Non-overlapping Schwarz algorithms for the incompressible Navier-Stokes equations with DDFV discretizations

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Workshop: Discrete Duality Finite Volume Method and Applications

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Contents

Introduction

Goal

Consider the incompressible Navier-Stokes problem:

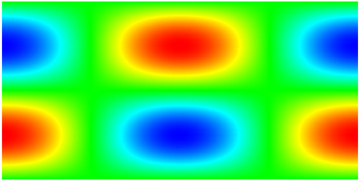
$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mathrm{div}(\sigma(\mathbf{u}, \mathrm{p})) = \mathbf{f} & \text{in} \quad \Omega \times [0, T], \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega \times [0, T], \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega \times [0, T], \\ \mathbf{u}(0) = \mathbf{u}_{init} & \text{in} \quad \Omega, \end{cases}$$

with $\sigma(\mathbf{u}, \mathrm{p}) = \dfrac{2}{\mathrm{Re}} \mathrm{D}\mathbf{u} - \mathrm{pId}$ and $\mathrm{D}\mathbf{u} = \dfrac{1}{2}(\nabla \mathbf{u} + {}^t\nabla \mathbf{u})$.

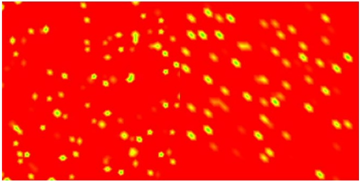
$\rightsquigarrow$ develop a non-overlapping iterative Schwarz algorithm with DDFV schemes

Example of Domain Decomposition algorithm

Limit problem:



Schwarz iterates:



Historical background

- Laplace problem:
  - Schwarz (1870) $\rightarrow$ overlapping
  - Lions (1990) $\rightarrow$ non-overlapping

- Isotropic diffusion :
  - Achdou-Japhet-Nataf-Maday (2002), Cautrès-Herbin-Hubert (2004), ...

- Anisotropic diffusion :
  - Gander-Halpern-Hubert-Krell (2018)

- Advection-diffusion-reaction :
  - Gander-Halpern (2007), Halpern-Hubert (2014)

- **Navier-Stokes equations** :
  - Finite differences : Blayo-Cherel-Rousseau (2016)
  - Finite elements : Lube-Müller-Otto (2001), Girault-Rivière-Wheeler (2005) ...

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$\Downarrow$

incompressibility constraint, local interface conditions

take convection into account, no restriction on Reynolds

Discrete Duality Finite Volume method

DDFV meshes

Primal mesh

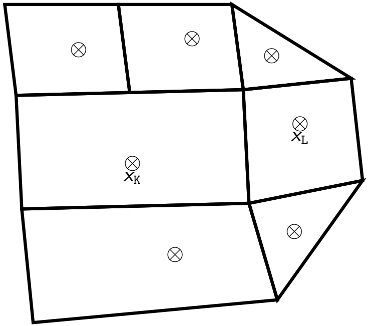
$\rightsquigarrow \mathbf{u}^{\mathfrak{M}} = (\mathbf{u}_{\kappa})_{\kappa \in \mathfrak{M}}$

Dual mesh

$\rightsquigarrow \mathbf{u}^{\mathfrak{M}^*} = (\mathbf{u}_{\kappa^*})_{\kappa^* \in \mathfrak{M}^*}$

Diamond mesh

$\rightsquigarrow \nabla^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}}, \; \mathrm{p}^{\mathfrak{D}}$



Our unknowns are:
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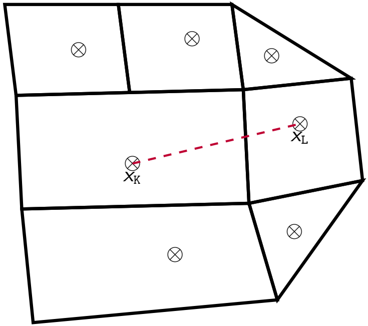

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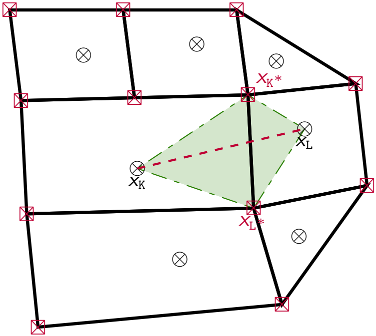

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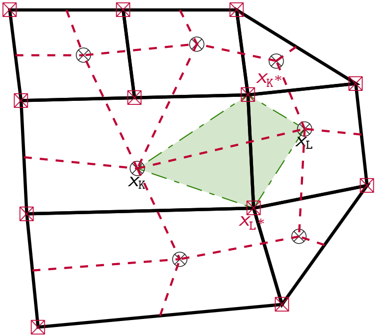

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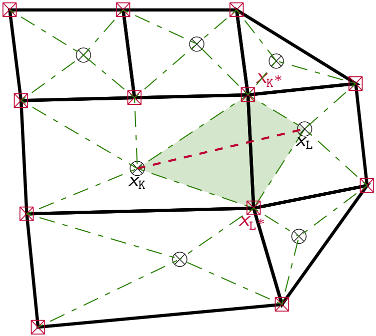

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$\mathbf{u}^{\mathfrak{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*})$ and $\mathrm{p}^{\mathfrak{D}}$

DDFV operators(1/2)

Discrete gradient

The operator $\nabla^{\mathfrak{D}} : (\mathbb{R}^2)^{\mathfrak{T}} \mapsto (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$ where

$$\nabla^{D}\mathbf{u}^{\mathfrak{T}}(x_L - x_K) = \mathbf{u}_L - \mathbf{u}_K,$$
$$\nabla^{D}\mathbf{u}^{\mathfrak{T}}(x_{L^*} - x_{K^*}) = \mathbf{u}_{L^*} - \mathbf{u}_{K^*}.$$

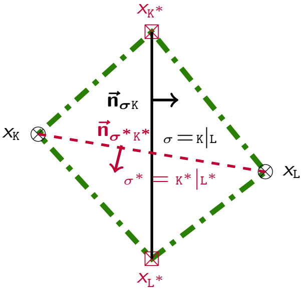
$$\nabla^{D}\mathbf{u}^{\mathfrak{T}} = \frac{1}{2m_D}\left[m_\sigma(\mathbf{u}_L - \mathbf{u}_K) \otimes \vec{\mathbf{n}}_{\sigma K} + m_{\sigma^*}(\mathbf{u}_{L^*} - \mathbf{u}_{K^*}) \otimes \vec{\mathbf{n}}_{\sigma^* K^*}\right].$$

[S. Krell, *Stabilized DDFV schemes for the incompressible Navier-Stokes equations*, 2011]

$\rightsquigarrow \mathrm{div}^{D}\mathbf{u}^{\mathfrak{T}} = \mathrm{Tr}(\nabla^{D}\mathbf{u}^{\mathfrak{T}}).$

