Semi-implicit high-resolution numerical schemes for some conservation laws and level set problems

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- Content
  - some representative models of hyperbolic equations
  - some representative high ”-order” or “-resolution” numerical methods
  - some semi-implicit or compact implicit schemes
  - coupled space-time discretizations

Motivation and background

- Representative models of hyperbolic equations
  - linear nonconservative advection equation
    - level set methods
  - linear conservative advection equation
    - transport equation
  - scalar nonlinear conservative equation
    - Burgers’ equation
  - hyperbolic system
    - shallow water equations (Euler equation)

Motivation and background

- Representative high ”-order” or “-resolution” numerical methods
  - up to **3rd order accurate** with a possible extension of the accuracy
  - **“essentially non-oscillatory”** high-resolution schemes
  - **unconditionally stable**

Motivation and background

$$\partial_t u + \vec{V}(x, y, t) \cdot \nabla u = 0$$

$$V_1(x, y, t) = -4\sin^2(2\pi x)\sin(2\pi y)\cos(2\pi y)\cos(\pi t)$$

$$V_2(x, y, t) = 4\sin^2(2\pi y)\sin(2\pi x)\cos(2\pi x)\cos(\pi t)$$

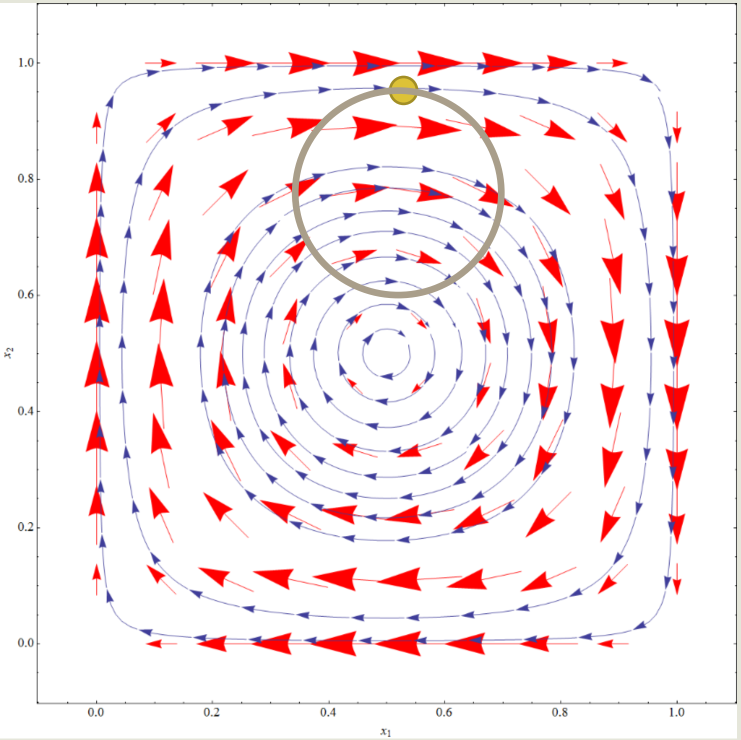
- Trajectories

$$X'(t) = V_1(X(t), Y(t), t)\,,\ \ X(0) = X^0$$

$$Y'(t) = V_2(X(t), Y(t), t)\,,\ \ Y(0) = Y^0$$

- Solution is constant along trajectories

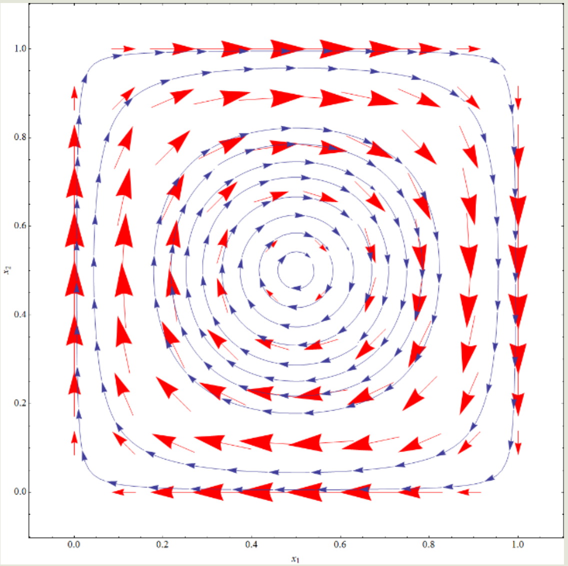
$$\frac{d}{dt}u(X(t), Y(t), t) = 0$$

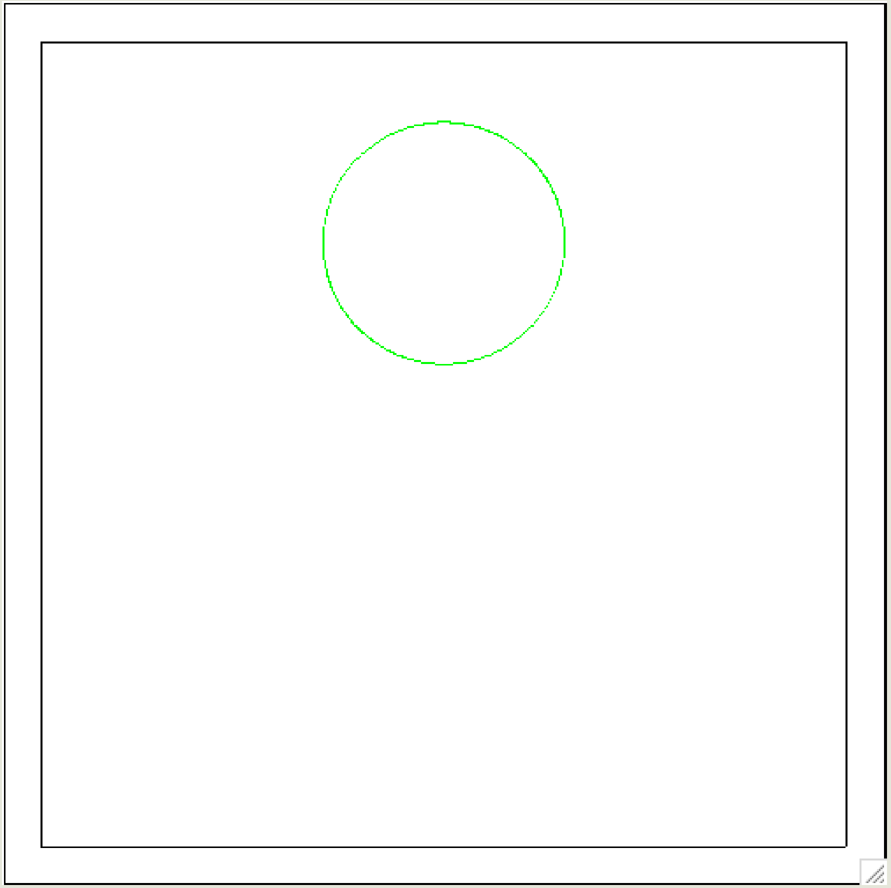


Motivation and background

$$\partial_t u + \vec{V}(x,y,t) \cdot \nabla u = 0$$

- Single vortex benchmark





P. Frolkovič, K. Mikula: *High resolution flux-based level set method*. SIAM J. Sci. Comp., 2007

Motivation and background

$$\partial_t u + \vec{V}(x, y, t) \cdot \nabla u = 0$$

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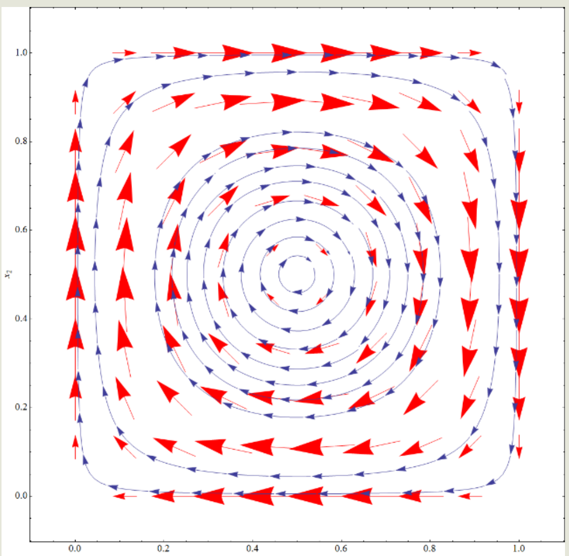
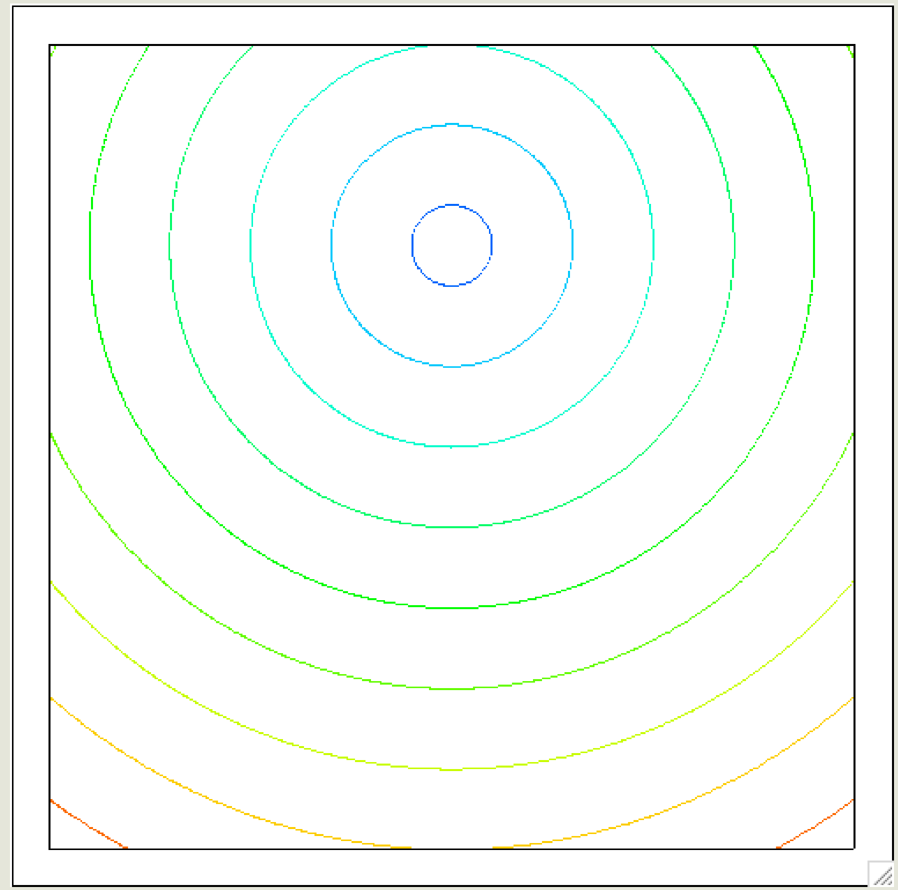


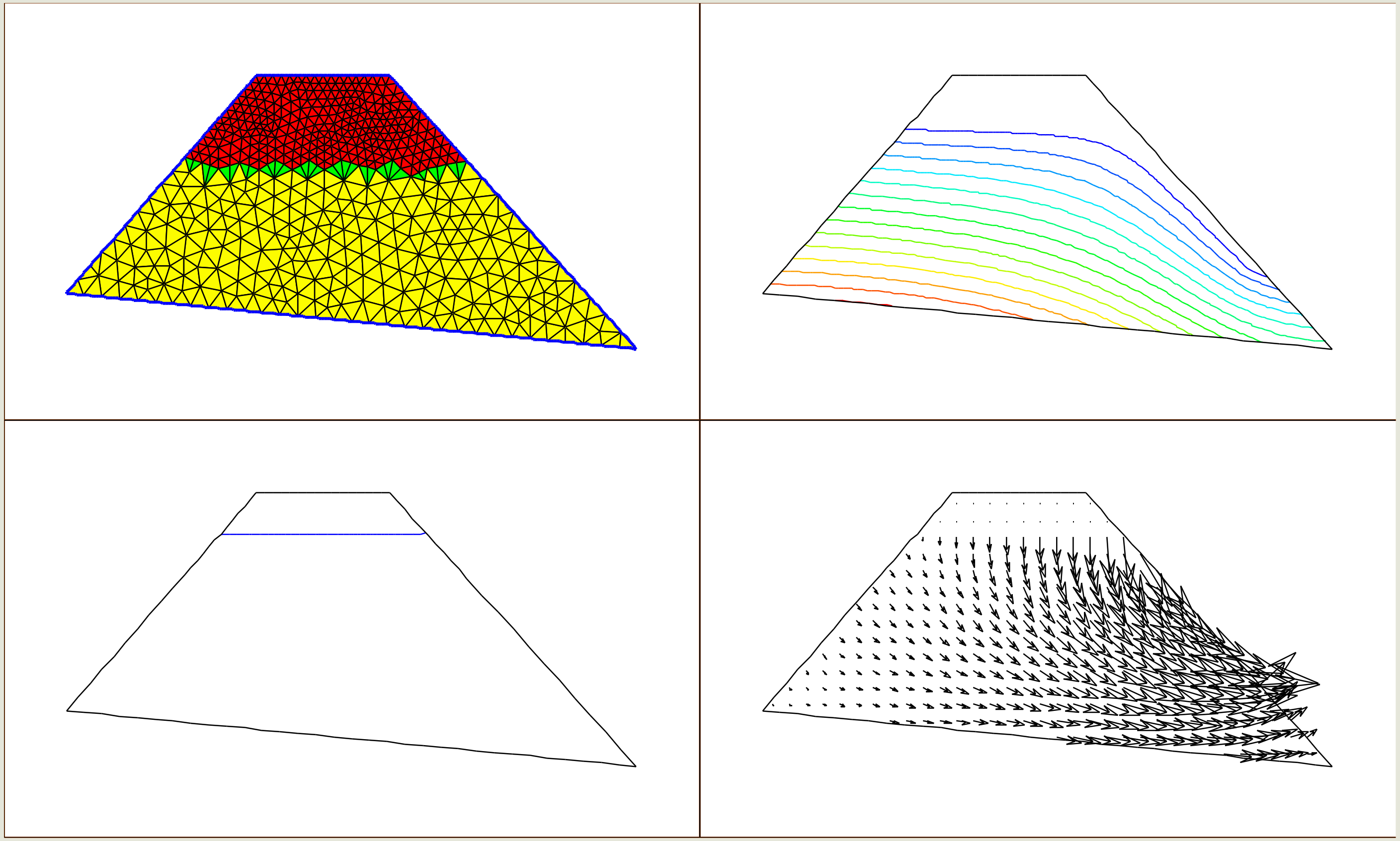


P. Frolkovič, K. Mikula: *High resolution flux-based level set method*. SIAM J. Sci. Comp., 2007

Motivation and background

$$\partial_t u + \vec{V}(x, y, t) \cdot \nabla u = 0$$

- Groundwater flow with free boundary



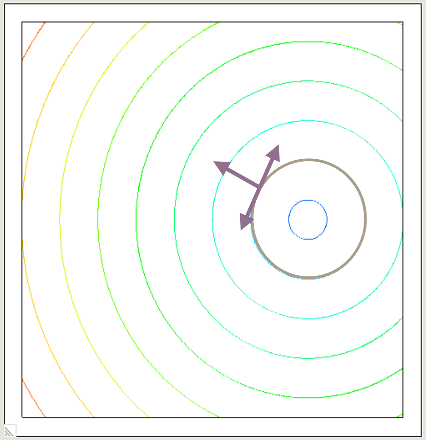
Frolkovič, P. (2012). *Application of level set method for groundwater flow with moving boundary.* Advances in water resources, 47, 56-66.

