_1}}\mathbf{t_1} \cdot \mathbf{s} + \frac{\partial T}{\partial \mathbf{t_2}}\mathbf{t_2} \cdot \mathbf{s}$$

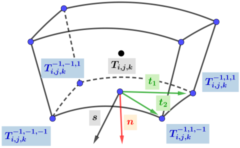
and we obtain

$$\frac{\partial T}{\partial \mathbf{n}}\mathbf{n} \cdot \mathbf{s} + \frac{\partial T}{\partial \mathbf{t_1}}\mathbf{t_1} \cdot \mathbf{s} + \frac{\partial T}{\partial \mathbf{t_2}}\mathbf{t_2} \cdot \mathbf{s} = -\delta g. \qquad (12)$$

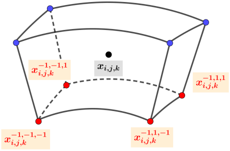
The central finite volume scheme

- We calculate unit vectors $\mathbf{n}$, $\mathbf{t_1}$ and $\mathbf{t_2}$ using the corresponding coordinates

a)



b)



$$\begin{aligned}
\mathbf{n} &= \frac{\mathbf{x}_{i-1,j,k} - \mathbf{x}_{i,j,k}}{\left|\mathbf{x}_{i-1,j,k} - \mathbf{x}_{i,j,k}\right|}, \\
\mathbf{t_1} &= \frac{\mathbf{x}_{i,j,k}^{-1,1,1} - \mathbf{x}_{i,j,k}^{-1,-1,-1}}{\left|\mathbf{x}_{i,j,k}^{-1,1,1} - \mathbf{x}_{i,j,k}^{-1,-1,-1}\right|}, \\
\mathbf{t_2} &= \frac{\mathbf{x}_{i,j,k}^{-1,1,-1} - \mathbf{x}_{i,j,k}^{-1,-1,1}}{\left|\mathbf{x}_{i,j,k}^{-1,1,-1} - \mathbf{x}_{i,j,k}^{-1,-1,1}\right|},
\end{aligned} \tag{13}$$

The central finite volume scheme

- and in a similar way we approximate the derivatives in the direction of the vectors (12)

$$\begin{aligned} \frac{\partial T}{\partial \mathbf{n}} &= \frac{T_{i-1,j,k} - T_{i,j,k}}{|\mathbf{x}_{i-1,j,k} - \mathbf{x}_{i,j,k}|}, \\ \frac{\partial T}{\partial \mathbf{t_1}} &= \frac{T_{i,j,k}^{-1,1,1} - T_{i,j,k}^{-1,-1,-1}}{|\mathbf{x}_{i,j,k}^{-1,1,1} - \mathbf{x}_{i,j,k}^{-1,-1,-1}|}, \\ \frac{\partial T}{\partial \mathbf{t_2}} &= \frac{T_{i,j,k}^{-1,1,-1} - T_{i,j,k}^{-1,-1,1}}{|\mathbf{x}_{i,j,k}^{-1,1,-1} - \mathbf{x}_{i,j,k}^{-1,-1,1}|}, \end{aligned} \qquad (14)$$

where we calculate the values of $T_{i,j,k}^{p,q,r}$ as

$$T_{i,j,k}^{p,q,r} = \frac{1}{8} \sum_{(l,m,n)\in B(p,q,r)} T_{i+l,j+m,k+n}. \qquad (15)$$

The central finite volume scheme

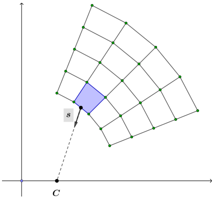
- We substitute these approximations into (12) to obtain an approximating equation for the BC (5) in the form

$$\nabla T \cdot \mathbf{s} \approx \frac{T_{i-1,j,k} - T_{i,j,k}}{|\mathbf{x}_{i-1,j,k} - \mathbf{x}_{i,j,k}|}\mathbf{n}\cdot\mathbf{s} + \frac{T_{i,j,k}^{-1,1,1} - T_{i,j,k}^{-1,-1,-1}}{|\mathbf{x}_{i,j,k}^{-1,1,1} - \mathbf{x}_{i,j,k}^{-1,-1,-1}|}\mathbf{t_1}\cdot\mathbf{s} + \frac{T_{i,j,k}^{-1,1,-1} - T_{i,j,k}^{-1,-1,1}}{|\mathbf{x}_{i,j,k}^{-1,1,-1} - \mathbf{x}_{i,j,k}^{-1,-1,1}|}\mathbf{t_2}\cdot\mathbf{s} = -\delta g. \quad (16)$$

- At the end, we add the equation into our system of linear equations and solve it.