$\rightsquigarrow \mathrm{D}^{D}\mathbf{u}^{\mathfrak{T}} = \dfrac{\nabla^{D}\mathbf{u}^{\mathfrak{T}} + {}^t(\nabla^{D}\mathbf{u}^{\mathfrak{T}})}{2}.$

$\rightsquigarrow \sigma^{D}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{D}) = \dfrac{2}{\mathrm{Re}}\mathrm{D}^{D}\mathbf{u}^{\mathfrak{T}} - \mathrm{p}^{D}\mathrm{Id}.$



DDFV operators (2/2)

Discrete divergence

$\mathbf{div}^{\mathfrak{T}} : \xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \mapsto \mathbf{div}^{\mathfrak{T}}\xi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{T}}$ where:

$$\mathbf{div}^{\mathtt{K}}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathtt{K}}} \sum_{\sigma \subset \partial \mathtt{K}} m_\sigma \xi^{\mathtt{D}} \vec{\mathbf{n}}_{\sigma \mathtt{K}}, \qquad \forall \mathtt{K} \in \mathfrak{M}$$

$$\mathbf{div}^{\mathtt{K}^*}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathtt{K}^*}} \sum_{\sigma^* \subset \partial \mathtt{K}^*} m_{\sigma^*} \xi^{\mathtt{D}} \vec{\mathbf{n}}_{\sigma^* \mathtt{K}^*}, \qquad \forall \mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$$

Discrete duality property

- On the continuous level: $\displaystyle\int_\Omega \mathrm{div}\xi \cdot \mathbf{u} = -\int_\Omega \xi : \nabla \mathbf{u} + \int_{\partial\Omega} \xi \vec{\mathbf{n}} \cdot \mathbf{u}$
- On the discrete level:

$$[[\mathbf{div}^{\mathfrak{T}}\xi^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = -(\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}} + (\gamma^{\mathfrak{D}}(\xi^{\mathfrak{D}})\vec{\mathbf{n}}, \gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}))_{\partial\Omega}$$

[S. Krell, *Stabilized DDFV schemes for the incompressible Navier-Stokes equations*, 2011]

DDFV operators (2/2)

Discrete divergence

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$$\mathbf{div}^{\mathtt{K}^*}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathtt{K}^*}} \sum_{\sigma^* \subset \partial \mathtt{K}^*} m_{\sigma^*} \xi^{\mathtt{D}} \vec{\mathbf{n}}_{\sigma^* \mathtt{K}^*}, \qquad \forall \mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$$

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- On the discrete level:

$$[[\mathbf{div}^{\mathfrak{T}}\xi^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = -(\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}} + (\gamma^{\mathfrak{D}}(\xi^{\mathfrak{D}})\vec{\mathbf{n}}, \gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}))_{\partial\Omega}$$

[S. Krell, *Stabilized DDFV schemes for the incompressible Navier-Stokes equations*, 2011]

Brezzi-Pitkäranta stabilization term

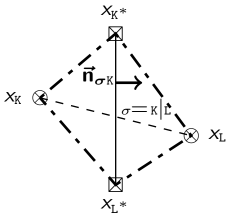
$$\Delta^{\mathtt{D}} \mathrm{p}^{\mathfrak{D}} = \frac{1}{m_{\mathtt{D}}} \sum_{\mathfrak{s}=\mathtt{D}|\mathtt{D}' \in \mathcal{E}_{\mathtt{D}}} \frac{\mathrm{d}_{\mathtt{D}}^2 + \mathrm{d}_{\mathtt{D}'}^2}{\mathrm{d}_{\mathtt{D}}^2} (\mathrm{p}^{\mathtt{D}'} - \mathrm{p}^{\mathtt{D}}) \qquad \forall \mathtt{D} \in \mathfrak{D}$$

DDFV discretization for Navier-Stokes problem with B-schemes

DDFV for Navier-Stokes : $(\mathcal{P})$

At each time step we solve:

$$\begin{cases} m_{\mathrm{K}} \dfrac{\mathbf{u}_{\mathrm{K}} - \bar{\mathbf{u}}_{\mathrm{K}}}{\delta t} + \displaystyle\sum_{\sigma \subset \partial \mathrm{K}} m_\sigma \mathcal{F}_{\sigma \mathrm{K}} = m_{\mathrm{K}} \mathbf{f}_{\mathrm{K}} & \forall \mathrm{K} \in \mathfrak{M} \\ m_{\mathrm{K}^*} \dfrac{\mathbf{u}_{\mathrm{K}^*} - \bar{\mathbf{u}}_{\mathrm{K}^*}}{\delta t} + \displaystyle\sum_{\sigma^* \subset \partial \mathrm{K}^*} m_{\sigma^*} \mathcal{F}_{\sigma^* \mathrm{K}^*} = m_{\mathrm{K}^*} \mathbf{f}_{\mathrm{K}^*} & \forall \mathrm{K}^* \in \mathfrak{M}^* \\ \mathrm{div}^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}) - \beta h_{\mathfrak{T}}^2 \Delta^{\mathrm{D}} \mathrm{p}^{\mathfrak{D}} = 0 & \forall \mathrm{D} \in \mathfrak{D} \end{cases}$$



with $\mathbf{u}^{\partial \mathfrak{M}} = \mathbf{u}^{\partial \mathfrak{M}^*} = 0$ and $\sum_{\mathrm{D} \in \mathfrak{D}} m_{\mathrm{D}} \mathrm{p}^{\mathrm{D}} = 0$.

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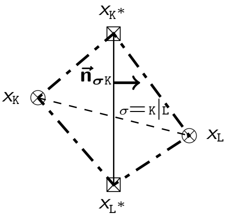

with $\mathbf{u}^{\partial \mathfrak{M}} = \mathbf{u}^{\partial \mathfrak{M}^*} = 0$ and $\sum_{\mathrm{D} \in \mathfrak{D}} m_{\mathrm{D}} \mathrm{p}^{\mathrm{D}} = 0$.