Motivation and background
□ level set methods to track an interface

- the (advection dominated) level set equation

$$\partial_t u + \vec{V} \cdot \nabla u + s \frac{\nabla u}{|\nabla u|} \cdot \nabla u - \epsilon \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) |\nabla u| = 0$$

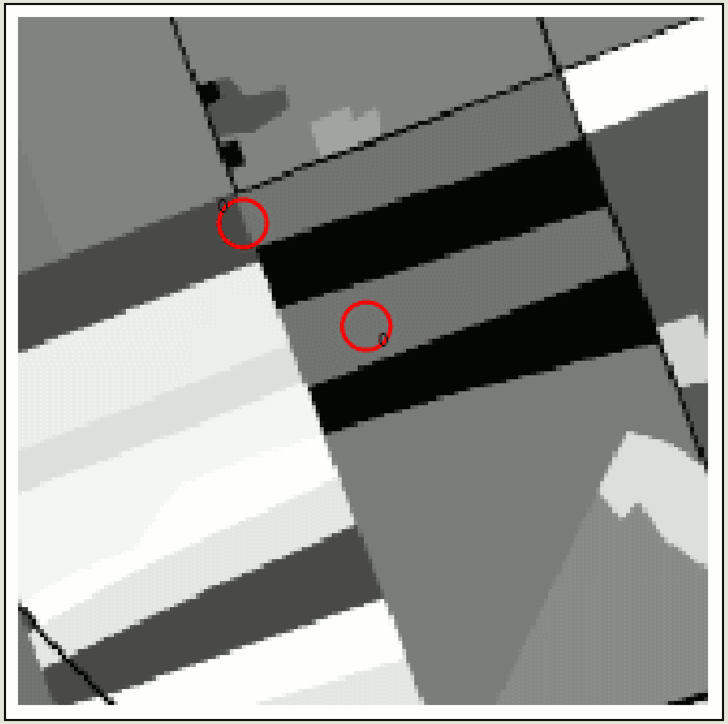
- the interface can move because of
  - external velocity
  - speed in normal direction
  - curvature



Sethian, J. A. (1999). *Level set methods and fast marching methods: evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science*.
Osher, S., & Fedkiw, R. (2002). *Level set methods and dynamic implicit surfaces*.

Motivation and background
□ level set methods to track an interface

- left: the interface



right: the distance function

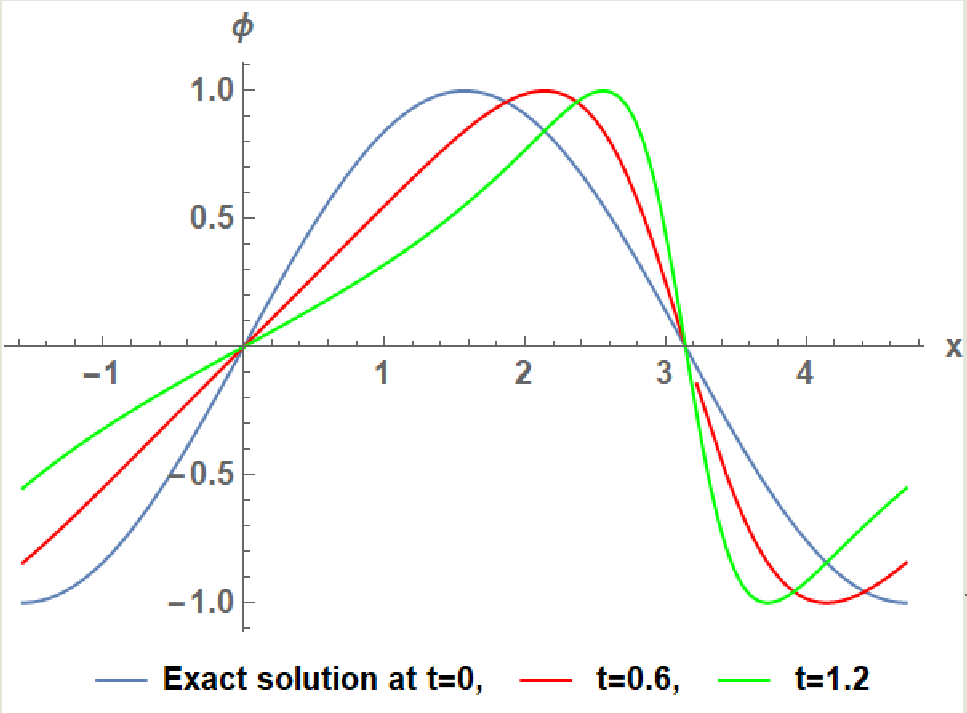
P. Frolkovič, K. Mikula, J. Urbán: **Semi-implicit** finite volume level set method for **advective** motion of interfaces **in normal direction**. Appl. Num. Meth. 2015

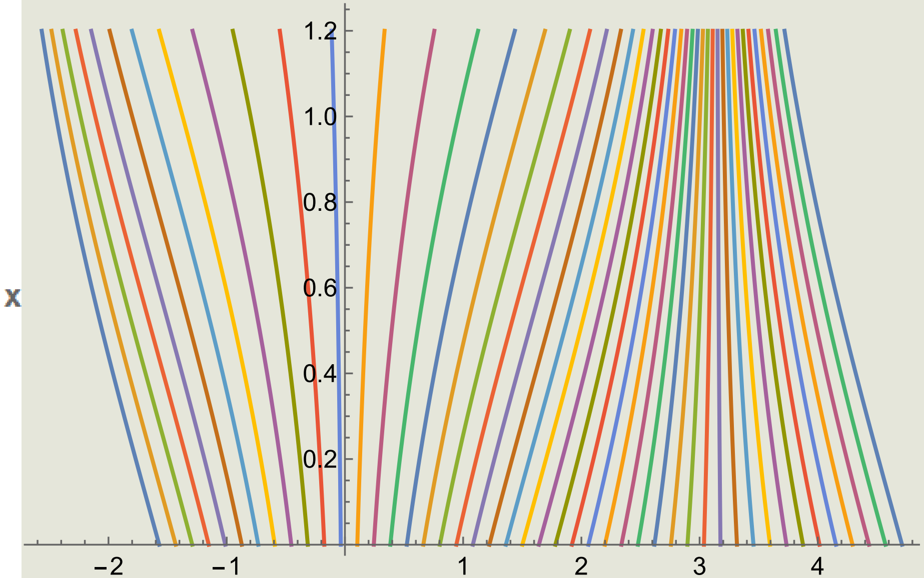
Motivation and background

$$\partial_t u + v(x)\partial_x u = 0$$

- Linear nonconservative advection equation

$$\partial_t u + \sin(x)\partial_x u = 0\,, \quad u(x,0) = \sin(x)$$





characteristic curves in (x,t) plane

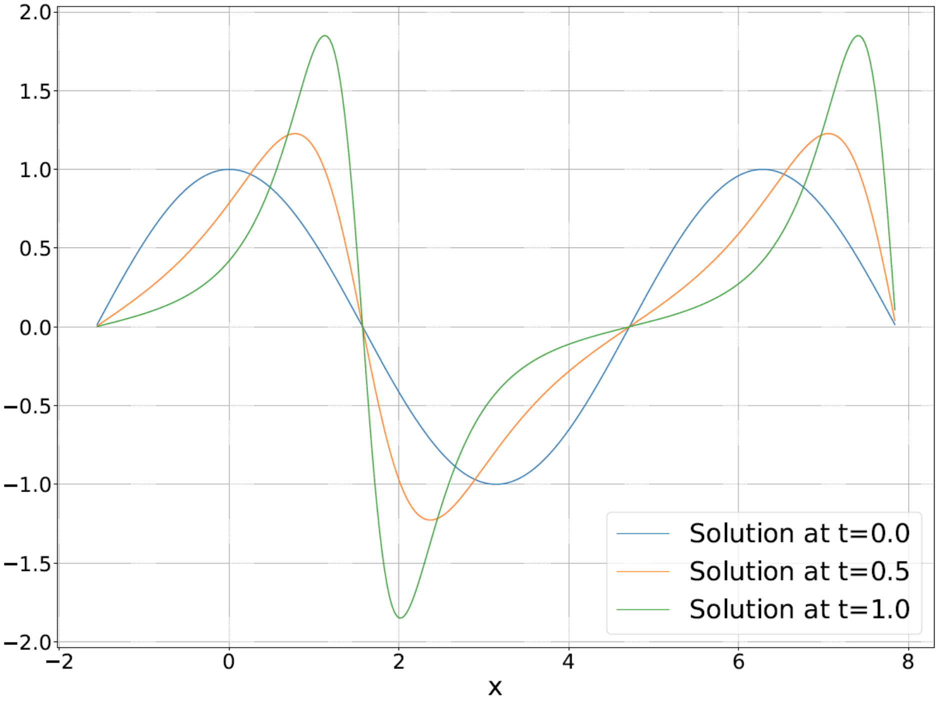
Frolkovič, P., Krišková, S., Rohová, M., Žeravý, M. (2022). *Semi-implicit methods for advection equations with explicit forms of numerical solution.* JJIAM, 1-25.

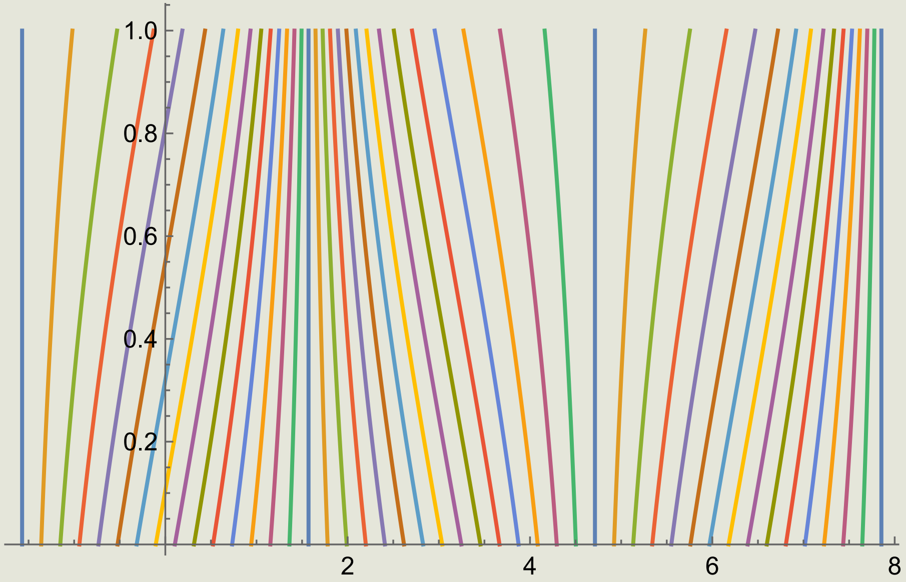
Motivation and background

$$\partial_t u + \partial_x (v(x)\, u) = 0$$

- Linear conservative advection equation

$$\partial_t u + \partial_x(\cos(x) u) = 0\,, \quad u(x,0) = \cos(x)$$





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Motivation and background

$$\partial_t u + \partial_x (v(x)\, u) = 0$$

- Non-conservative advection equation

$$\partial_t u + v(x)\partial_x u = 0$$

  - typically, only problems with continuous solutions are considered
  - the derivatives can be discontinuous
  - relation to conservative equation:

$$\partial_t(\partial_x u) + \partial_x(v(x)\partial_x u) = 0$$

- Conservative advection equation
  - discontinuous solutions must be considered here

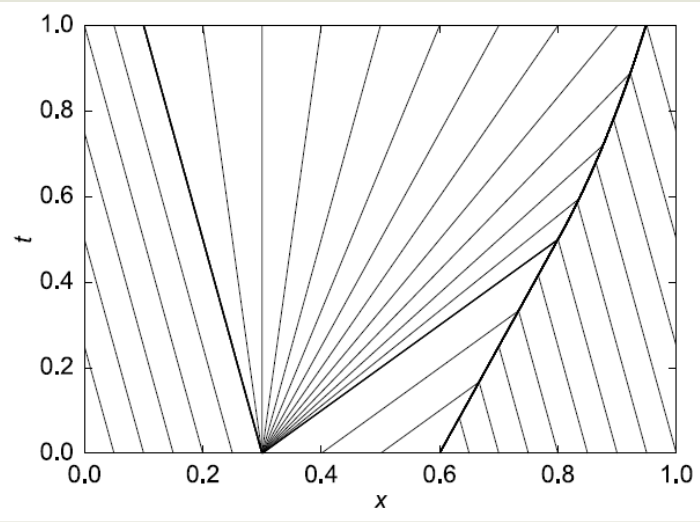
Osher, S., & Sethian, J. A. (1988). *Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations.* JCP, 79(1), 12-49.

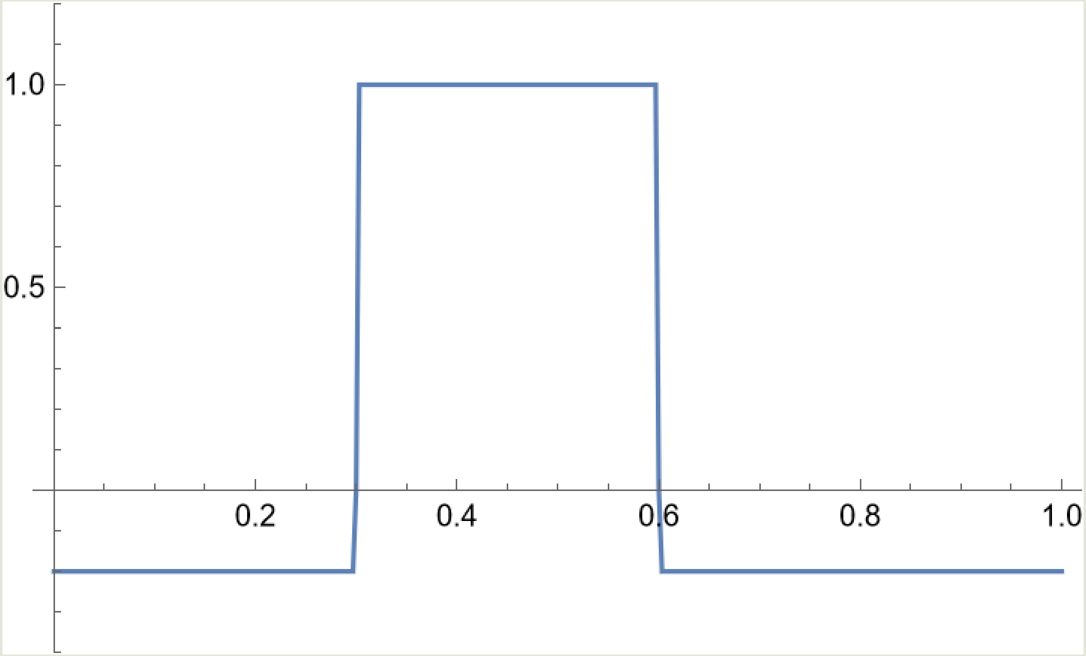
Motivation and background

$$\partial_t u + \partial_x f(u) = 0$$

- Burgers' equation (conservative and nonconservative form)

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0 \qquad \partial_t u + u\partial_x u = 0$$



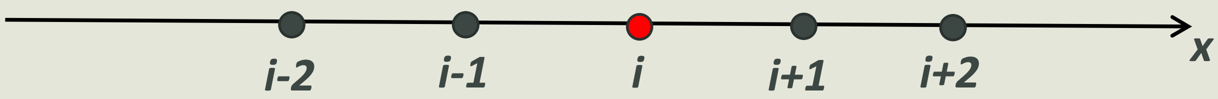


Lozano, Aslam: *Implicit fast sweeping method for hyperbolic systems of conservation laws.* J. Comp. Phys., 2021.