Numerical experiments - Testing experiment No. 1

- The computational domain: $\langle 0, \pi\rangle \times \langle 0, \pi\rangle \times \langle 1m, 2m\rangle$.

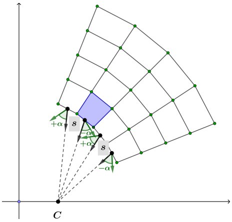


- The C point was located at $(0.1, -0.2, -0.1)$, the Dirichlet BC (6) was specified in the form $T^\star = 1/r$, where r is distance from C, and the oblique derivative BC (5) as $-1/r^2$.

$n_1 \times n_2 \times n_3$	$\|\|T^\star - T\|\|_{L_2(\Omega)}$	EOC
2×2×4	$6.74805.10^{-2}$	-
4×4×8	$9.00317.10^{-3}$	2.90597
8×8×16	$1.54266.10^{-3}$	2.54502
16×16×32	$3.01950.10^{-4}$	2.35328
32×32×64	$0.67123.10^{-5}$	2.16928

Numerical experiments - Testing experiment No. 2

- The computational domain was the same as in the previous experiment, but we further rotated the oblique vector by $\pm 20^{\circ}$.



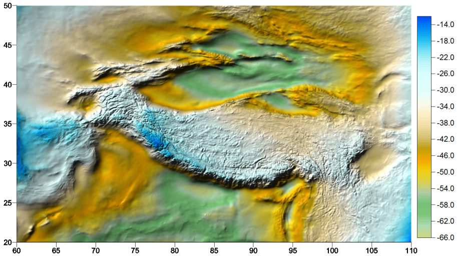
$n_1 \times n_2 \times n_3$	$\|\|T^{\star} - T\|\|_{L_2(\Omega)}$	EOC
2×2×4	$6.43828.10^{-2}$	-
4×4×8	$8.14779.10^{-3}$	2.98220
8×8×16	$1.34261.10^{-3}$	2.60137
16×16×32	$2.44307.10^{-4}$	2.45827
32×32×64	$0.52002.10^{-5}$	2.23204

Numerical experiments - Testing experiment No. 3

- Himalayas region: $B \in \langle 20.0°, 50.0° \rangle$, $L \in \langle 60.0°, 110.0° \rangle$
- The bottom boundary was approximated by the WGS84 ellipsoid, and the upper boundary was at the height of 240 km.
- To calculate the oblique vector we used heights generated from SRTM30.
- The number of divisions was $900 \times 1500 \times 1200$, which corresponds to the size of the finite volume $5' \times 5' \times 200m$.
- All BCs were generated from the EGM2008 model that is the Earth Gravitational Model based on spherical harmonics up to degree 2160.
- For a comparison, we performed the same computation with the Neumann BC applied on the bottom boundary.

Numerical experiments - Testing experiment No. 3

- The quasigeoidal heights in the Himalayas



Residuals [GPU]	Neumann BC	Oblique derivative BC
Min. value	-0.227	-0.087
Mean value	0.009	0.004
Max. value	0.332	0.095
STD	0.031	0.017
RMS	0.032	0.018

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The upwind finite volume scheme

- The main idea of this approach is that we think of the BC (5) as a stationary advection equation for the unknown disturbing potential and we approximate it by the first-order upwind method.

The upwind finite volume scheme

- The main idea of this approach is that we think of the BC (5) as a stationary advection equation for the unknown disturbing potential and we approximate it by the first-order upwind method.
- We apply the divergence operator to $T(\mathbf{x})\mathbf{s}(\mathbf{x})$

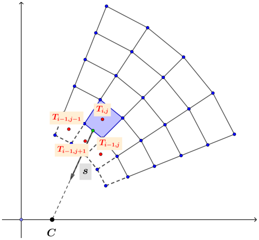
$$\nabla \cdot (T(\mathbf{x})\mathbf{s}(\mathbf{x})) = T(\mathbf{x})\nabla \cdot \mathbf{s}(\mathbf{x}) + \nabla T(\mathbf{x}) \cdot \mathbf{s}(\mathbf{x}). \qquad (17)$$