The fluxes are a sum of a "diffusion" and a "convection" term:

$$m_\sigma \mathcal{F}_{\sigma \mathrm{K}} = m_\sigma (\mathcal{F}^d_{\sigma \mathrm{K}} + \mathcal{F}^c_{\sigma \mathrm{K}}) \approx \int_\sigma \sigma(\mathbf{u}, \mathrm{p}) \cdot \vec{\mathbf{n}} + \int_\sigma (\mathbf{u} \cdot \vec{\mathbf{n}}) \mathbf{u}$$

- The **diffusion fluxes**: $m_\sigma \mathcal{F}^d_{\sigma \mathrm{K}} = -m_\sigma \sigma^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma \mathrm{K}}$.
- The **convection fluxes**, with $B : \mathbb{R} \to \mathbb{R}^+$:

$$m_\sigma \mathcal{F}^c_{\sigma \mathrm{K}} = m_\sigma F_{\sigma \mathrm{K}} \left(\frac{\mathbf{u}_{\mathrm{K}} + \mathbf{u}_{\mathrm{L}}}{2} \right) + m_\sigma B\left(F_{\sigma \mathrm{K}}\right) (\mathbf{u}_{\mathrm{K}} - \mathbf{u}_{\mathrm{L}}),$$

[C. Chainais, J. Droniou, *Finite volume schemes for non-coercive elliptic problems with neumann boundary conditions*, 2011]

Well-posedness of $(\mathcal{P})$

Theorem (Well-posedness)

Let $\beta > 0$.
Assume that B be is an even Lipschitz continuous function such that $B(s) \geq 0$, $\forall s \in \mathbb{R}$. Then the scheme $(\mathcal{P})$ is well-posed.

- If $B(s) = 0 \Rightarrow$ *centered* discretization
- If $B(s) = \frac{1}{2}|s| \Rightarrow$ *upwind* discretization

Generalization of the result of [S. Krell, *Stabilized DDFV schemes for the incompressible Navier-Stokes equations*, 2011]

Non-overlapping DDFV Schwarz algorithm for Navier-Stokes problem

The Navier-Stokes problem

- Find $\mathbf{u} : \Omega_T \to \mathbb{R}^2$ and $\mathrm{p} : \Omega_T \to \mathbb{R}$ such that:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \mathrm{div}(\sigma(\mathbf{u}, \mathrm{p})) = \mathbf{f} & \text{in} \quad \Omega_T = \Omega \times [0, T] \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega_T \end{cases}$$

with $T > 0$, $\mathbf{u} = 0$ on $\partial\Omega$, $\mathbf{u}_0 = \mathbf{u}_{init} \in (L^\infty(\Omega))^2$.

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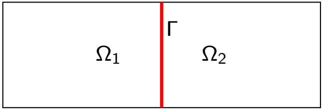
- Domain decomposition:



$\Omega = \Omega_1 \cup \Omega_2$

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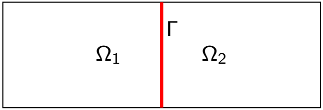


$\Omega = \Omega_1 \cup \Omega_2$

- Transmission conditions on Γ, ℓ iteration index:

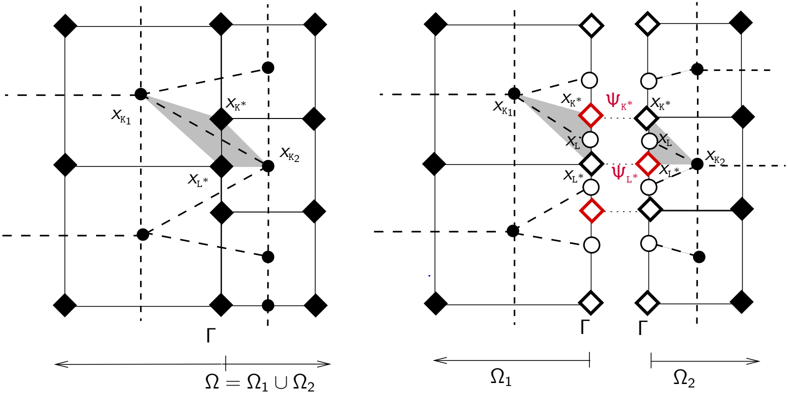
$$\sigma(\mathbf{u}_j^\ell, \mathrm{p}_j^\ell) \cdot \vec{\mathbf{n}}_j - \frac{1}{2}(\mathbf{u}_j^\ell \cdot \vec{\mathbf{n}}_j)(\mathbf{u}_j^\ell) + \lambda \mathbf{u}_j^\ell$$

$$= \sigma(\mathbf{u}_i^{\ell-1}, \mathrm{p}_i^{\ell-1}) \cdot \vec{\mathbf{n}}_i - \frac{1}{2}(\mathbf{u}_i^{\ell-1} \cdot \vec{\mathbf{n}}_i)(\mathbf{u}_i^{\ell-1}) + \lambda \mathbf{u}_i^{\ell-1}$$

$$\mathrm{div}(\mathbf{u}_j^\ell) + \alpha \mathrm{p}_j^\ell = -\mathrm{div}(\mathbf{u}_i^{\ell-1}) + \alpha \mathrm{p}_i^{\ell-1}$$

where $\vec{\mathbf{n}}_j$ is the outer normal to Ω_j, $\lambda, \alpha > 0$.

DDFV on composite meshes



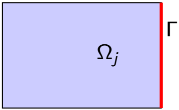
DDFV meshes.

[M.J. Gander L. Halpern, F. Hubert, S. Krell, *Optimized Schwarz Methods for Anisotropic Diffusion with DDFV discretizations*, 2018]

DDFV scheme for the subdomain problem

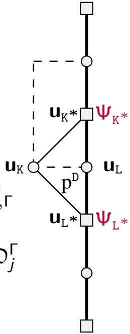
We define the DDFV discretization for the transmission conditions, to which we refer by

$$\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}_{\mathfrak{T}_j}, \mathrm{p}_{\mathfrak{D}_j}, \Psi_{\mathfrak{T}_j}, \mathbf{f}_{\mathfrak{T}}, \mathbf{h}_{\mathfrak{T}_j}, g_{\mathfrak{D}_j}) = 0$$



the following system:

$$\begin{cases} m_{\mathrm{K}} \dfrac{\mathbf{u}_{\mathrm{K}} - \bar{\mathbf{u}}_{\mathrm{K}}}{\delta t} + \displaystyle\sum_{\sigma \subset \partial \mathrm{K}} m_\sigma \mathcal{F}_{\sigma \mathrm{K}} = m_{\mathrm{K}} \mathbf{f}_{\mathrm{K}} & \forall \mathrm{K} \in \mathfrak{M}_j \\ m_{\mathrm{K}^*} \dfrac{\mathbf{u}_{\mathrm{K}^*} - \bar{\mathbf{u}}_{\mathrm{K}^*}}{\delta t} + \displaystyle\sum_{\sigma^* \subset \partial \mathrm{K}^*} m_{\sigma^*} \mathcal{F}_{\sigma^* \mathrm{K}^*} = m_{\mathrm{K}^*} \mathbf{f}_{\mathrm{K}^*} & \forall \mathrm{K}^* \in \mathfrak{M}_j^* \\ m_{\mathrm{K}^*} \dfrac{\mathbf{u}_{\mathrm{K}^*} - \bar{\mathbf{u}}_{\mathrm{K}^*}}{\delta t} + \displaystyle\sum_{\sigma^* \subset \partial \mathrm{K}^*} m_{\sigma^*} \mathcal{F}_{\sigma^* \mathrm{K}^*} + m_{\partial\Omega \cap \partial \mathrm{K}^*} \Psi_{\mathrm{K}^*} = m_{\mathrm{K}^*} \mathbf{f}_{\mathrm{K}^*} & \forall \mathrm{K}^* \in \partial\mathfrak{M}_{j,\Gamma}^* \\ \mathrm{div}^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}) - \beta h_{\mathfrak{T}}^2 \Delta^{\mathrm{D}} \mathrm{p}^{\mathfrak{D}} = 0 & \forall \mathrm{D} \in \mathfrak{D}_j \setminus \mathfrak{D}_j^\Gamma \end{cases}$$



with $\mathbf{u}^{\partial\mathfrak{M}_{j,D}} = 0$ and $\mathbf{u}^{\partial\mathfrak{M}_{j,D}^*} = 0$, plus the *transmission conditions* on Γ.