Method of Lines

$$\partial_t u + V(x)\partial_x u = 0$$

- <u>**Spatial discretization with FDM using (at most) 5 points stencils**</u>



- Notation

$$x_i - x_{i-1} \equiv h\,,\;\; V_i := V(x_i)\,,\;\; u_i := u(x_i,\cdot)\,,\;\; U_i \approx u_i\,,\;\ldots$$

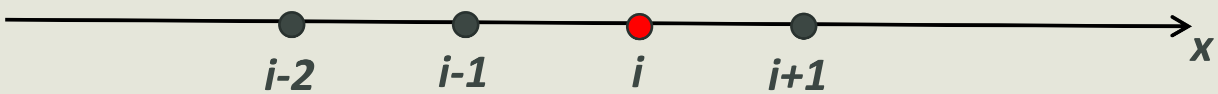
- **Stable** upwind finite differences to approximate $\partial_x u(x_i,\cdot)$

order	$V_i > 0$		$V_i < 0$
1st	$\frac{U_i - U_{i-1}}{h}$		$\frac{U_{i+1} - U_i}{h}$
2nd	$\frac{3U_i - 4U_{i-1} + U_{i-2}}{2h}$	$\frac{U_{i+1} - U_{i-1}}{2h}$	$\frac{-3U_i + 4U_{i+1} - U_{i+2}}{2h}$

Method of Lines

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- **Spatial upwind discretization with FDM using 4 points stencils**



- **Stable** finite differences to approximate $\partial_x u(x_i, \cdot)$

order	$V_i > 0$	
1st	$\frac{U_i - U_{i-1}}{h}$	
2nd	$\frac{3U_i - 4U_{i-1} + U_{i-2}}{2h}$	$\frac{U_{i+1} - U_{i-1}}{2h}$

- **Stable** parametric 2nd order accurate finite difference

$$\omega \frac{3U_i - 4U_{i-1} + U_{i-2}}{2h} + (1-\omega)\frac{U_{i+1} - U_{i-1}}{2h}\,, \quad \omega \geq 0$$

  - it is 3rd order accurate for $\omega = 1/3$

Method of Lines

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- **First order accurate and stable upwind space discretization**

$$\partial_x u_i \approx \frac{U_i - U_{i-1}}{h}$$

- **Method Of Lines (MOL)**

$$U_i' = -V_i \frac{U_i - U_{i-1}}{h}\,, \quad U_i = U_i(t) \approx u(x_i, t)$$

  - **first order accurate explicit (forward) Euler method**

$$U_i^n = U_i^{n-1} - \tau V_i \frac{U_i^{n-1} - U_{i-1}^{n-1}}{h}\,, \quad U_i^n \approx u(x_i, t^n)\,,\ \tau := t^n - t^{n-1}\,,\ n = 1, 2, \dots$$

  - **first order accurate implicit (backward) Euler method**

$$U_i^n = U_i^{n-1} - \tau V_i \frac{U_i^n - U_{i-1}^n}{h}$$

Method of Lines

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- **First order accurate and stable space discretization**

$$\partial_x u_i \approx \frac{U_i - U_{i-1}}{h}$$

- **Method Of Lines (MOL)**

$$U_i' = -V_i \frac{U_i - U_{i-1}}{h}\,, \quad U_i = U_i(t) \approx u(x_i, t)$$

  - **first order accurate explicit (forward) Euler method (Courant numbers)**

$$U_i^n = \left(1 - \frac{\tau V_i}{h}\right) U_i^{n-1} + \frac{\tau V_i}{h} U_{i-1}^{n-1}\,, \quad C_i := \frac{\tau V_i}{h}$$

  - **first order accurate implicit (backward) Euler method**

$$U_i^n = U_i^{n-1} - \tau V_i \frac{U_i^n - U_{i-1}^n}{h}$$

Method of Lines

$$\partial_t u + V(x)\partial_x u = 0\,, \qquad V(x) > 0$$

- **First order accurate and stable space discretization**

$$\partial_x u_i \approx \frac{U_i - U_{i-1}}{h}$$

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  - **first order accurate explicit (forward) Euler method (Courant numbers)**

$$U_i^n = (1 - C_i)\, U_i^{n-1} + C_i U_{i-1}^{n-1}\,, \qquad C_i := \frac{\tau V_i}{h} \qquad \Rightarrow \qquad 0 \le C_i \le 1$$

  - **first order accurate implicit (backward) Euler method**

$$U_i^n = U_i^{n-1} - \tau V_i \frac{U_i^n - U_{i-1}^n}{h}$$

Method of Lines

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

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$$\partial_x u_i \approx \frac{U_i - U_{i-1}}{h}$$

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$$U_i^n = (1 - C_i)\,U_i^{n-1} + C_i U_{i-1}^{n-1}\,, \quad C_i := \frac{\tau V_i}{h} \quad \Rightarrow \quad 0 \le C_i \le 1$$

  - **first order accurate implicit (backward) Euler method (explicitly solvable!)**

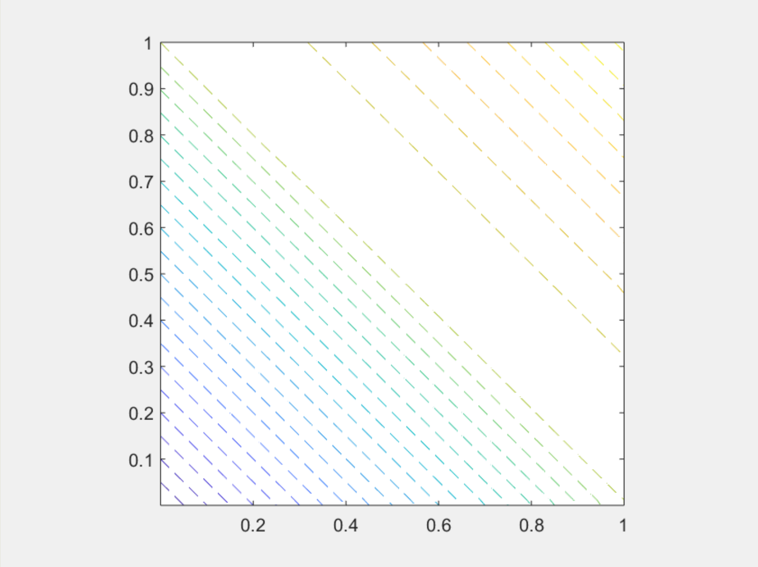
$$(1 + C_i)\,U_i^n = U_i^{n-1} + C_i U_{i-1}^n\,,\ i = 1, 2, \dots\,, \quad U_0^n \text{ is given}\,,\ \ 0 \le C_i$$

Motivations for implicit schemes
□ Illustrative example

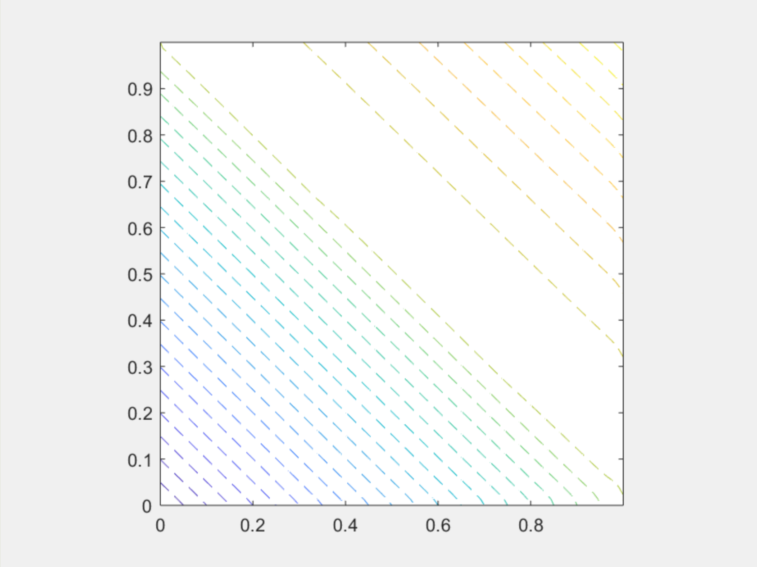
- Fully explicit time discretization of hyperbolic problems
  - powerful if the stability restriction on time step is appropriate

$$\boxed{\tau \sim \frac{\min h}{\max |s|}}$$

- Difficulties that can arise in some numerical simulations
  - large variation of the grid size or the velocity …



fully **explicit** scheme

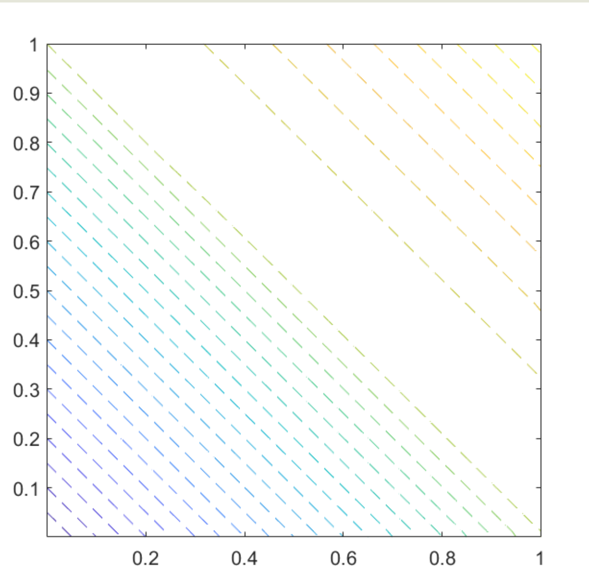


semi-implicit scheme

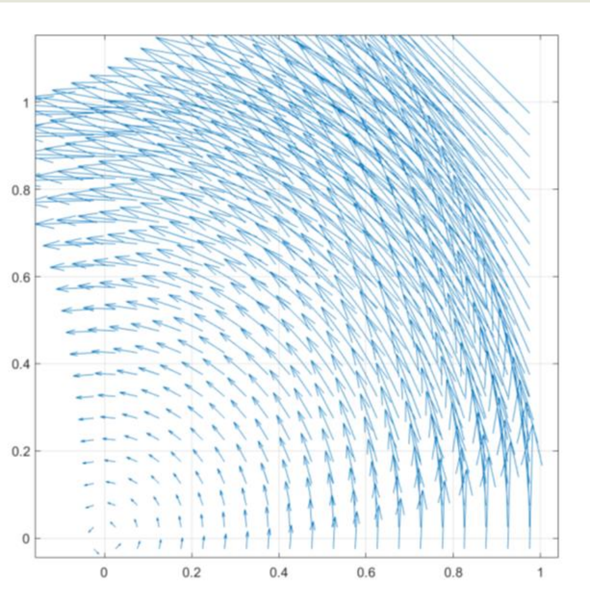
Motivations for implicit schemes

$$\partial_t u + \vec{V}(x,y) \cdot \nabla u = 0$$

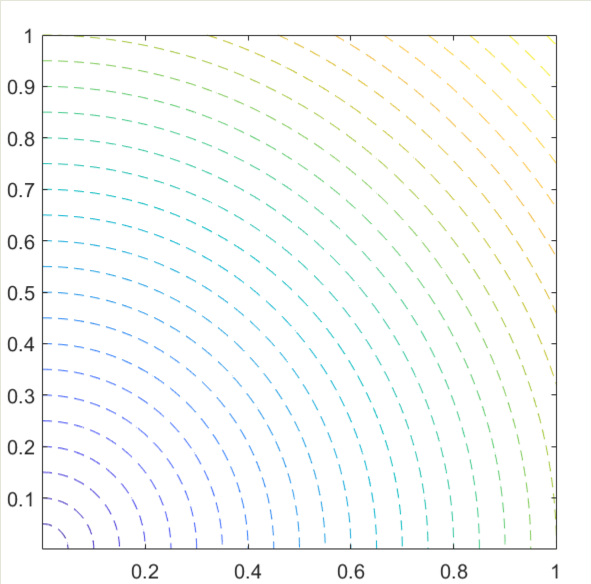
- a linear advection equation with a rotational velocity
- a stationary solution has circular isocontours
- the magnitude of the velocity varies exponentially along the radius
- the time dependent solution is fixed at inflow boundaries



initial function



velocity field



stationary solution

Motivations for implicit schemes

- Fully explicit time discretization of hyperbolic problems
  - powerful if the stability restriction on time step is appropriate

$$\boxed{\tau \sim \frac{\min h}{\max |s|}}$$

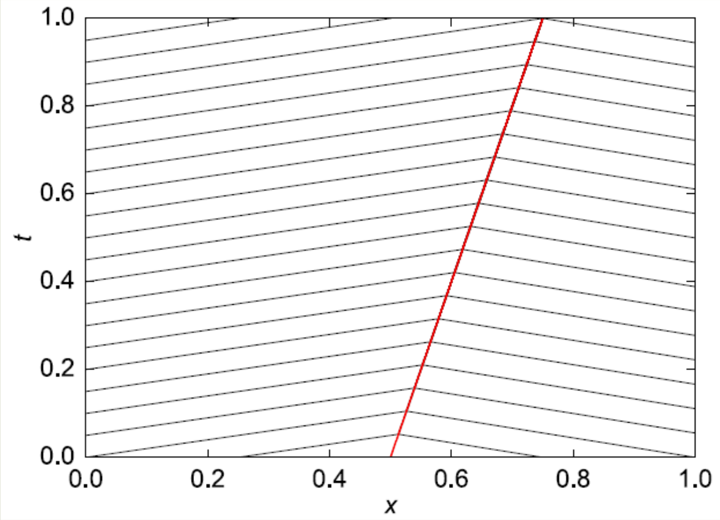
- Difficulties that can arise in some numerical simulations
  - large variation of the grid size or the velocity
  - unfitted grids with arbitrary small "cut cells"
  - large speed components for "uninteresting" features
  - …
- Our aim
  - high-order and high-resolution unconditionally stable (semi- or compact-) implicit numerical methods for some hyperbolic problems with advantageous properties for algebraic solvers

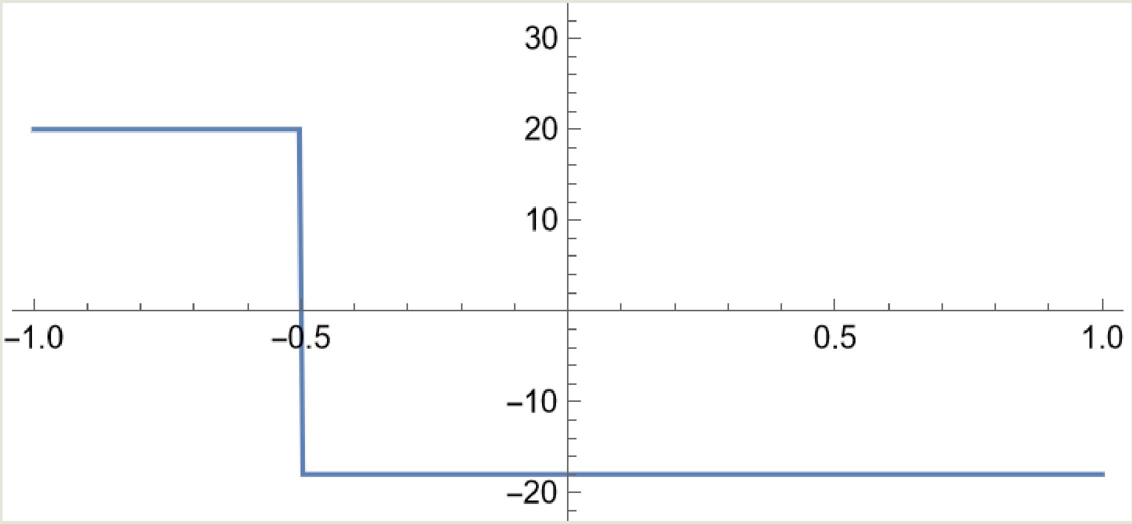
Motivations for implicit schemes

$$\partial_t u + \partial_x f(u) = 0$$

- Burgers' equation and slowly moving shock

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0 \qquad \partial_t u + u\partial_x u = 0$$





- in this example implicit scheme can take 20 times larger time step

Lozano, Aslam: *Implicit fast sweeping method for hyperbolic systems of conservation laws.* J. Comp. Phys., 2021.