- From the equation (17), we express $\nabla T(\mathbf{x}) \cdot \mathbf{s}(\mathbf{x})$ and substitute in BC (5)

$$\nabla \cdot (T(\mathbf{x})\mathbf{s}(\mathbf{x})) - T(\mathbf{x})\nabla \cdot \mathbf{s}(\mathbf{x}) = \delta g(\mathbf{x}). \qquad (18)$$

The upwind finite volume scheme

- We add one row of finite volumes below the bottom boundary



and integrate the equation (18) over one of the added volumes $V_{i,j,k}$ (to simplify, we will not write $(\mathbf{x})$)

$$\int_{V_{i,j,k}} \nabla \cdot (T\mathbf{s})\,\mathrm{d}V - \int_{V_{i,j,k}} T\nabla \cdot \mathbf{s}\,\mathrm{d}V = \int_{V_{i,j,k}} \delta g\,\mathrm{d}V. \qquad (19)$$

The upwind finite volume scheme

- We assume a constant value of T in the finite volume $V_{i,j,k}$ and denote it by $T_{i,j,k}$, and integrate the left-hand side of (19)

$$\sum_{(p,q,r)\in N_1} \int_{e_{i,j,k}^{p,q,r}} T \, \mathbf{s}\cdot\mathbf{n}_{i,j,k}^{p,q,r} \, \mathrm{d}S - \sum_{(p,q,r)\in N_1} T_{i,j,k} \int_{e_{i,j,k}^{p,q,r}} \mathbf{s}\cdot\mathbf{n}_{i,j,k}^{p,q,r} \, \mathrm{d}S = \int_{V_{i,j,k}} \delta g \, \mathrm{d}V.$$

The upwind finite volume scheme

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$$\sum_{(p,q,r)\in N_1} \int_{e^{p,q,r}_{i,j,k}} T\ \mathbf{s}\cdot\mathbf{n}^{p,q,r}_{i,j,k}\,\mathrm{d}S - \sum_{(p,q,r)\in N_1} T_{i,j,k} \int_{e^{p,q,r}_{i,j,k}} \mathbf{s}\cdot\mathbf{n}^{p,q,r}_{i,j,k}\,\mathrm{d}S = \int_{V_{i,j,k}} \delta g\,\mathrm{d}V.$$

- Let $T^{p,q,r}_{i,j,k}$ denote the constant approximation of the solution on the boundary $e^{p,q,r}_{i,j,k}$ and volume size $V_{i,j,k}$ as $m(V_{i,j,k})$, we get

$$\begin{aligned}\sum_{(p,q,r)\in N_1} T^{p,q,r}_{i,j,k} \int_{e^{p,q,r}_{i,j,k}} \mathbf{s}\cdot\mathbf{n}^{p,q,r}_{i,j,k}\,\mathrm{d}S \quad &- \quad \sum_{(p,q,r)\in N_1} T_{i,j,k} \int_{e^{p,q,r}_{i,j,k}} \mathbf{s}\cdot\mathbf{n}^{p,q,r}_{i,j,k}\,\mathrm{d}S \\ &= \quad \delta g\ m(V_{i,j,k}).\end{aligned}$$

The upwind finite volume scheme

- When we set

$$s_{i,j,k}^{p,q,r} = \int_{e_{i,j,k}^{p,q,r}} \mathbf{s} \cdot \mathbf{n}_{i,j,k}^{p,q,r} \, \mathrm{d}S \approx m(e_{i,j,k}^{p,q,r}) \, \mathbf{s} \cdot \mathbf{n}_{i,j,k}^{p,q,r}, \tag{20}$$

we obtain

$$\sum_{(p,q,r) \in N_1} s_{i,j,k}^{p,q,r} (T_{i,j,k}^{p,q,r} - T_{i,j,k}) = \delta g \; m(V_{i,j,k}). \tag{21}$$

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- Let

$$T_{i,j,k}^{p,q,r} = T_{i,j,k}, \quad \text{if} \quad s_{i,j,k}^{p,q,r} > 0, \tag{22}$$

$$T_{i,j,k}^{p,q,r} = T_{i+p,j+q,k+r}, \quad \text{if} \quad s_{i,j,k}^{p,q,r} < 0, \tag{23}$$

It corresponds to inflow when $s_{i,j,k}^{p,q,r} < 0$ and outflow when $s_{i,j,k}^{p,q,r} > 0$.