Transmission conditions

- Transmission conditions on Γ at the continuous level:

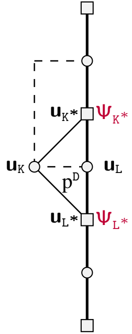
$$\sigma(\mathbf{u},\mathrm{p})\cdot\vec{\mathbf{n}} - \frac{1}{2}(\mathbf{u}\cdot\vec{\mathbf{n}})\mathbf{u} + \lambda\mathbf{u} = \mathbf{h}$$
$$\mathrm{div}(\mathbf{u}) + \alpha\mathrm{p} = \mathbf{g}$$

- Discrete transmission conditions :

$$-\mathcal{F}_{\sigma\mathtt{K}} + \frac{1}{2}F_{\sigma\mathtt{K}}\mathbf{u}_{\mathtt{L}} + \lambda\mathbf{u}_{\mathtt{L}} = \mathbf{h}_{\mathtt{L}} \qquad \forall\sigma\in\partial\mathfrak{M}_{j,\Gamma}$$
$$-\Psi_{\mathtt{K}^*} + \frac{1}{2}(\bar{\mathbf{u}}_{\mathtt{K}^*}\cdot\vec{\mathbf{n}}_{\sigma\mathtt{K}})\,\mathbf{u}_{\mathtt{K}^*} + \lambda\mathbf{u}_{\mathtt{K}^*} = \mathbf{h}_{\mathtt{K}^*} \qquad \forall\mathtt{K}^*\in\partial\mathfrak{M}^*_{j,\Gamma}$$
$$\mathrm{div}^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}}) - \beta h^2_{\mathfrak{T}}\Delta^{\mathtt{D}}\mathrm{p}^{\mathfrak{D}} + \alpha\mathrm{p}^{\mathtt{D}} = g_{\mathtt{D}} \qquad \forall\mathtt{D}\in\mathfrak{D}^{\Gamma}_j$$

with $\lambda,\alpha>0$ and the flux:

$$m_\sigma\mathcal{F}_{\sigma\mathtt{K}} = \underbrace{-m_\sigma\sigma^{\mathtt{D}}(\mathbf{u}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}})\,\vec{\mathbf{n}}_{\sigma\mathtt{K}}}_{m_\sigma\mathcal{F}^d_{\sigma\mathtt{K}}} \underbrace{+m_\sigma F_{\sigma\mathtt{K}}\left(\frac{\mathbf{u}_{\mathtt{K}}+\mathbf{u}_{\mathtt{L}}}{2}\right) + m_\sigma B\left(F_{\sigma\mathtt{K}}\right)(\mathbf{u}_{\mathtt{K}}-\mathbf{u}_{\mathtt{L}})}_{m_\sigma\mathcal{F}^c_{\sigma\mathtt{K}}},$$



Theorem

The scheme $\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}_{\mathfrak{T}_j},p_{\mathfrak{D}_j},\Psi_{\mathfrak{T}_j},\mathbf{f}_{\mathfrak{T}},\mathbf{h}_{\mathfrak{T}_j},g_{\mathfrak{D}_j})=0$ *is well-posed.*

Iterative domain decomposition solver

1. Choose $\mathbf{h}^0_{\mathfrak{T}_j} \in \mathbb{R}^{\partial\mathfrak{M}_{j,\Gamma}\cup\partial\mathfrak{M}^*_{j,\Gamma}}$ and $g^0_{\mathfrak{D}_j} \in \mathbb{R}^{\mathfrak{D}_j}$.

2. Compute $(\mathbf{u}^\ell_{\mathfrak{T}_j}, \mathrm{p}^\ell_{\mathfrak{D}_j}, \Psi^\ell_{\mathfrak{T}_j}) \in \mathbb{R}^{\mathfrak{T}_j} \times \mathbb{R}^{\mathfrak{D}_j} \times \mathbb{R}^{\partial\mathfrak{M}^*_{j,\Gamma}}$ solution to

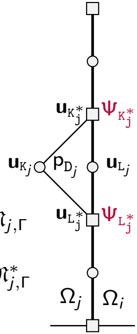
$$\mathcal{L}^{\mathfrak{T}_j,\mu}_{\Omega_j,\Gamma}(\mathbf{u}^\ell_{\mathfrak{T}_j}, \mathrm{p}^\ell_{\mathfrak{D}_j}, \Psi^\ell_{\mathfrak{T}_j}, \mathbf{f}_{\mathfrak{T}_j}, \mathbf{h}^{\ell-1}_{\mathfrak{T}_j}, g^{\ell-1}_{\mathfrak{D}^\Gamma_j}) = 0.$$

3. Compute the new values of $\mathbf{h}^\ell_{\mathfrak{T}_J}$ and of $g^\ell_{\mathfrak{D}^\Gamma_j}$ by:

$$\mathbf{h}^\ell_{\mathrm{L_j}} = \widetilde{\mathcal{F}}^\ell_{\sigma\mathrm{K}_i} - \frac{1}{2} F_{\sigma\mathrm{K}_i}\mathbf{u}^\ell_{\mathrm{L_i}} + \lambda\,\mathbf{u}^\ell_{\mathrm{L_i}}, \qquad \forall \mathrm{L_j} \in \partial\mathfrak{M}_{j,\Gamma}$$

$$\mathbf{h}^\ell_{\mathrm{K}^*_\mathrm{j}} = \Psi^\ell_{\mathrm{K}^*_\mathrm{i}} - \frac{1}{2}(\bar{\mathbf{u}}_{\mathrm{K}^*_\mathrm{i}} \cdot \vec{\mathbf{n}}_{\sigma\mathrm{K}})\,\mathbf{u}^\ell_{\mathrm{K}^*_\mathrm{i}} + \lambda\,\mathbf{u}^\ell_{\mathrm{K}^*_\mathrm{i}}, \qquad \forall \mathrm{K}^*_\mathrm{j} \in \partial\mathfrak{M}^*_{j,\Gamma}$$

$$g^\ell_{\mathrm{D}_j} = \frac{1}{m_{\mathrm{D}_j}}\left(-m_{\mathrm{D}_i}\,\mathrm{div}^{\mathrm{D}_i}(\mathbf{u}^\ell_{\mathfrak{T}_i}) + \beta m_{\mathrm{D}_i} h^2_{\mathfrak{T}}\Delta^{\mathrm{D}_i}\mathrm{p}^\ell_{\mathfrak{D}_i} + \alpha\, m_{\mathrm{D}_i}\,\mathrm{p}^\ell_{\mathrm{D}_i}\right), \qquad \forall \mathrm{D}_j \in \mathfrak{D}^\Gamma_j$$