Motivations for implicit schemes

$$\partial_t u + \partial_x f(u) = 0$$

- Applications having slow and fast processes:
  - "all Mach number flows" [Degond, Tang 2011]
    - "one is interested in the material not the noise"
  - problems close to equilibrium
    - kinetic equations with linear advection [Pieraccini, Puppo 2012]
  - compressible reactive flows (detonation initiation problem [Lozano, Aslam 2021])
  - relaxation schemes for PDEs [Jin, Xin 1995], …

S. Jin and Z. Xin. *The relaxation schemes for systems of conservation laws in arbitrary space dimensions.* Communications on Pure and Applied Mathematics, 48(3):235–276, **1995.**

P. Degond and M. Tang. *All speed scheme for the low Mach number limit of the isentropic Euler equations.* Comm. Computat. Phys., 10(1):1–31, **2011**

S. Pieraccini and G. Puppo. *Microscopically implicit–macroscopically explicit schemes for the BGK equation.* J. Comput. Phys., 231:299–327, **2012**

Lozano, Aslam: *Implicit fast sweeping method for hyperbolic systems of conservation laws.* J. Comp. Phys., **2021**.

- a "black-box implicit solver" ?!

Motivations for implicit schemes

$$\partial_t u + \partial_x f(u) = 0$$

- **<u>Third order accurate space discretization</u>**

$$\partial_x f_i \approx \frac{2f_{i+1} + 3f_i - 6f_{i-1} + f_{i-2}}{6h} \qquad \omega = 1/3$$

- **<u>Method Of Lines (MOL)</u>**

$$U_i' = -\frac{2f(U_{i+1}) + 3f(U_i) - 6f(U_{i-1}) + f(U_{i-2})}{6h}\,, \quad U_i = U_i(t) \approx u(x_i, t)$$

  - third order accurate time discretization?!
    - spatial and temporal discretization are fully decoupled

- Properties
  - nonlinear ODEs
  - “ + ” : available library of high-quality ODE solvers, simplicity
  - “ - ” : ODE solver must be very robust with almost no knowledge on RHS
    - Strong Stability Preserving (SSP) methods are currently used

Motivations for implicit schemes

$$\partial_t u + \partial_x f(u) = 0$$

- S. Gottlieb, *Strong Stability Preserving Time Discretizations,* 2015

SSP methods give you a guarantee of provable nonlinear stability for any convex functional, any starting values, any nonlinear, non autonomous equations provided only that forward Euler strong stability is satisfied.

This is a very strong property, so it should not be surprising that it is associated with time-step bounds and order barriers:

- Explicit SSP Runge–Kutta methods have order $p \leq 4$ and $\mathcal{C}_{\text{eff}} \leq 1$.
- Implicit SSP Runge–Kutta methods have order $p \leq 6$ and step size restriction seems to be $\mathcal{C}_{\text{eff}} \leq 2$.

Motivations for implicit schemes

$$\partial_t u + \partial_x f(u) = 0$$

- S. Gottlieb, D. Ketcheson, and C. W. Shu. *Strong Stability Preserving Runge-Kutta and Multistep Time Discretizations*. World Scientific, Singapore, 2011

with effective CFL$= 0.33$, which is known as the Shu-Osher method [111]. And the 4ª order method with 10 stages and effective CFL$= 0.6$,

$$u' = f(u)$$

$$\begin{aligned}
u^{(1)} &= u^n + \frac{1}{6}\Delta t f(u^n) \\
u^{(i+1)} &= u^{(i)} + \frac{1}{6}\Delta t f(u^{(i)}), \quad i = 1, 2, 3 \\
u^{(5)} &= \frac{3}{5}u^n + \frac{2}{5}(u^{(4)} + \frac{1}{6}\Delta t f(u^{(4)})) \\
u^{(i+1)} &= u^{(i)} + \frac{1}{6}\Delta t f(u^{(i)}), \quad i = 5, 6, 7, 8 \\
u^{n+1} &= \frac{1}{25}u^n + \frac{9}{25}(u^{(4)} + \frac{1}{6}\Delta t f(u^{(4)})) + \frac{3}{5}(u^{(9)} + \frac{1}{6}\Delta t f(u^{(9)})).
\end{aligned}$$

This method belongs to the family of Strong Stability Preserving Runge-Kutta schemes (SSPRK) see [48].

Motivations for implicit schemes

$$\partial_t u + \partial_x f(u) = 0$$

- Lax-Wendroff or Cauchy-Kovalewskaya procedure or 2-derivative RK or ...

$$\partial_t u = -\partial_x f(u)\,, \quad \partial_{tt} u = -\partial_{tx} f(u) = \dots$$

- Taylor series

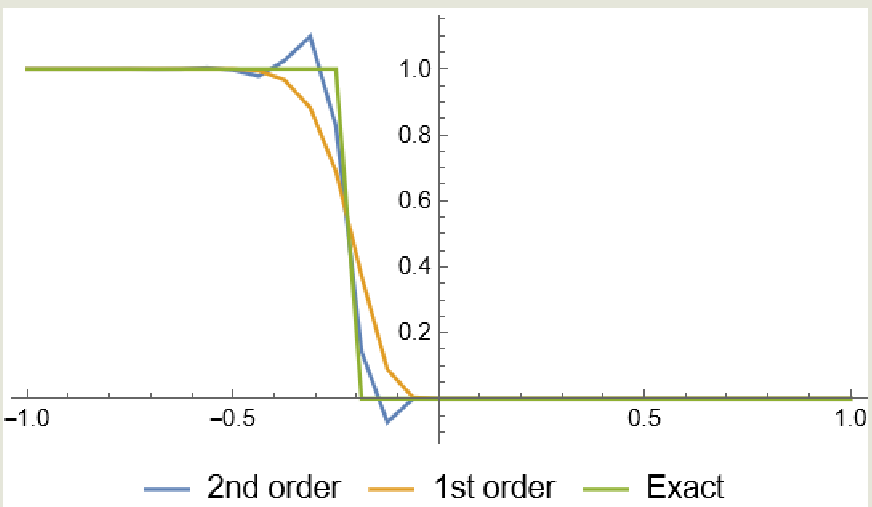
$$\begin{aligned} u(t^{n-1}) &= u(t^n) - \tau\partial_t u(t^n) + 0.5\tau^2 \partial_{tt} u(t^n) + \mathcal{O}(\tau^3) \\ u(t^{n-1}) &= u(t^n) + \tau\partial_x f(u(t^n)) + 0.5\tau^2\,(?!) + \mathcal{O}(\tau^3) \end{aligned}$$

- One-step time discretizations “aware of space discretization” for hyp. eq.
  - ADER (Titarev, Toro 2002)
  - Lax-Wendroff type (Qiu, Shu 2003, Zorio, Baeza, Mulet 2017, …)
    - CAT (Compact Approximate Taylor) (Carrillo, Parés 2019, ...)
  - Local space-time FV & DG (Dumbser, Enaux, Toro 2008)
  - Two-derivative (RK) methods (Chan, Tsai 2010, Tsai 2014, Li 2019, …]

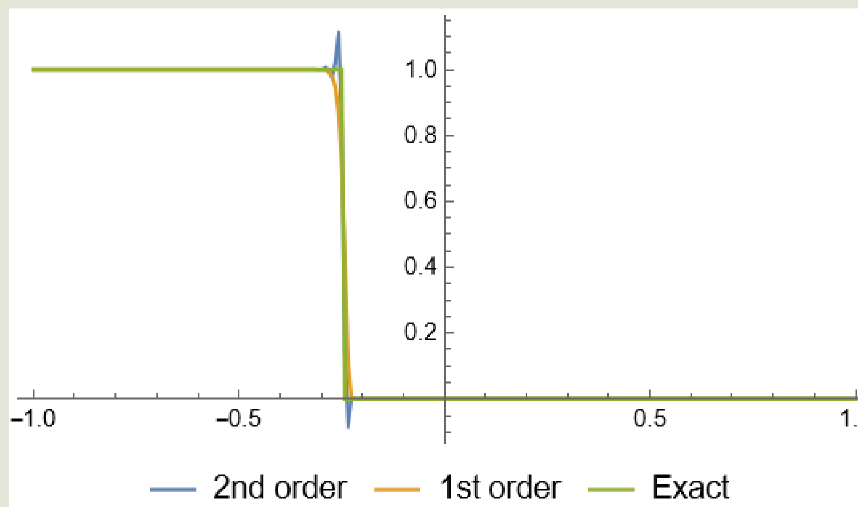
Motivations for implicit schemes

$$\partial_t u + \partial_x f(u) = 0$$

- Second (and high) order schemes
  - according to the Godunov theorem, it is not possible to devise any linear numerical scheme that is better than first order accurate and monotone. The only way to circumvent the theorem is the design of nonlinear scheme.
  - finite differences with fixed stencils are not suitable



coarse mesh computations



three times uniformly refined grid

Motivations for implicit schemes

$$\partial_t u + \partial_x f(u) = 0$$

- Second (and high) order schemes
  - according to the Godunov theorem, it is not possible to devise any linear numerical scheme that is better than first order accurate and monotone. The only way to circumvent the theorem is the design of nonlinear scheme.
  - finite differences with fixed stencils are not suitable

- Remember the parametric 2nd order accurate finite difference:

$$\partial_x^{\omega,2} f_i := \omega \frac{3f_i - 4f_{i-1} + f_{i-2}}{2h} + (1-\omega)\frac{f_{i+1} - f_{i-1}}{2h}$$

  - we device schemes with the parameter depending on the solution
  - criteria for "good numerical schemes":
    - accuracy (consistency)
    - stability
    - solvability

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- Next:
  - derivation of 1st , 2nd and 3rd order (semi-)implicit schemes
  - a short review of previous published efforts
- Why to call it "semi-implicit" schemes for level set methods?
  - "opposite to fully implicit the semi-implicit numerical schemes require the solution of linear systems of equations for the computation of the numerical solution with no Newton iteration." [Boscarino et.al., 2016]
  - "semi-implicit additive schemes in which the two (RK) tableau correspond respectively to an explicit and an implicit scheme" [Boscarino et.al., 2016]

$$\partial_t u + \vec{V} \cdot \nabla u + s\frac{\nabla u}{|\nabla u|} \cdot \nabla u - \epsilon \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) |\nabla u| = 0$$

  - no such arguments for conservation laws, see later

Boscarino, S., Filbet, F., & Russo, G. (2016). *High order semi-implicit schemes for time dependent partial differential equations*. Journal of Scientific Computing, 68(3), 975-1001.

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \qquad V(x) > 0$$

- **Fully implicit 1st order accurate schemes**

$$u_i^n := u(x_i, t^n)\,,\ \partial_t u_i^n := \partial_t u(x_i, t^n)\,,\ V_i := V(x_i)\,,\ U_i^n \approx u_i^n\,, \ldots$$

  - “Lax-Wendroff procedure”

$$\partial_t u_i^n = -V_i \partial_x u_i^n$$

  - Taylor series

$$u_i^{n-1} \quad = \quad u_i^n \quad - \quad \tau \partial_t u_i^n \quad + \quad \mathcal{O}(\tau^2)$$

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- **Fully implicit 1st order accurate schemes**

$$u_i^n := u(x_i, t^n)\,,\ \partial_t u_i^n := \partial_t u(x_i, t^n)\,,\ V_i := V(x_i)\,,\ U_i^n \approx u_i^n\,,\ \ldots$$

  - "Lax-Wendroff procedure"

$$\color{red}{\partial_t u_i^n} = -V_i \color{blue}{\partial_x u_i^n}$$

  - Taylor series

$$\begin{aligned} u_i^{n-1} &= u_i^n - \tau \partial_t u_i^n + \mathcal{O}(\tau^2) \\ u_i^{n-1} &= u_i^n + \tau V_i \color{blue}{\partial_x u_i^n} + \mathcal{O}(\tau^2) \end{aligned}$$

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \qquad V(x) > 0$$

- **Fully implicit 1st order accurate schemes**

$$u_i^n := u(x_i, t^n)\,,\ \partial_t u_i^n := \partial_t u(x_i, t^n)\,,\ V_i := V(x_i)\,,\ U_i^n \approx u_i^n\,,\ \ldots$$

  - "Lax-Wendroff procedure"

$$\partial_t u_i^n = -V_i \partial_x u_i^n$$

  - Taylor series

$$\begin{aligned} u_i^{n-1} &= u_i^n - \tau\partial_t u_i^n + \mathcal{O}(\tau^2) \\ u_i^{n-1} &= u_i^n + \tau V_i \partial_x u_i^n + \mathcal{O}(\tau^2) \end{aligned}$$

  - 1st order accurate finite difference approximation

$$U_i^{n-1} = U_i^n + \tau V_i \frac{U_i^n - U_{i-1}^n}{h} \quad \Rightarrow \ U_i^n = \frac{1}{1 + C_i}\left(U_i^{n-1} + C_i U_{i-1}^n\right)\,,\ C_i := \frac{\tau V_i}{h}$$

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- **Fully implicit 2nd order accurate schemes**

$$u_i^n := u(x_i, t^n)\,,\ \partial_t u_i^n := \partial_t u(x_i, t^n)\,,\ V_i := V(x_i)\,,\ U_i^n \approx u_i^n\,,\ \ldots$$

  - Lax-Wendroff or Cauchy-Kovalewskaya procedure

$$\partial_{xt} u_i^n = -\partial_x (V_i \partial_x u_i^n)\,, \quad \partial_{tt} u_i^n = -V_i \partial_{tx} u_i^n \;\Rightarrow\; \partial_{tt} u_i^n = V_i \partial_x (V_i \partial_x u_i^n)$$

  - Taylor series

$$\begin{aligned} u_i^{n-1} &= u_i^n - \tau \partial_t u_i^n + 0.5\tau^2 \partial_{tt} u_i^n + \mathcal{O}(\tau^3) \\ u_i^{n-1} &= u_i^n + \tau V_i \partial_x u_i^n + 0.5\tau^2 V_i \partial_x (V_i \partial_x u_i^n) + \mathcal{O}(\tau^3) \end{aligned}$$

  - The 1st term must be approximated by a 2nd order scheme, e.g.,

$$2h\partial_x u \approx 2h\partial_x^{\omega,2} U_i := \omega(3U_i - 4U_{i-1} + U_{i-2}) + (1-\omega)(U_{i+1} - U_{i-1})$$

Frolkovič, P., & Mikula, K. (2018). *Semi-implicit second order schemes for numerical solution of level set advection equation on Cartesian grids.* Applied Mathematics and Computation, 329, 129-142.