The upwind finite volume scheme

- After inserting equations (22) - (23) to (21) we get the equation for the approximation of the BC (5)

$$\sum_{(p,q,r)\in N_1^{in}} s_{i,j,k}^{p,q,r}(T_{i+p,j+q,k+r} - T_{i,j,k}) = \delta g\ m(V_{i,j,k}), \qquad (24)$$

where N_1^{in} denotes neighbours on inflow boundaries of the finite volume $V_{i,j,k}$, i.e., where $s_{i,j,k}^{p,q,r} < 0$.

The upwind finite volume scheme

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Droniou J, Medl'a M, Mikula K: Design and analysis of finite volume methods for elliptic equations with oblique derivatives; application to Earth gravity field modelling. Journal of Computational Physics, s. 2019

Numerical experiments - Testing experiment No. 1

- The computational domain was the same as in the previous approach, but we have more shifted $C = (0.3, -0.2, 0.1)$ and the oblique vector modified by an angle $\pm 20°$.

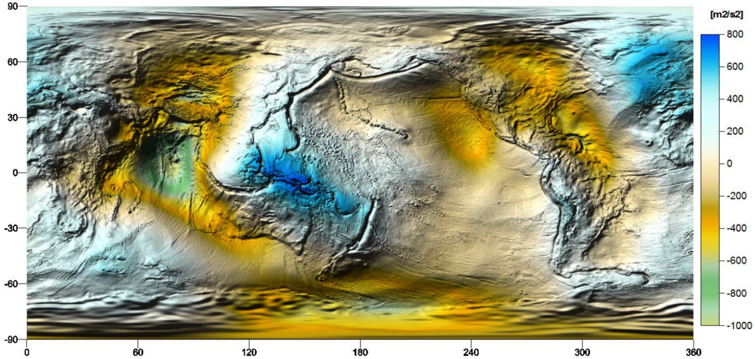
Upwind scheme				
$n_1 \times n_2 \times n_3$	$\|T^* - T\|_{L_2(\Omega)}$	EOC	$\|T^* - T\|_{MAX(\Gamma)}$	EOC
8×8×4	0.177728	-	0.362022	-
16×16×8	0.059441	1.58	0.177806	1.03
32×32×16	0.022542	1.39	0.083563	1.08
64×64×32	0.010819	1.05	0.041756	1.00
128×128×64	0.005143	1.07	0.019506	1.13
Central scheme				
$n_1 \times n_2 \times n_3$	$\|T^* - T\|_{L_2(\Omega)}$	EOC	$\|T^* - T\|_{MAX(\Gamma)}$	EOC
8×8×4	0.061529	-	0.3511	-
16×16×8	0.146351	-1.25	0.209212	0.75
32×32×16	0.058753	1.31	0.050549	2.05
64×64×32	0.008090	2.86	0.053722	2.64
128×128×64	0.004520	0.83	0.024245	0.84

Numerical experiments - Testing experiment No. 2

- Global gravity field modelling - reconstruction of the EGM2008 model.
- The bottom boundary was approximated by the WGS84 ellipsoid.
- The upper boundary was at the height of $240\ km$.
- The SRTM30PLUS model was used to calculate the oblique vector.
- All BCs were generated from the EGM2008 model, and the obtained numerical solution was compared to EGM2008.
- The computational domain was divided in L, B, H directions into $n_1 \times n_2 \times n_3$ parts:
  a) $540 \times 270 \times 75$ (i.e. volume size: $40' \times 40' \times 3200\,m$),
  b) $1080 \times 540 \times 150$ (i.e. volume size: $20' \times 20' \times 1600\,m$),
  c) $2160 \times 1080 \times 300$ (i.e. volume size: $10' \times 10' \times 800\,m$),
  d) $4320 \times 2160 \times 600$ (i.e. volume size: $5' \times 5' \times 400\,m$).