Convergence study of the First Schwarz algorithm

Let $(\mathcal{S})$ be the First Schwarz algorithm. Then:

$$(\mathcal{S}) \xrightarrow{\ell\to\infty} (\widetilde{\mathcal{P}})$$

where $(\widetilde{\mathcal{P}})$ is problem $(\mathcal{P})$ with modified fluxes on the interface:

$$\bullet\, m_\sigma \widetilde{\mathcal{F}}_{\sigma\mathrm{K}} = \underbrace{-m_\sigma \sigma^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\,\vec{\mathbf{n}}_{\sigma\mathrm{K}}}_{m_\sigma \mathcal{F}^d_{\sigma\mathrm{K}}} + \underbrace{m_\sigma F_{\sigma\mathrm{K}}\left(\frac{\mathbf{u}_{\mathrm{K}}+\mathbf{u}_{\mathrm{L}}}{2}\right) + m_\sigma \widetilde{B}\,(F_{\sigma\mathrm{K}})\,(\mathbf{u}_{\mathrm{K}}-\mathbf{u}_{\mathrm{L}})}_{m_\sigma \widetilde{\mathcal{F}}^c_{\sigma\mathrm{K}}}$$

$$\bullet\, m_{\sigma^*} \widetilde{\mathcal{F}}_{\sigma^*\mathrm{K}^*} = \underbrace{-m_{\sigma^*} \sigma^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\,\vec{\mathbf{n}}_{\sigma^*\mathrm{K}^*}}_{m_{\sigma^*} \mathcal{F}^d_{\sigma^*\mathrm{K}^*}} + \underbrace{m_{\sigma^*} F_{\sigma^*\mathrm{K}^*}\left(\frac{\mathbf{u}_{\mathrm{K}^*}+\mathbf{u}_{\mathrm{L}^*}}{2}\right) + m_{\sigma^*} \widetilde{B}\,(F_{\sigma^*\mathrm{K}^*})\,(\mathbf{u}_{\mathrm{K}^*}-\mathbf{u}_{\mathrm{L}^*})}_{m_{\sigma^*} \widetilde{\mathcal{F}}^c_{\sigma^*\mathrm{K}^*}}$$

Theorem (Convergence of the First Schwarz algorithm)

♦ *Suppose* $m_{\mathrm{D}} = 2m_{\mathrm{D}_i} = 2m_{\mathrm{D}_j}$.
▶ *The solution of the First DDFV Schwarz algorithm* $(\mathcal{S})$ *converges when* $\ell \to \infty$ *to the solution of the DDFV scheme* $(\widetilde{\mathcal{P}})$ *on* Ω .

[T. Goudon, S. Krell, G. Lissoni, *Convergence study of a DDFV scheme for the Navier- Stokes equations arising in the domain decomposition setting*, 2020]

Convergence study of the Second Schwarz algorithm

Let $(\bar{\mathcal{S}})$ be the second Schwarz algorithm. Then:

$$(\bar{\mathcal{S}}) \xrightarrow{\ell\to\infty} (\mathcal{P})$$

where $(\bar{\mathcal{S}})$ is algorithm $(\mathcal{S})$ with modified fluxes on the interface:

$$\bullet\, m_\sigma \bar{\mathcal{F}}_{\sigma\mathrm{K}} = \underbrace{-m_\sigma \sigma^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma\mathrm{K}}}_{m_\sigma \mathcal{F}^d_{\sigma\mathrm{K}}} + \underbrace{m_\sigma F_{\sigma\mathrm{K}}\left(\frac{\mathbf{u}_\mathrm{K}+\mathbf{u}_\mathrm{L}}{2}\right) + m_\sigma \bar{B}\,(F_{\sigma\mathrm{K}})\,(\mathbf{u}_\mathrm{K}-\mathbf{u}_\mathrm{L})}_{m_\sigma \bar{\mathcal{F}}^c_{\sigma\mathrm{K}}}$$

$$\bullet\, m_{\sigma^*} \bar{\mathcal{F}}_{\sigma^*\mathrm{K}^*} = \underbrace{-m_{\sigma^*} \sigma^{\mathrm{D}}(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}})\, \vec{\mathbf{n}}_{\sigma^*\mathrm{K}^*}}_{m_{\sigma^*} \mathcal{F}^d_{\sigma^*\mathrm{K}^*}} + \underbrace{m_{\sigma^*} F_{\sigma^*\mathrm{K}^*}\left(\frac{\mathbf{u}_{\mathrm{K}^*}+\mathbf{u}_{\mathrm{L}^*}}{2}\right)}_{m_{\sigma^*} \bar{\mathcal{F}}^c_{\sigma^*\mathrm{K}^*}}$$

Theorem (Convergence of the Second Schwarz algorithm)

♦ *Suppose* $m_\mathrm{D} = 2m_{\mathrm{D}_i} = 2m_{\mathrm{D}_j}$.

▶ *The solution of the Second DDFV Schwarz algorithm* $(\bar{\mathcal{S}})$ *converges when* $\ell \to \infty$ *the solution of the classical DDFV scheme* $(\mathcal{P})$ *on* Ω*, with the choice* $B = \frac{1}{2}|s|$ *on the primal mesh and* $B = 0$ *on the dual mesh.*

Numerical results

Numerical tests

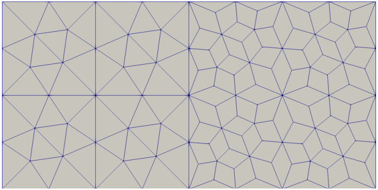
We consider the following exact solutions to the Navier-Stokes problem:

$$\mathbf{u}(t,x,y) = \begin{pmatrix} -2\pi\cos(\pi x)\sin(2\pi y)\exp(-5\eta t\pi^2), \\ \pi\sin(\pi x)\cos(2\pi y)\exp(-5\eta t\pi^2) \end{pmatrix},$$

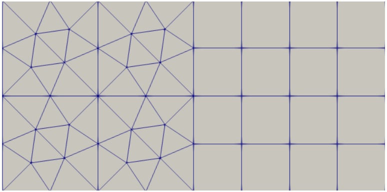
$$\mathrm{p}(t,x,y) = -\frac{\pi^2}{4}(4\cos(2\pi x) + \cos(4\pi y))\exp(-10t\eta\pi^2).$$

The algorithms, in all the following simulations, are initialized with initial random guesses $\mathbf{h}^0_{\mathfrak{T}_j}$ and $g^0_{\mathfrak{D}_j}$ for $j = 1, 2$. The time step is $\delta t = 10^{-4}$ and $B = \frac{1}{2}|s|$.
As a stopping criterion, we impose:

$$\max\left(\|\mathbf{e}^{\ell}_{\mathfrak{T}_j}\|_2, \|\Pi^{\ell}_{\mathfrak{D}_j}\|_2\right) < 10^{-6}$$



Mesh^1_1.