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- **Fully implicit 2nd order accurate schemes**

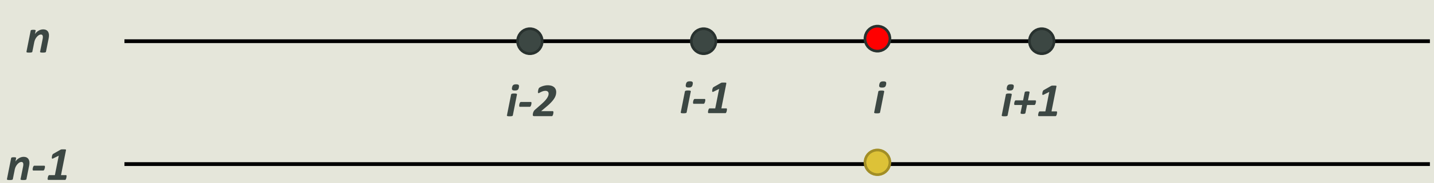
$$u_i^n := u(x_i, t^n)\,,\ \partial_t u_i^n := \partial_t u(x_i, t^n)\,,\ V_i := V(x_i)\,,\ U_i^n \approx u_i^n\,,\ \ldots$$

  - Lax-Wendroff or Cauchy-Kovalewskaya procedure

$$\partial_{xt} u_i^n = -\partial_x(V_i \partial_x u_i^n)\,, \quad \partial_{tt} u_i^n = -V_i \partial_{tx} u_i^n \ \Rightarrow \ \partial_{tt} u_i^n = V_i \partial_x (V_i \partial_x u_i^n)$$

  - Taylor series

$$\begin{aligned} u_i^{n-1} &= u_i^n - \tau \partial_t u_i^n + 0.5\tau^2 \partial_{tt} u_i^n + \mathcal{O}(\tau^3) \\ u_i^{n-1} &= u_i^n + \tau V_i \partial_x u_i^n + 0.5\tau^2 V_i \partial_x (V_i \partial_x u_i^n) + \mathcal{O}(\tau^3) \end{aligned}$$



Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- **Semi-implicit 2nd order accurate schemes**
  - Lax-Wendroff procedure $\qquad u_i^n := u(x_i, t^n)\,,\ U_i^n \approx u_i^n\,,\ \text{etc.}$

$$\partial_t u_i^n = -V_i \partial_x u_i^n\,, \quad \partial_{tt} u_i^n = -V_i \partial_{tx} u_i^n\,, \quad \partial_{tx} u_i^n = \partial_x (V_i \partial_x u_i^n)$$

  - Taylor series

$$\begin{aligned} u_i^{n-1} &= u_i^n - \tau \partial_t u_i^n + 0.5\tau^2 \partial_{tt} u_i^n + \mathcal{O}(\tau^3) \\ u_i^{n-1} &= u_i^n + \tau V_i \partial_x u_i^n - 0.5\tau^2 V_i \partial_{tx} u_i^n + \mathcal{O}(\tau^3) \end{aligned}$$

  - 2nd order accurate FD schemes for the spatial derivatives:

$$2h\partial_x^{\omega,2} U_i := \omega(3U_i - 4U_{i-1} + U_{i-2}) + (1-\omega)(U_{i+1} - U_{i-1})$$

    - standard Crank-Nicolson (trapezoidal) scheme can be obtained

$$U_i^{n-1} = U_i^n + \tau V_i \partial_x^{\omega,2} U_i^n - 0.5\tau^2 V_i \frac{\partial_x^{\omega,2} U_i^n - \partial_x^{\omega,2} U_i^{n-1}}{\tau}$$

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \qquad V(x) > 0$$

- **Semi-implicit 2nd order accurate schemes**
  - Lax-Wendroff procedure $\quad u_i^n := u(x_i, t^n)\,,\ U_i^n \approx u_i^n\,,\ \text{etc.}$

$$\partial_t u_i^n = -V_i \partial_x u_i^n\,, \quad \partial_{tt} u_i^n = -V_i \partial_{tx} u_i^n\,, \quad \partial_{tx} u_i^n = \partial_x (V_i \partial_x u_i^n)$$

  - Taylor series

$$\begin{aligned} u_i^{n-1} &= u_i^n - \tau \partial_t u_i^n + 0.5\tau^2 \partial_{tt} u_i^n + \mathcal{O}(\tau^3) \\ u_i^{n-1} &= u_i^n + \tau V_i \partial_x u_i^n - 0.5\tau^2 V_i \partial_{tx} u_i^n + \mathcal{O}(\tau^3) \end{aligned}$$

  - 2nd order accurate FD schemes for the spatial derivatives:

$$2h\partial_x^{\omega,2} U_i := \omega(3U_i - 4U_{i-1} + U_{i-2}) + (1-\omega)(U_{i+1} - U_{i-1})$$

    - a Crank-Nicolson (trapezoidal) scheme can be obtained

$$U_i^{n-1} = U_i^n + 0.5\tau V_i \partial_x^{\omega,2} U_i^n + 0.5\tau V_i \partial_x^{\omega,2} U_i^{n-1}$$

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

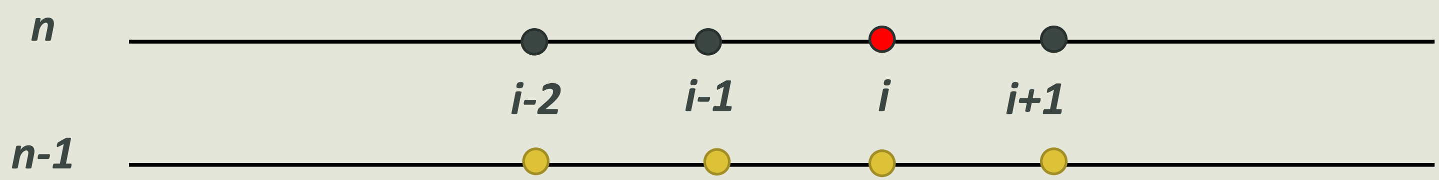
- **Semi-implicit 2nd order accurate schemes**
  - Lax-Wendroff procedure $\qquad u_i^n := u(x_i, t^n)\,,\ U_i^n \approx u_i^n\,,\ \text{etc.}$

$$\partial_t u_i^n = -V_i \partial_x u_i^n\,, \quad \partial_{tt} u_i^n = -V_i \partial_{tx} u_i^n\,, \quad \partial_{tx} u_i^n = \partial_x (V_i \partial_x u_i^n)$$

  - Taylor series

$$\begin{aligned} u_i^{n-1} &= u_i^n - \tau \partial_t u_i^n + 0.5\tau^2 \partial_{tt} u_i^n + \mathcal{O}(\tau^3) \\ u_i^{n-1} &= u_i^n + \tau V_i \partial_x u_i^n - 0.5\tau^2 V_i \partial_{tx} u_i^n + \mathcal{O}(\tau^3) \end{aligned}$$

  - 2nd order accurate FD schemes for the spatial derivatives:



$$U_i^{n-1} = U_i^n + 0.5\tau V_i \partial_x^{\omega,2} U_i^n + 0.5\tau V_i \partial_x^{\omega,2} U_i^{n-1}$$

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- **Semi-implicit 2nd order accurate schemes**
  - Lax-Wendroff procedure

$$\partial_t u_i^n = -V_i \partial_x u_i^n\,, \quad \partial_{tt} u_i^n = -V_i \partial_{tx} u_i^n\,, \quad \partial_{tx} u_i^n = \partial_x (V_i \partial_x u_i^n)$$

  - Taylor series

$$\begin{aligned} u_i^{n-1} &= u_i^n - \tau \partial_t u_i^n + 0.5\tau^2 \partial_{tt} u_i^n + \mathcal{O}(\tau^3) \\ u_i^{n-1} &= u_i^n + \tau V_i \partial_x u_i^n - 0.5\tau^2 V_i \partial_{tx} u_i^n + \mathcal{O}(\tau^3) \end{aligned}$$

  - non-standard implicit scheme

$$U_i^{n-1} = U_i^n + \tau V_i \partial_x^{\omega,2} U_i^n - 0.5\tau^2 V_i \frac{\partial_x^{\omega,1} U_i^n - \partial_x^{\omega,1} U_i^{n-1}}{\tau}$$

$$h\partial_x^{\omega,1} U_i := \omega(U_i - U_{i-1}) + (1-\omega)(U_{i+1} - U_i)$$

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- **Semi-implicit 2nd order accurate schemes**
  - Lax-Wendroff procedure

$$\partial_t u_i^n = -V_i \partial_x u_i^n\,, \quad \partial_{tt} u_i^n = -V_i \partial_{tx} u_i^n\,, \quad \partial_{tx} u_i^n = \partial_x (V_i \partial_x u_i^n)$$

  - Taylor series

$$\begin{aligned} u_i^{n-1} &= u_i^n - \tau \partial_t u_i^n + 0.5\tau^2 \partial_{tt} u_i^n + \mathcal{O}(\tau^3) \\ u_i^{n-1} &= u_i^n + \tau V_i \partial_x u_i^n - 0.5\tau^2 V_i \partial_{tx} u_i^n + \mathcal{O}(\tau^3) \end{aligned}$$

  - not so standard implicit scheme

$$U_i^{n-1} = U_i^n + \tau V_i \partial_x^{\omega,2} U_i^n - 0.5\tau^2 V_i \frac{\partial_x^{\omega,1} U_i^n - \partial_x^{\omega,1} U_i^{n-1}}{\tau}$$

$$\boxed{\partial_x^{\omega,2} U_i - 0.5\partial_x^{\omega,1} U_i = U_i - U_{i-1} - 0.5\partial_x^{\omega,1} U_{i-1}}$$

$$h\partial_x^{\omega,1} U_i := \omega(U_i - U_{i-1}) + (1-\omega)(U_{i+1} - U_i)$$

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \qquad V(x) > 0$$

- <u>**Semi-implicit 2nd order accurate schemes**</u>
  - Lax-Wendroff procedure

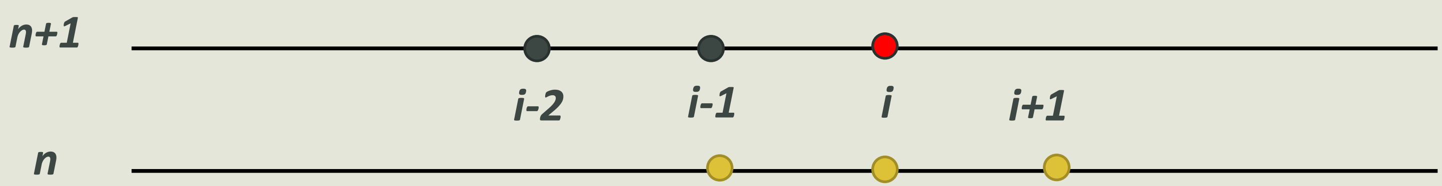
$$\partial_t u_i^n = -V_i \partial_x u_i^n\,, \quad \partial_{tt} u_i^n = -V_i \partial_{tx} u_i^n\,, \quad \partial_{tx} u_i^n = \partial_x (V_i \partial_x u_i^n)$$

  - Taylor series

$$\begin{aligned} u_i^{n-1} &= u_i^n - \tau \partial_t u_i^n + 0.5\tau^2 \partial_{tt} u_i^n + \mathcal{O}(\tau^3) \\ u_i^{n-1} &= u_i^n + \tau V_i \partial_x u_i^n - 0.5\tau^2 V_i \partial_{tx} u_i^n + \mathcal{O}(\tau^3) \end{aligned}$$

  - not so standard implicit scheme

$$U_i^{n-1} = U_i^n + C_i(U_i^n - U_{i-1}^n) - 0.5\tau V_i(\partial_x^{\omega,1} U_{i-1}^n + \partial_x^{\omega,1} U_i^{n-1})$$



Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- <u>**Semi-implicit 3rd order accurate schemes**</u>
  - Lax-Wendroff procedure

$$\partial_t u_i^n = -V_i \partial_x u_i^n\,, \quad \partial_{tt} u_i^n = -V_i \partial_{tx} u_i^n\,, \quad \partial_{ttt} u_i^n = V_i \partial_{tx}(V_i \partial_x u_i^n)$$

  - Taylor series

$$u_i^{n-1} = u_i^n \quad - \quad \tau \partial_t u_i^n \quad + \quad \frac{\tau^2}{2}\partial_{tt} u_i^n \quad - \quad \frac{\tau^3}{6}\partial_{ttt} u_i^n \quad + \quad \mathcal{O}(\tau^4)$$

$$u_i^{n-1} = u_i^n \quad + \quad \tau V_i \partial_x u_i^n \quad - \quad \frac{\tau^2 V_i}{2}\partial_{tx} u_i^n \quad - \quad \frac{\tau^3 V_i}{6}\partial_{tx}(V_i \partial_x u_i^n) \quad + \quad \mathcal{O}(\tau^4)$$

  - one shall apply the 3rd, 2nd, and 1st order approximation of the terms, respectively. Moreover:

$$\partial_t u_i^n = \frac{u_i^n - u_i^{n-1}}{\tau} - \frac{\tau}{2} V_i \partial_{tx} u_i^n + \mathcal{O}(\tau^2)$$

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- fully implicit scheme for $\omega = 0$

$$U_i^n + \tau V_i \frac{U_{i+1}^n - U_{i-1}^n}{2h} + \frac{\tau^2 V_i}{2} \frac{V_i(U_{i+1}^n - U_i^n) - V_{i-1}(U_i^n - U_{i-1}^n)}{h^2} = U_i^{n-1}$$

- semi-implicit scheme for $\omega = 0$

$$U_i^n + \frac{1}{2}\tau V_i \frac{U_i^n - U_{i-1}^n}{h} = U_i^{n-1} - \frac{1}{2}\tau V_i \frac{U_{i+1}^{n-1} - U_i^{n-1}}{h}$$

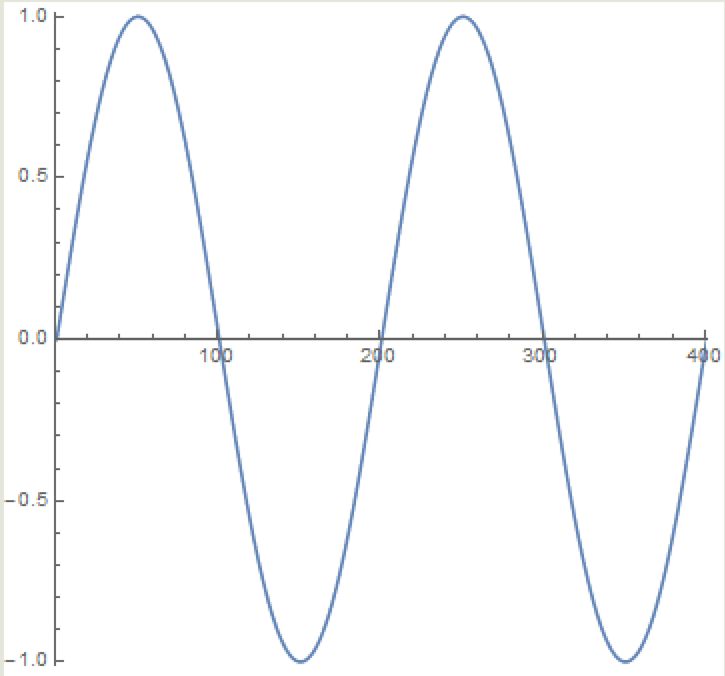
- advantages of semi-implicit scheme against fully-implicit scheme:
  - simpler algebraic system to solve (in fact one forward substitution is enough)
  - simpler stability analysis (only one value of the velocity used)
  - 2nd order accuracy also for dimension-by-dimension usage (not for FI)
  - unconditional stability for all omega (only for $\omega \leq 1/2$ for FI scheme)