Numerical experiments - Testing experiment No. 2

Resol.	Min. res.	Max. res.	Mean res.	STD (total)	STD (Sea)	STD (Land)
$40' \times 40'$	-78.910	80.426	-0.392	5.238	4.771	6.228
$20' \times 20'$	-46.584	27.558	-0.273	1.948	1.489	2.750
$10' \times 10'$	-22.011	7.954	-0.265	0.904	0.327	1.578
$5' \times 5'$	-13.926	7.932	-0.114	0.558	0.183	0.991



Numerical experiments - Testing experiment No. 2

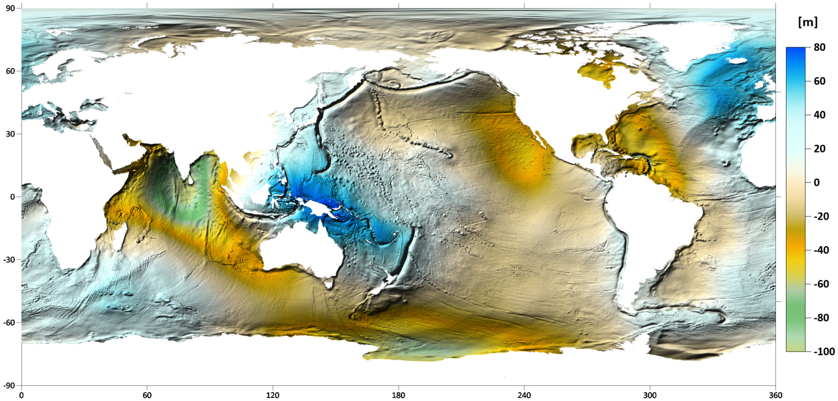


Figure: The mean sea surface

Numerical experiments - Testing experiment No. 2

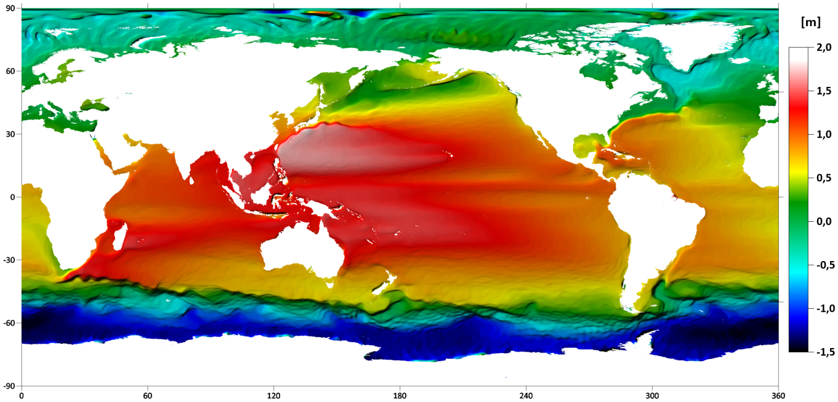


Figure: Mean dynamic topography: mean sea surface - geoid

Numerical experiments - Testing experiment No. 2

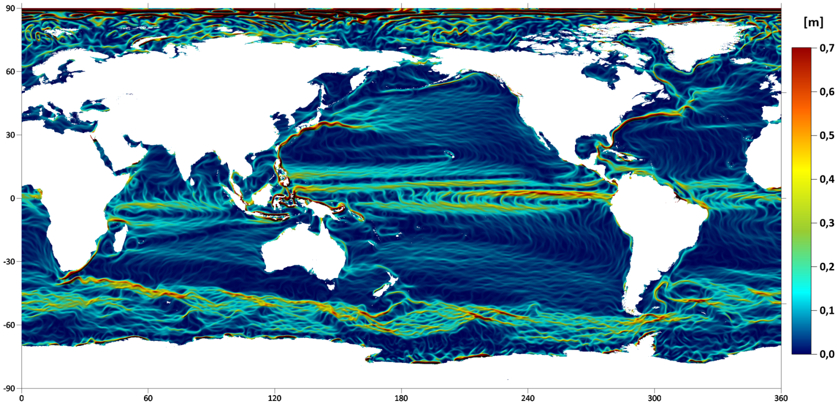


Figure: A model of sea currents

Conclusion

The value of the W_0 was officially adopted as a constant at the 26^{th} General Assembly of the International Union of Geodesy and Geophysics in Prague in the summer of 2015. Scientists led by Dr. Laura Sánchez determined that the value of the Earth's gravity potential, which best describes the mean sea surface, is

$$W_0 = 62\,636\,853.4\, m^2 s^{-2}.$$

One of the four groups that cooperated on this research was from the Department of the Mathematics and descriptive geometry at Slovak University of Technology.

Thank you for your attention.