Mesh^2_1.

Convergence of the algorithms

- $(\mathcal{S})$ and $(\bar{\mathcal{S}})$ have the same behavior $\rightarrow$ Focus on $(\mathcal{S})$

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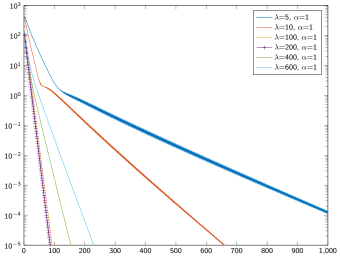
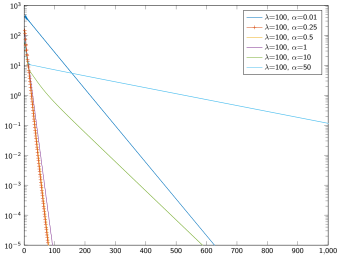
Fix $\alpha = 1$, $\beta = 10^{-1}$, and Mesh_1^1



$\rightarrow$ $\boxed{\lambda_{opt} = 200}$

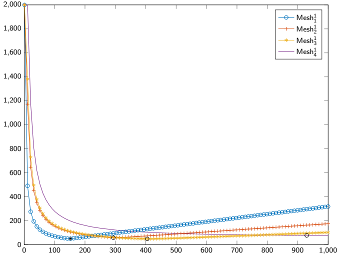
Fix $\lambda = 100$, $\beta = 10^{-1}$, and Mesh_1^1



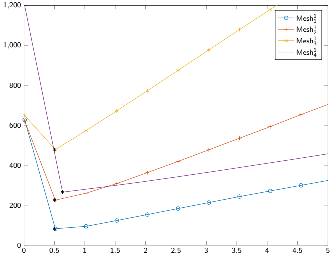
$\rightarrow$ $\boxed{\alpha_{opt} = 0.25}$

Comparison on mesh refinement

Optimization of λ and α on $(\text{Mesh}^1_m)_m,\ m = 1, 2, 3, 4$:



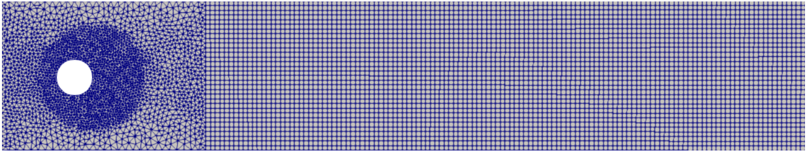
$\alpha = 1, \beta = 10^{-1}$



$\lambda = 100, \beta = 10^{-1}$

	Mesh$^1_1$	**Mesh$^1_2$**	**Mesh$^1_3$**	**Mesh$^1_4$**
λ	152.36	293.36	404.63	929.36
α	0.5	0.5	0.5	0.6

Simulation of a flow in a pipe



Left domain → 4464 cells. Right domain → 4096 cells.

- Initial condition: $\mathbf{u}_{init} = (0, 0)$
- Time-dependent inflow on $x = 0$ and outflow on $x = 2.2$ is:

$$\mathbf{g}_1 = 0.41^{-2} \sin(\pi t/8)(6y(0.41 - y), 0).$$

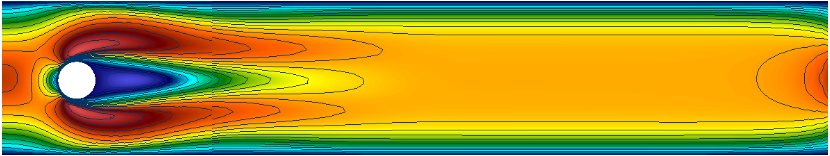
- Stopping criterion :

$$\max\left(\|\mathbf{e}^{\ell}_{\mathfrak{T}_j}\|_2, \|\Pi^{\ell}_{\mathfrak{D}_j}\|_2\right) < 10^{-3}$$

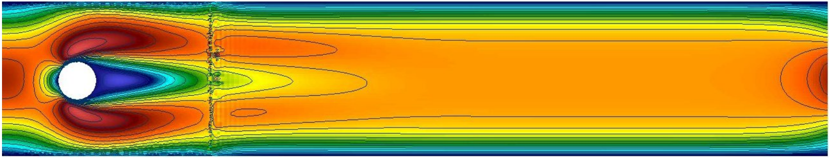
- $\eta = 10^{-3}\mathrm{m}^2\mathrm{s}^{-1}$, $\delta t = 0.0016$
- $0 \leq \mathrm{Re}(t) \leq 100$
- $\lambda = 200, \alpha = 1, \beta = 0.01$

Velocity profile comparison

First component of the velocity solution to the Navier-Stokes problem on Ω:



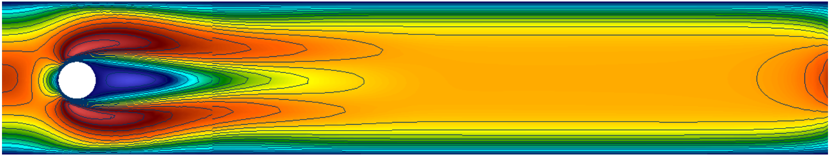
▸ Limit problem $(\widetilde{\mathcal{P}})$, at $T = 2s$.



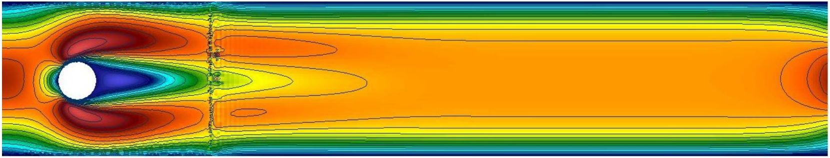
▸ Schwarz algorithm $(\mathcal{S})$ with $B = 0$, at $T = 2s$.
Convergence in 299 iterations (with $\lambda = 200, \alpha = 1, \beta = 0.01$).