Frolkovič, Mikula: *Semi-implicit second order schemes for numerical solution of level set advection equation on Cartesian grids*. Appl. Math. Comp. 2018

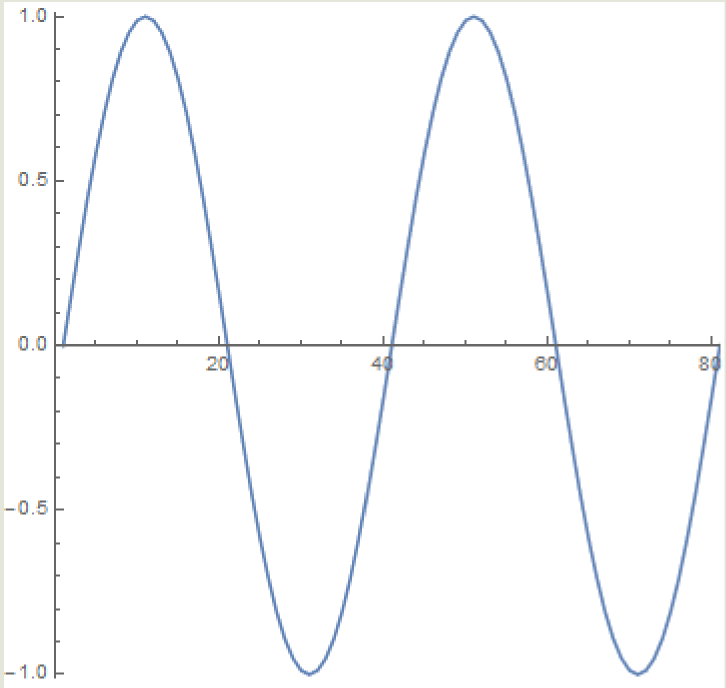
Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- **Illustration of instability for fully implicit scheme**



stable computation



unstable computation $\omega = 0$

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- Previous efforts
  - Inflow-Implicit / Outflow-Explicit [Mikula, Ohlberger, preprint 2010, FVCA 2011]

$$U_i^n + \frac{1}{2}\tau V_{i-1/2}\frac{U_i^n - U_{i-1}^n}{h} = U_i^{n-1} - \frac{1}{2}\tau V_{i+1/2}\frac{U_{i+1}^{n-1} - U_i^{n-1}}{h}$$

Mikula, K., Ohlberger, M., & Urbán, J. (2014). Inflow-implicit/outflow-explicit finite volume methods for solving advection equations. Applied Numerical Mathematics, 85

McCartin, B. J. (2005). The method of angled derivatives. Appl. Math. Comp., 170(1).

K.V. Roberts, N.O. Weiss (1966), Convective difference schemes, Math. Comput. 20

Saul'ev, V. K. E. (1963). *Solution of certain boundary-value problems on high-speed computers by the fictitious-domain method.* Sibirskii Matematicheskii Zhurnal, 4, 912-925.

Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- Previous efforts
  - Inflow-Implicit / Outflow-Explicit, constant velocity case $\quad 0 \le C$

$$(1+\frac{C}{2})U_i^n - \frac{C}{2}U_{i-1}^n = (1+\frac{C}{2})U_i^{n-1} - \frac{C}{2}U_{i+1}^{n-1}$$

  - Angled derivative scheme [Roberts, Weiss, 1966]

$$U_i^n = U_i^{n-1} - \frac{\frac{C}{2}}{1+\frac{C}{2}}(U_{i+1}^{n-1} - U_{i-1}^n) \qquad -1 \le C \le 2$$

Mikula, K., Ohlberger, M., & Urbán, J. (2014). Inflow-implicit/outflow-explicit finite volume methods for solving advection equations. Applied Numerical Mathematics, 85

McCartin, B. J. (2005). The method of angled derivatives. Appl. Math. Comp., 170(1).

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Semi-implicit method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \quad V(x) > 0$$

- Previous efforts
  - Inflow-Implicit / Outflow-Explicit, constant velocity case $0 \le C$

$$(1+\frac{C}{2})U_i^n - \frac{C}{2}U_{i-1}^n = (1+\frac{C}{2})U_i^{n-1} - \frac{C}{2}U_{i+1}^{n-1}$$

  - Duraisamy, Baeder [2007]

$$(1-\frac{C}{2})U_i^n + \frac{C}{2}U_{i+1}^n = (1-\frac{C}{2})U_i^{n-1} + \frac{C}{2}U_{i-1}^{n-1} \qquad 0 \le C$$

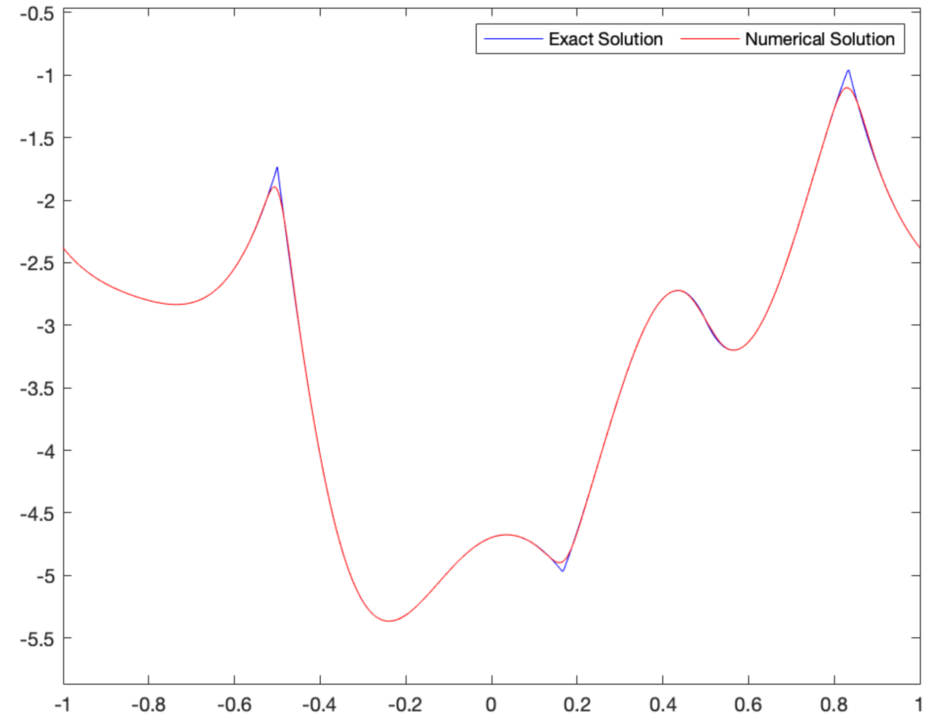
    - it can be derived analogously using “explicit form” of Taylor series [FM, 2016]

Frolkovič, P., & Mikula, K. (2016). *Higher order semi-implicit schemes for linear advection equation on Cartesian grids with numerical stability analysis.* arXiv:1611.04153.
Duraisamy, K., & Baeder, J. D. (2007). *Implicit scheme for hyperbolic conservation laws using nonoscillatory reconstruction in space and time***.** SIAM J. Sci. Comp., 29(6), 2607-2620.

Semi-implicit WENO method for the model equation

$$\partial_t u + \bar{V}\partial_x u = 0$$

- constant speed
- $\bar{C} \approx 5$
- $h = \dfrac{2}{640}$



Nikola Gajdošová

Jiang, G. S., & Peng: *Weighted ENO schemes for Hamilton--Jacobi equations*. SIAM Journal on Scientific computing, 2000.

Semi-implicit WENO method for the model equation

$$\partial_t u + V(x)\partial_x u = 0\,, \qquad V(x) > 0$$

- weighted ENO 3$^{\text{rd}}$ scheme : $(C_i = 1)$

$$U_i^n + (1-\omega)\frac{U_{i+1}^n - U_{i-1}^n}{2h} + \omega\frac{3U_i^n - 4U_{i-1}^n + U_{i-2}^n}{2h} - \frac{1}{2}(1-\omega)\frac{U_{i+1}^n - U_i^n}{h} - \frac{1}{2}\omega\frac{U_i^n - U_{i-1}^n}{h}$$
$$+\frac{1}{2}(1-\omega)\frac{U_{i+1}^{n-1} - U_i^{n-1}}{h} + \frac{1}{2}\omega\frac{U_i^{n-1} - U_{i-1}^{n-1}}{h} = U_i^{n-1}$$

- $\omega = 1/3 \quad \Rightarrow$ 3$^{\text{rd}}$ order accurate in space

$$\omega = \omega_i = \frac{1}{1+2r_i^n} \qquad\qquad r_i = \frac{\epsilon + U_i^{n-1} - U_{i-1}^{n-1} - U_{i-1}^n + U_{i-2}^n}{\epsilon + U_{i+1}^{n-1} - U_i^{n-1} - U_i^n + U_{i-1}^n}$$

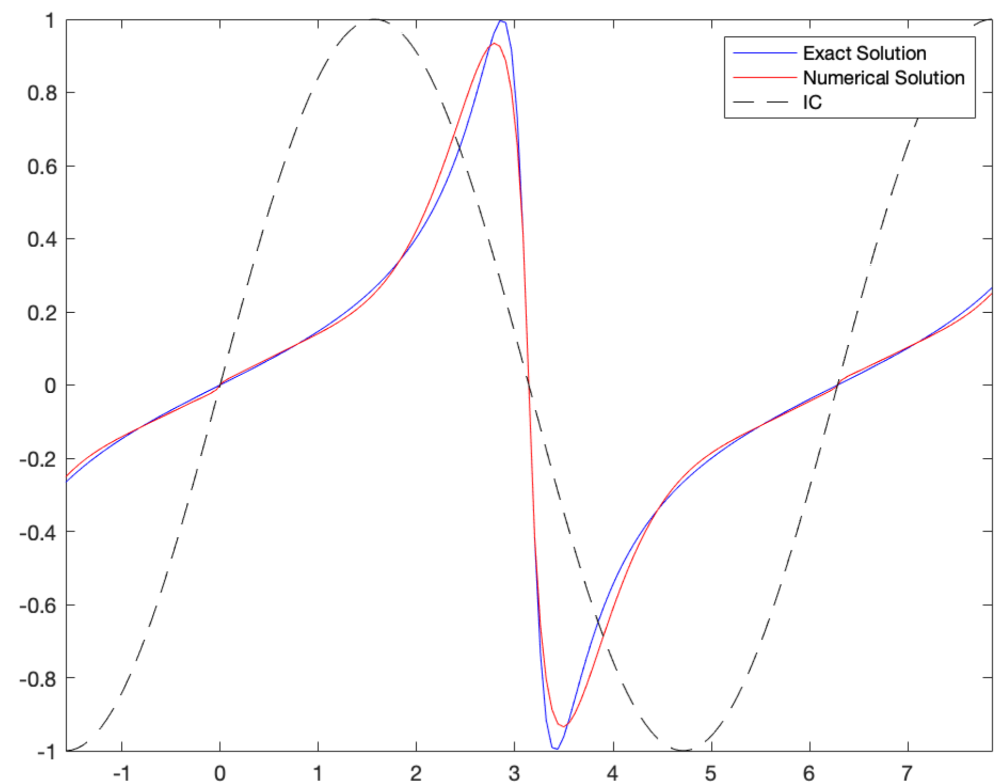
Jiang, Peng: *Weighted ENO schemes for Hamilton--Jacobi equations*. SIAM Journal on Scientific computing, 2000.

- 3$^{\text{rd}}$ order accuracy in time as shown before
- predictor-corrector procedure to resolve $r_i = r_i(U_i^n)$

Semi-implicit WENO method for the model equation

$$\partial_t u + V(x)\partial_x u = 0$$

- $V(x) = \sin(x)$
- $\max |C_i| \approx 34$
- $h = \dfrac{3\pi}{160}$

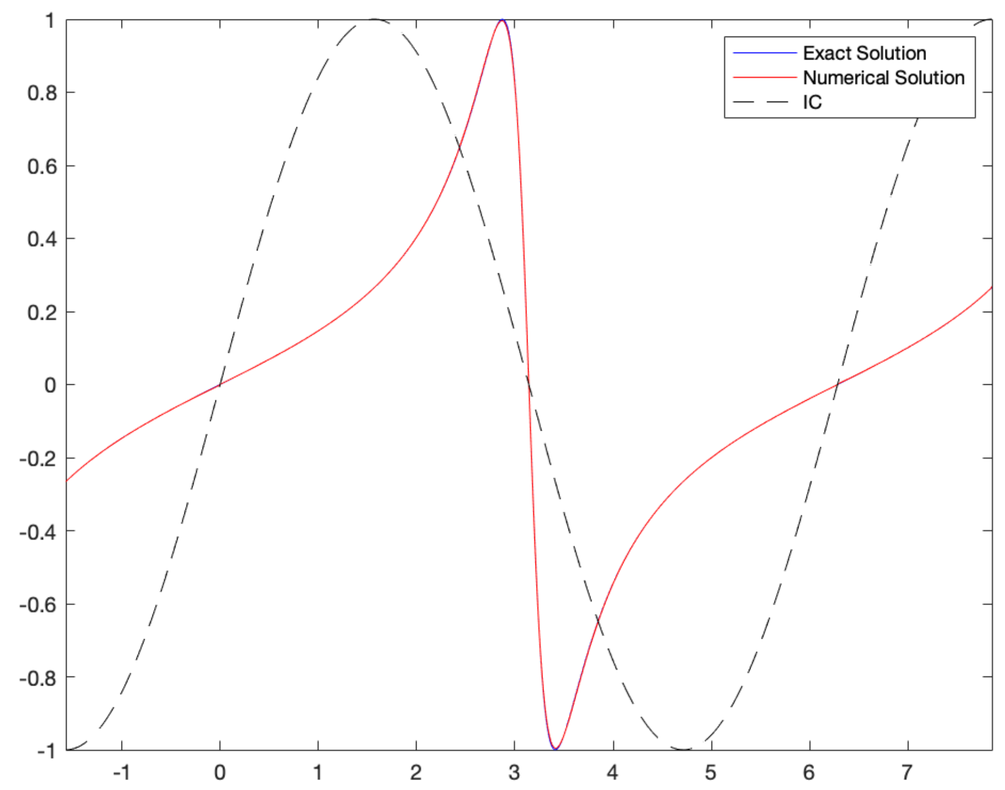


Nikola Gajdošová

Semi-implicit WENO method for the model equation

$$\partial_t u + V(x)\partial_x u = 0$$

- $V(x) = \sin(x)$
- $\max |C_i| \approx 34$
- $h = \dfrac{3\pi}{640}$
- EOC ≈ 3
- for each *n* the eqs solved with two sweeps
- the nonlinearity resolved with one correction
- EOC is preserved also for WENO variant