Velocity profile comparison

First component of the velocity solution to the Navier-Stokes problem on Ω:



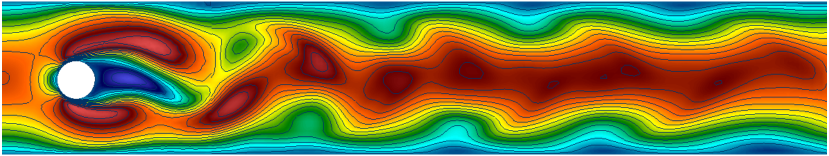
▸ Limit problem $(\widetilde{\mathcal{P}})$, at $T = 2s$.



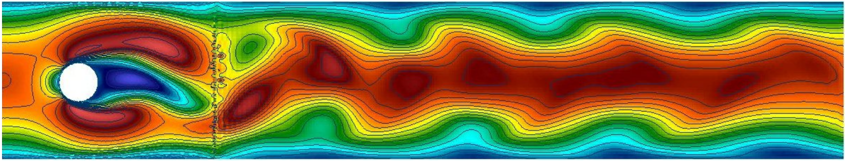
▸ Schwarz algorithm $(\mathcal{S})$ with $B = 0$, at $T = 2s$.
Convergence in 107 iterations (with $\lambda = 50, \alpha = 0.5, \beta = 0.01$).

Velocity profile comparison

First component of the velocity solution to the Navier-Stokes problem on Ω:



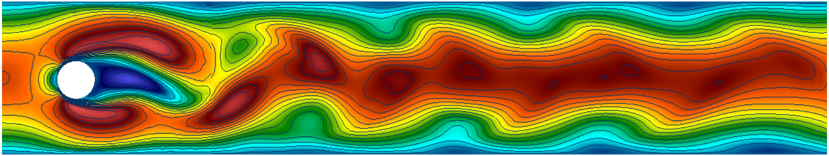
▸ Limit problem $(\widetilde{\mathcal{P}})$, at $T = 6s$.



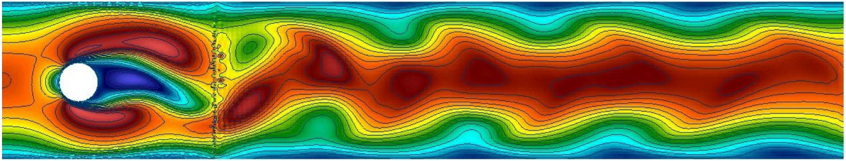
▸ Schwarz algorithm $(\mathcal{S})$ with $B = 0$, at $T = 6s$.
Convergence in 377 iterations (with $\lambda = 200, \alpha = 1, \beta = 0.01$).

Velocity profile comparison

First component of the velocity solution to the Navier-Stokes problem on Ω:

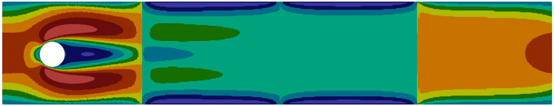


▸ Limit problem $(\widetilde{\mathcal{P}})$, at $T = 6s$.

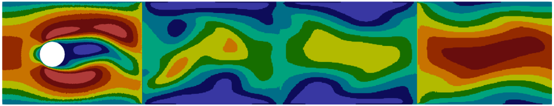


▸ Schwarz algorithm $(\mathcal{S})$ with $B = 0$, at $T = 6s$.
Convergence in 178 iterations (with $\lambda = 50, \alpha = 0.5, \beta = 0.01$).

Velocity profile comparison - multi domains



▸ Schwarz algorithm ($\mathcal{S}$) with $B = 0$, at $T = 6s$.
Convergence in 300 iterations (with $\lambda = 200, \alpha = 1, \beta = 0.01$).



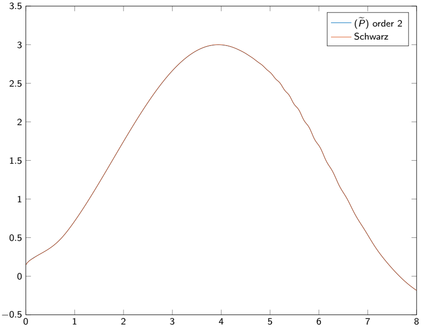
▸ Schwarz algorithm ($\mathcal{S}$) with $B = 0$, at $T = 6s$.
Convergence in 482 iterations (with $\lambda = 200, \alpha = 1, \beta = 0.01$).

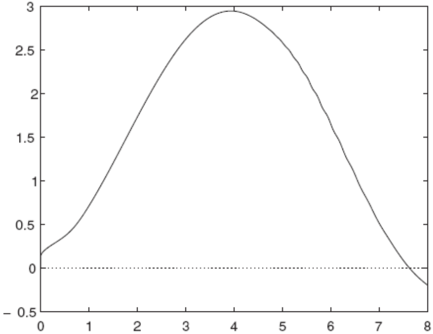
We compare the number of iterations at convergence at $T = 6s$ between the case of 2, 4 and 5 subdomains :

$\sharp$ subdomains	2	4	5
$\sharp$ iterations	377	482	663

Drag and lift coefficients

$$c_{d,max} = 2.9985, \quad c_{d,max}^{Schw} = 2.9999, \quad c_{d,max}^{ref} = 2.9775$$

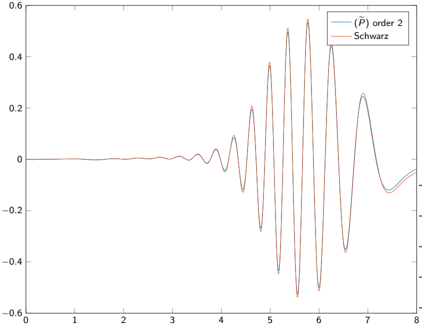


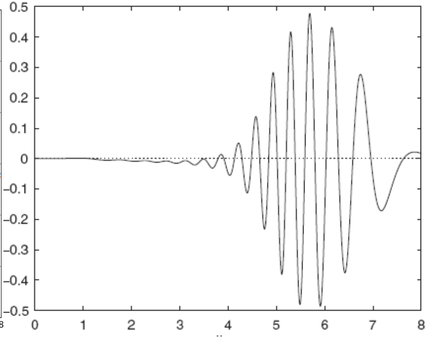


[J. Volker, *Reference values for drag and lift of a two-dimensional time-dependent flow around a cylinder*, 2004]

Drag and lift coefficients

$$c_{l,max} = 0.5183, \quad c_{l,max}^{Schw} = 0.5100, \quad c_{l,max}^{ref} = 0.5442$$





[V. John, *Reference values for drag and lift of a two-dimensional time-dependent flow around a cylinder*, 2004]

Conclusions and Perspectives

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  - general discretization of the Navier-Stokes problem
  - design of transmission conditions
  - design of two iterative solvers
  - convergence of the algorithms
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Grazie per l'attenzione!