Nikola Gajdošová

Semi-implicit WENO method for the model equation

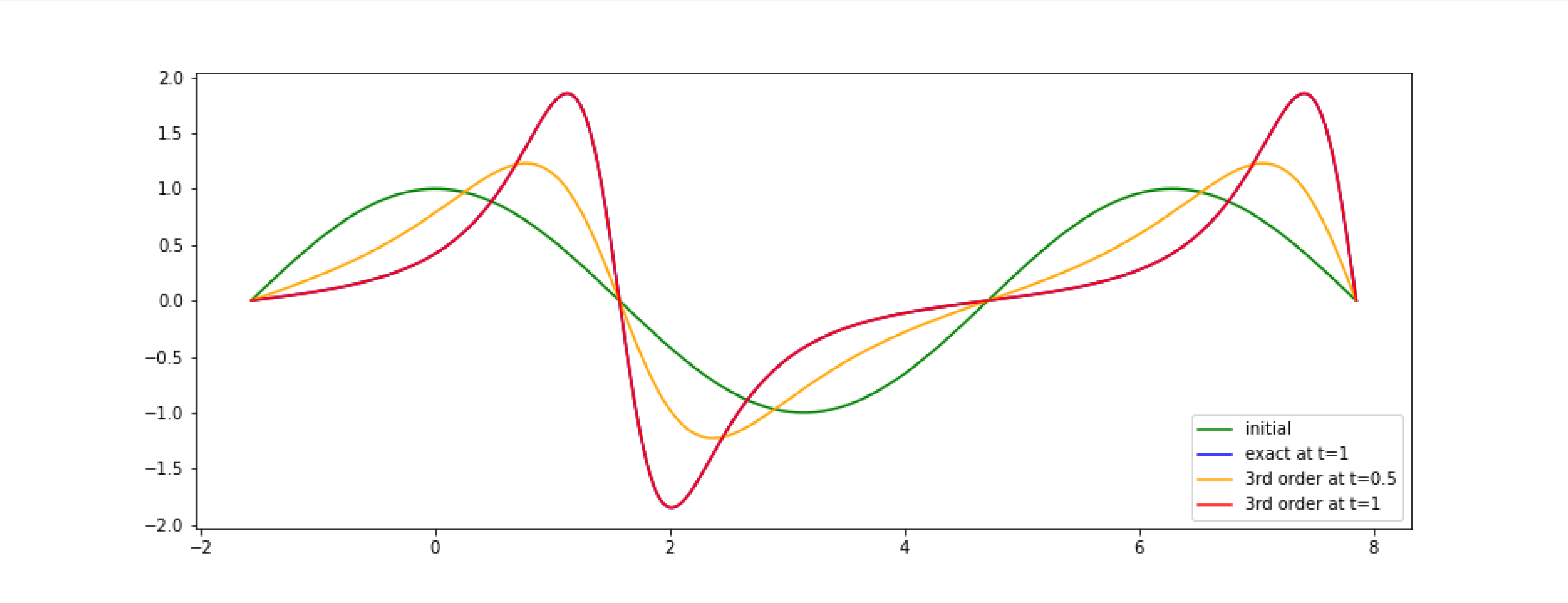
$$\partial_t u + V(x,y) \cdot \nabla u = 0$$

- Very recent results (with Nikola Gajdošová)
  - 3rd order accurate semi-implicit unconditionally stable scheme in 2D case for linear nonconservative advection equation is derived
    - no dimensional splitting is used
    - confirmed, e.g., for a rotation of a smooth initial profile

Compact implicit WENO method for the model equation

$$\partial_t u + \partial_x (V(x)u) = 0$$

- Very recent results (with Dagmar Žáková)
  - 3rd order accurate compact implicit unconditionally stable **finite volume scheme** in 1D case for linear conservative advection equation



TVD compact implicit conservative method

$$\partial_t u + \partial_x f(u) = 0\,,\;\; f'(u) \geq 0\,,\;\; u \in R$$

- Conservative FDM or FVM methods (numerical fluxes)

$$U_i^n + \frac{\tau}{h}\left(F^n_{i+1/2} - F^n_{i-1/2}\right) = U_i^{n-1}$$

- Fully implicit 1st order accurate FD scheme

$$U_i^n + \frac{\tau}{h}(f(U_i^n) - f(U_{i-1}^n)) = U_i^{n-1}$$

$$F^n_{i+1/2} = f_i^n := f(U_i^n)$$

TVD compact implicit conservative method

$$\partial_t u + \partial_x f(u) = 0\,,\ \ f'(u) \geq 0\,,\ \ u \in R$$

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$$U_i^n + \frac{\tau}{h} F_{i+1/2}^n = U_i^{n-1} + \frac{\tau}{h} F_{i-1/2}^n$$

- Fully implicit 1st order accurate FD scheme [Lozano, Alsam, 2021]

$$U_i^n + \frac{\tau}{h} f(U_i^n) = U_i^{n-1} + \frac{\tau}{h} f(U_{i-1}^n)\,,\ \ i = 1, 2, \dots, I$$

$$F_{i+1/2}^n = f_i^n := f(U_i^n)$$

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$$F_{i+1/2}^n = f_i^n := f(U_i^n)$$

- **Compact implicit 2nd order accurate conservative FD schemes**

$$F_{i+1/2}^n = f_i^n - 0.5 l_i \left((1-\omega_i)(f_i^n - f_{i+1}^{n-1}) + \omega_i (f_{i-1}^n - f_i^{n-1}) \right)$$

Lozano, Aslam: *Implicit fast sweeping method for hyperbolic systems of conservation laws.* Journal of Computational Physics, 2021.

Frolkovič, Žeravý: *High-resolution compact implicit numerical schemes for conservation laws.* 2022 (in rev.)

TVD semi-implicit conservative method

$$\partial_t u + \partial_x f(u) = 0\,,\, f'(u) > 0$$

- **Compact implicit 2nd order accurate conservative FD schemes**

$$U_i^n + \frac{\tau}{h} F_{i+1/2}^n = U_i^{n-1} + \frac{\tau}{h} F_{i-1/2}^n$$

$$F_{i+1/2}^n = f_i^n - 0.5 l_i \left((1-\omega_i)(f_i^n - f_{i+1}^{n-1}) + \omega_i (f_{i-1}^n - f_i^{n-1}) \right)$$

- **TVD scheme:** $$\sum |U_i^n - U_{i-1}^n| \le \sum |U_i^{n-1} - U_{i-1}^{n-1}|$$
- Sufficient conditions (e.g. for periodic BC)

$$U_i^n + c_{i-1}(U_i^n - U_{i-1}^n) = U_i^{n-1}\,, \quad c_{i-1} \ge 0$$

TVD semi-implicit conservative method

$$\partial_t u + \partial_x f(u) = 0\,,\, f'(u) > 0$$

- **Compact implicit 2nd order accurate conservative FD schemes**

$$U_i^n + \frac{\tau}{h} F_{i+1/2}^n = U_i^{n-1} + \frac{\tau}{h} F_{i-1/2}^n$$

$$F_{i+1/2}^n = f_i^n - 0.5 l_i \left((1-\omega_i)(f_i^n - f_{i+1}^{n-1}) + \omega_i (f_{i-1}^n - f_i^{n-1}) \right)$$

- **TVD scheme:** $r_i := \dfrac{f_{i-1}^n - f_i^{n-1}}{f_i^n - f_{i+1}^{n-1}}\,, \quad \Phi_i := 1 - \omega_i + \omega_i r_i$

$$\begin{aligned} F_{i+1/2}^n &= f_i^n - \tfrac{l_i}{2} \tfrac{\Phi_i}{r_i} (f_{i-1}^n - f_i^{n-1}) \\ F_{i-1/2}^n &= f_{i-1}^n - \tfrac{l_{i-1}}{2} \Phi_{i-1} (f_{i-1}^n - f_i^{n-1}) \end{aligned}$$

$$f_{i-1}^n - f_i^{n-1} = f_i^n - f_i^{n-1} - (f_i^n - f_{i-1}^n)$$

TVD semi-implicit conservative method

$$\partial_t u + \partial_x f(u) = 0\,,\, f'(u) > 0$$

- **Compact implicit 2nd order accurate conservative FD schemes**

$$U_i^n + \frac{\tau}{h} F_{i+1/2}^n = U_i^{n-1} + \frac{\tau}{h} F_{i-1/2}^n$$

$$F_{i+1/2}^n = f_i^n - 0.5 l_i \left((1-\omega_i)(f_i^n - f_{i+1}^{n-1}) + \omega_i (f_{i-1}^n - f_i^{n-1}) \right)$$

- **TVD scheme**

$$\left(1 - \frac{\tau}{h} \frac{f_i^n - f_i^{n-1}}{U_i^n - U_i^{n-1}} \frac{1}{2} \left(l_i \frac{\Phi_i}{r_i} - l_{i-1} \Phi_{i-1} \right) \right) (U_i^n - U_i^{n-1})$$

$$+ \left(1 + \frac{1}{2} \left(l_i \frac{\Phi_i}{r_i} - l_{i-1} \Phi_{i-1} \right) \right) \frac{f_i^n - f_{i-1}^n}{U_i^n - U_{i-1}^n} (U_i^n - U_{i-1}^n) = 0$$

TVD semi-implicit conservative method

$$\partial_t u + \bar{V} \partial_x u = 0 \,, \bar{V} > 0$$

- **Compact implicit 2nd order accurate conservative FD schemes**

$$U_i^n + C U_{i+1/2}^n = U_i^{n-1} + C U_{i-1/2}^n$$

$$U_{i+1/2}^n = U_i^n - 0.5 l_i \left((1-\omega_i)(U_i^n - U_{i+1}^{n-1}) + \omega_i (U_{i-1}^n - U_i^{n-1}) \right)$$

- **TVD scheme**

$$\left(1 - C \frac{1}{2} \left(l_i \frac{\Phi_i}{r_i} - l_{i-1} \Phi_{i-1} \right) \right) \left(U_i^n - U_i^{n-1} \right)$$

$$+ \left(1 + \frac{1}{2} \left(l_i \frac{\Phi_i}{r_i} - l_{i-1} \Phi_{i-1} \right) \right) C \left(U_i^n - U_{i-1}^n \right) = 0$$

TVD semi-implicit conservative method

$$\partial_t u + \partial_x f(u) = 0\,,\, f'(u) > 0$$

- **Compact implicit 2nd order accurate conservative FD schemes**
  - "standard" limiters can be used for maximal Courant number $C \leq 1$
  - the following choice appears convenient

$$\Phi_i = \begin{cases} 2 & 2 \leq r_i \\ \frac{-1}{C} & r_i \leq -\frac{1}{C} \\ r & \text{otherwise} \end{cases}$$

  - limiting in time must be used [Duraisamy & Baeder]

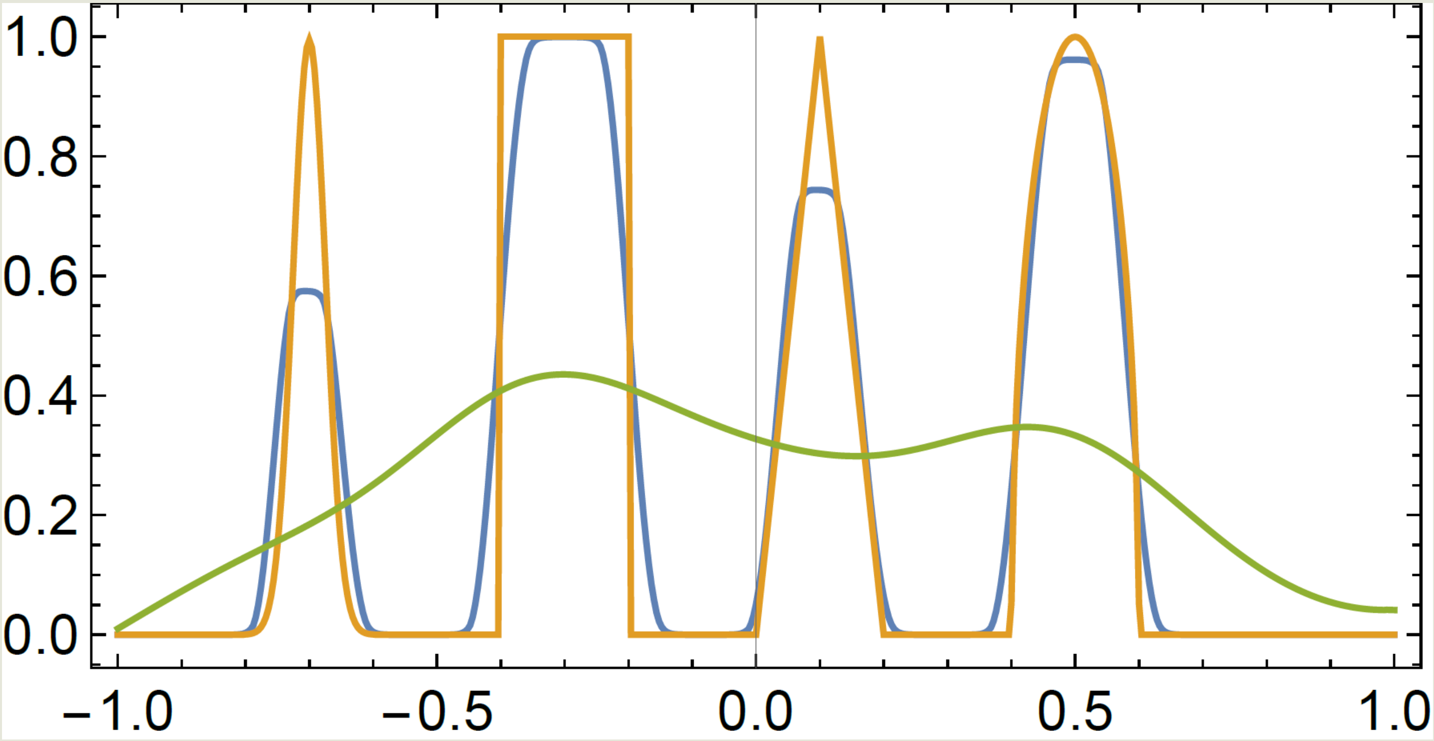
$$l_i = \min\{1, \max\{0, \frac{r_i}{\Phi_i}\left(\frac{2}{C} + l_{i-1}\Phi_{i-1}\right)\}\}$$

Frolkovič, Žeravý: *High-resolution compact implicit numerical schemes for conservation laws.*
Duraisamy & Baeder: *Implicit Scheme For Hyperbolic Conservation Laws Using Nonoscillatory Reconstruction In Space And Time*. SIAM Journal on Scientific computing, 2000.