Brezzi-Pitkäranta stabilization

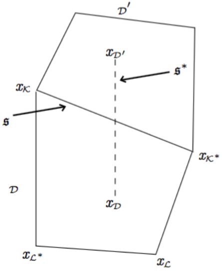
Brezzi-Pitkaranta stabilization term

$$\Delta^{\mathrm{D}} p^{\mathfrak{D}} = \frac{1}{m_{\mathrm{D}}} \sum_{\mathfrak{s}=\mathrm{D}|\mathrm{D}' \in \mathcal{E}_{\mathrm{D}}} \frac{\mathrm{d}_{\mathrm{D}}^2 + \mathrm{d}_{\mathrm{D}'}^2}{\mathrm{d}_{\mathrm{D}}^2} (\mathrm{p}^{\mathrm{D}'} - \mathrm{p}^{\mathrm{D}}) \quad \forall \mathrm{D} \in \mathfrak{D}$$

$$\int_{\mathrm{D}} \Delta \mathrm{p} = \sum_{\mathfrak{s}=\mathrm{D}|\mathrm{D}' \in \mathcal{E}_{\mathrm{D}}} \int_{\mathfrak{s}} \nabla \mathrm{p} \cdot \vec{\mathbf{n}}_{\mathfrak{s}\mathrm{D}}$$

$$\int_{\mathrm{D}} \Delta \mathrm{p} \sim \sum_{\mathfrak{s}=\mathrm{D}|\mathrm{D}' \in \mathcal{E}_{\mathrm{D}}} m_{\mathfrak{s}} \frac{\mathrm{p}(x_{\mathrm{D}'}) - \mathrm{p}(x_{\mathrm{D}})}{\mathrm{d}_{\mathrm{D}',\mathrm{D}}}$$

$$\int_{\mathrm{D}} \Delta \mathrm{p} \sim \sum_{\mathfrak{s}=\mathrm{D}|\mathrm{D}' \in \mathcal{E}_{\mathrm{D}}} (\mathrm{p}(x_{\mathrm{D}'}) - \mathrm{p}(x_{\mathrm{D}}))$$



Non linear convection term

To discretize :

$$\int_\sigma (\mathbf{u}^n \cdot \vec{\mathbf{n}}_{\sigma \mathrm{K}})\mathbf{u}^{n+1}$$

We impose:

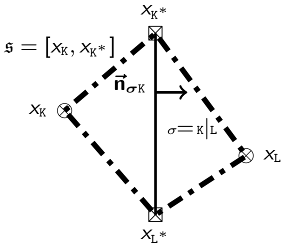
$$F_{\sigma \mathrm{K}}(\mathbf{u}^n) = -\sum_{\mathfrak{s} \in \mathfrak{S}_\mathrm{K} \cap \mathcal{E}_\mathrm{D}} G_{\mathfrak{s},\mathrm{D}}(\mathbf{u}^n) \quad \text{if } \sigma \in \mathcal{E}$$

where

$$G_{\mathfrak{s},\mathrm{D}}(\mathbf{u}^n) = m_\mathfrak{s} \frac{\mathbf{u}^n_\mathrm{K} + \mathbf{u}^n_{\mathrm{K}^*}}{2} \cdot \vec{\mathbf{n}}_{\mathfrak{s}\mathrm{D}} \rightsquigarrow \int_\mathfrak{s} \mathbf{u}^n \cdot \vec{\mathbf{n}}_{\mathfrak{s}\mathrm{D}}.$$

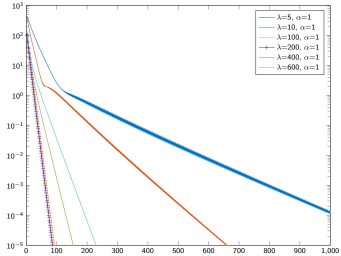
We have conservativity:

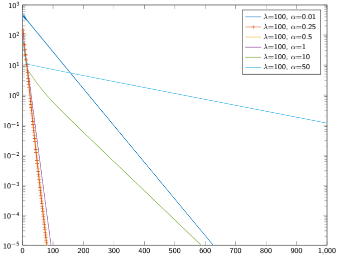
$$F_{\sigma \mathrm{K}} = -F_{\sigma \mathrm{L}}, \quad \forall \sigma = \mathrm{K}|\mathrm{L}$$



Comparison between First and Second algorithm

First Schwarz algorithm, $(\mathcal{S})$:



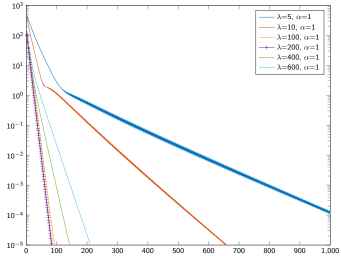


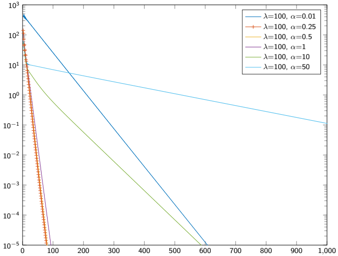
$\lambda_{\mathbf{opt}}$	200
α	1

λ	100
$\alpha_{\mathbf{opt}}$	0.25

Comparison between First and Second algorithm

Second Schwarz algorithm, $(\bar{\mathcal{S}})$:



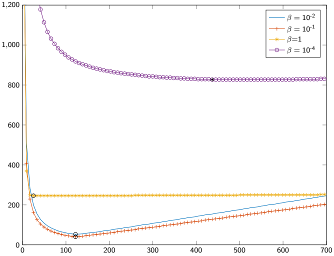


$\lambda_{\textbf{opt}}$	200
α	1

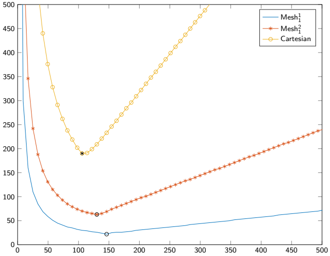
λ	100
$\alpha_{\textbf{opt}}$	0.25

Influence on λ of β and the mesh type

Optimization of λ on Mesh_1^2:



$\alpha = 1$



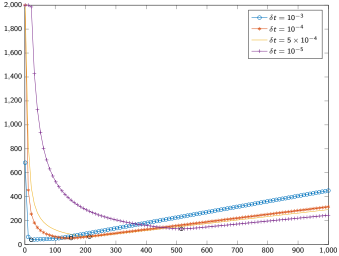
$\alpha = 1,\ \beta = 10^{-1}$

β	10^{-4}	10^{-2}	10^{-1}	1
λ	436.81	122	122	25.2
# iter	818	53	40	246

	Mesh_1^1	**Mesh_1^2**	**Cartesian**
λ	146.2	130.1	105.91

Influence on λ of δt

Optimization of λ for $\alpha = 1, \beta = 10^{-1}$ with different time steps:



	$\delta t = 10^{-3}$	$\delta t = 10^{-4}$	$\delta t = 5 \times 10^{-4}$	$\delta t = 10^{-5}$
λ	21.18	146.2	212.9	515.63