Numerical experiments

$$\partial_t u + \partial_x f(u) = 0\,, \quad f(u) = u$$

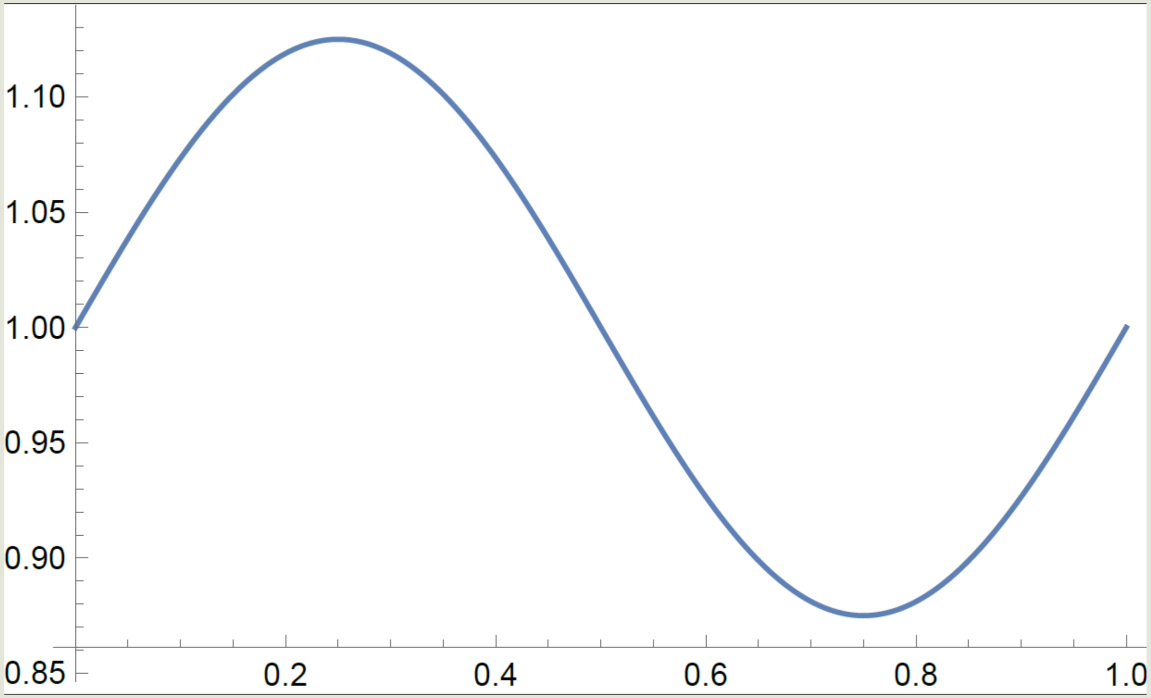
- **<u>Benchmark for TVD property</u>**
  - Max CN=4, h=2/500, 125 & 250 time steps, periodic solution
  - orange=exact, blue=2nd order, green=1st order



Numerical experiments

$$\partial_t u + \partial_x f(u) = 0\,, \qquad f(u) = \frac{u^2}{2}$$

- **Smooth solution with the initial function 1+Sin[2πx]/8**
  - Max CN=4.5, omega=1, T=1, h=1/80
  - initial condition



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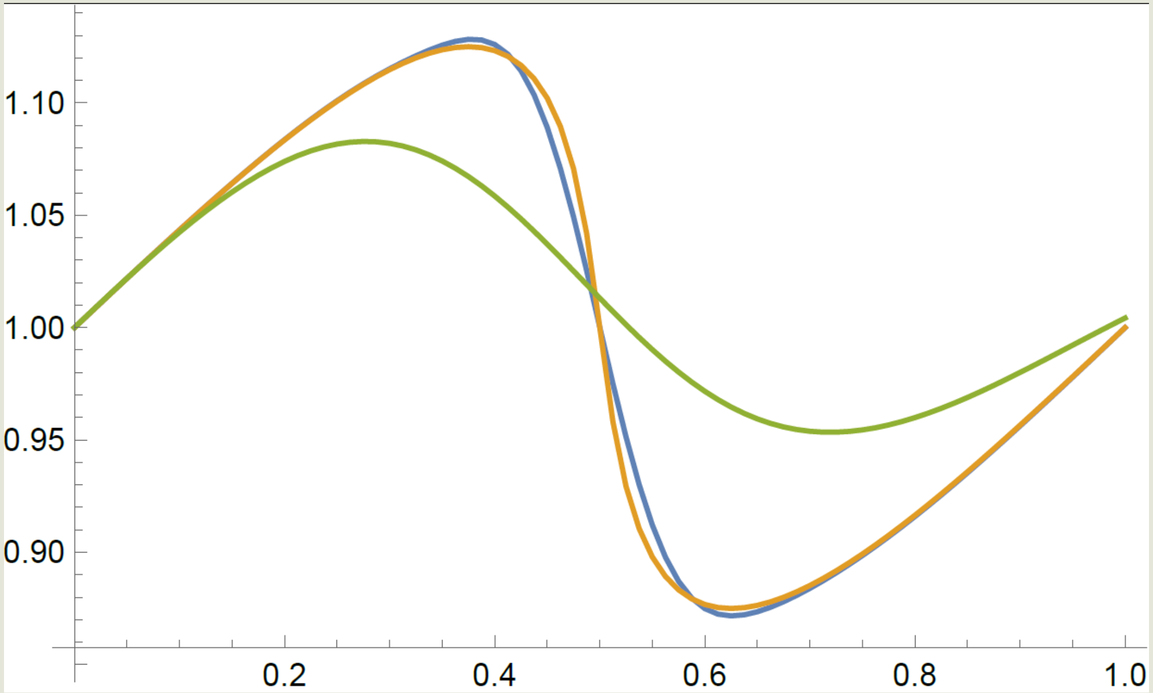
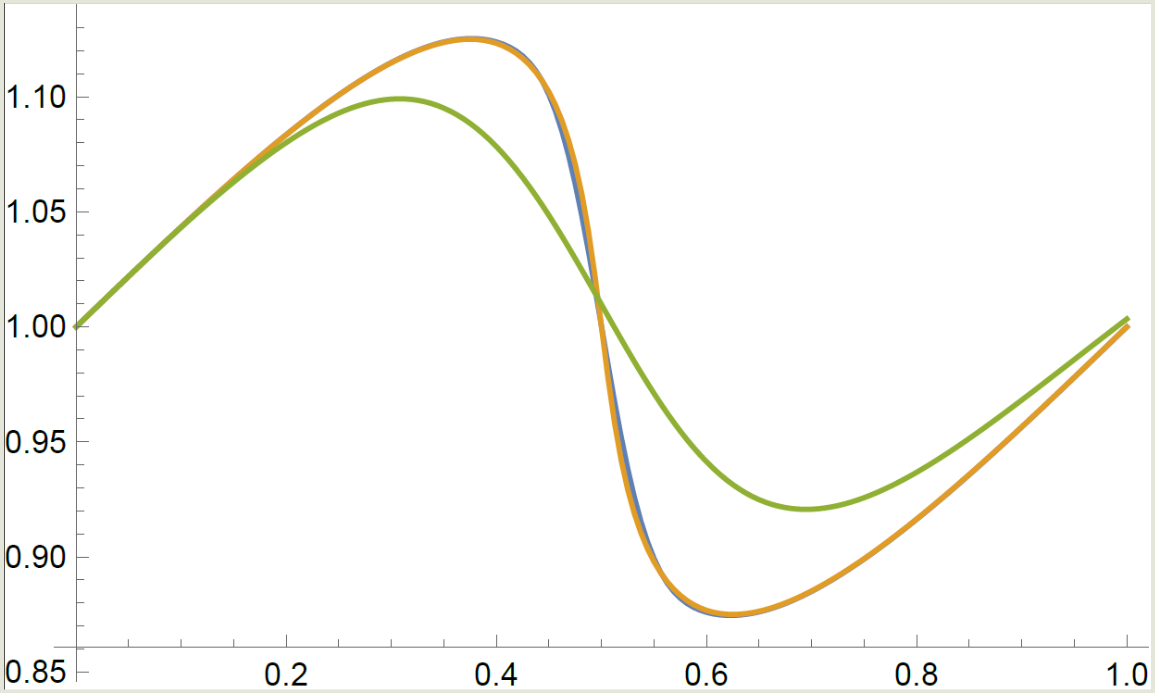


Numerical experiments

$$\partial_t u + \partial_x f(u) = 0\,, \qquad f(u) = \frac{u^2}{2}$$

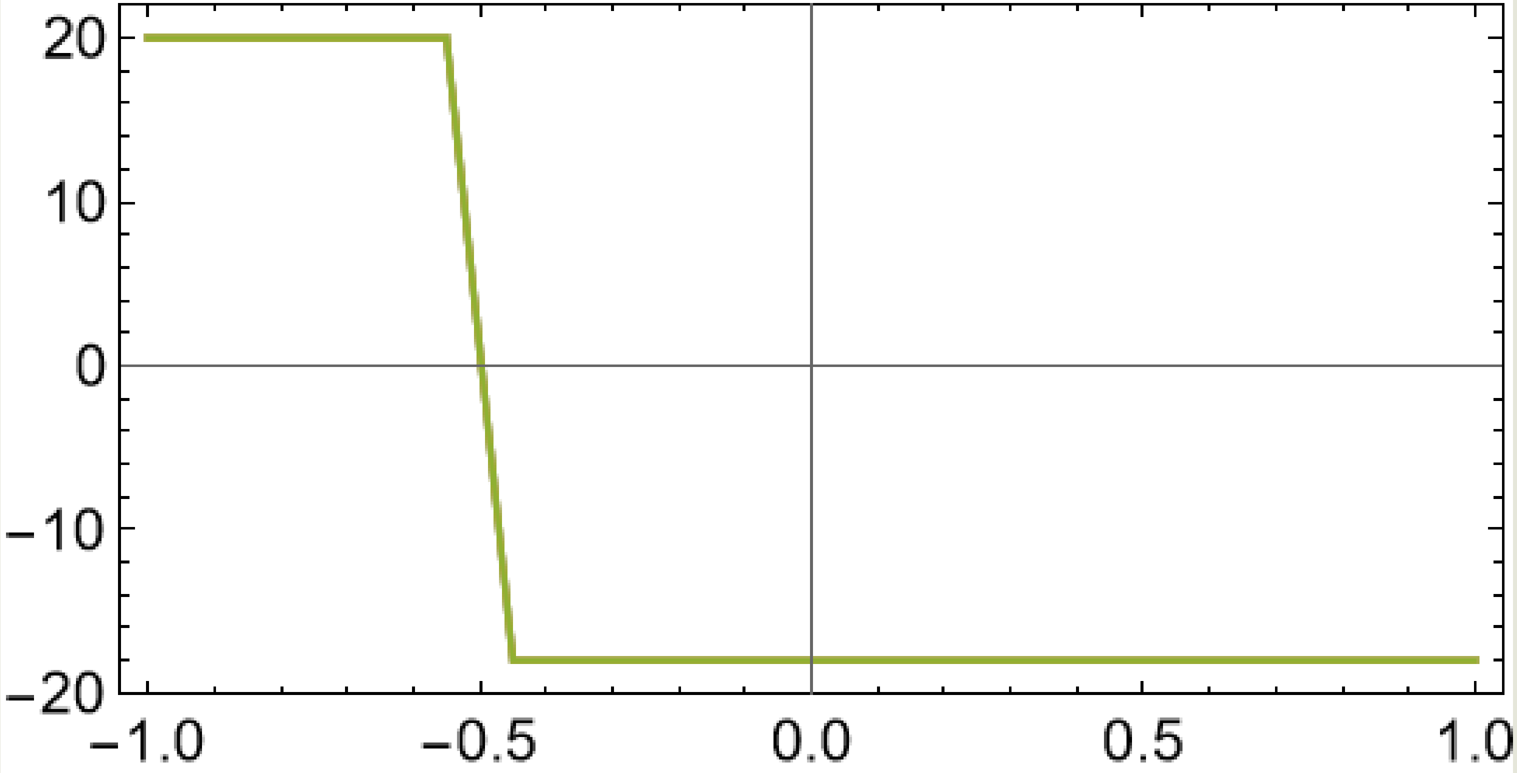
- **Smooth solution with the initial function 1+Sin[2πx]/8**
  - Max CN=4.5, omega=1, T=1, h=1/160
  - orange=exact, blue=2nd order, green=1st order
  - EOC=2.08



Numerical experiments

$$\partial_t u + \partial_x f(u) = 0\,, \quad f(u) = \frac{u^2}{2}$$

- **Slowly moving shock [Lozano, Aslam, 2021]**
  - Max CN=10., T=1, h=2/20
  - orange=exact, blue=2nd order, green=1st order



Numerical experiments

$$\partial_t u + \partial_x f(u) = 0\,, \qquad f(u) = \frac{u^2}{2}$$

- **Slowly moving shock**
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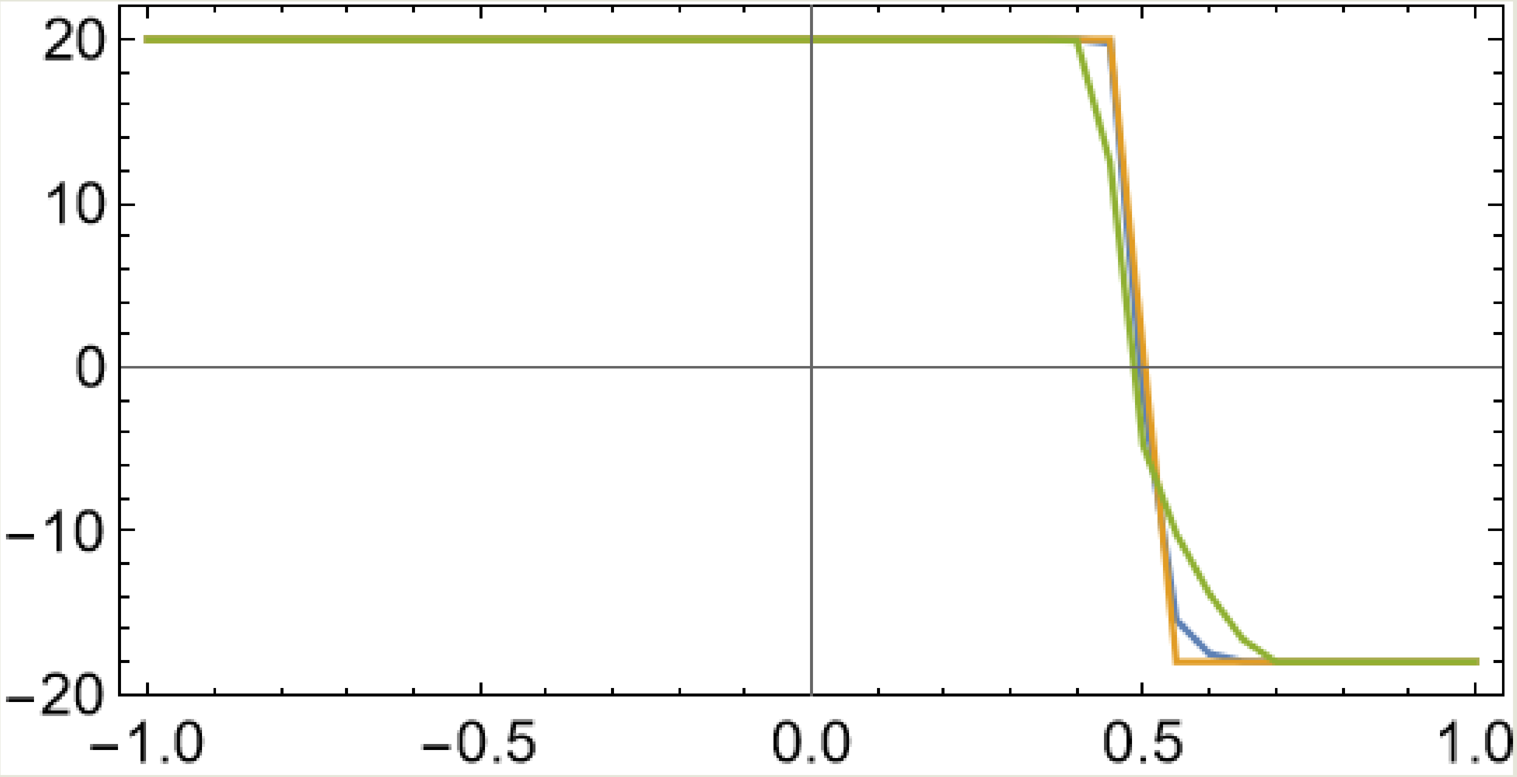
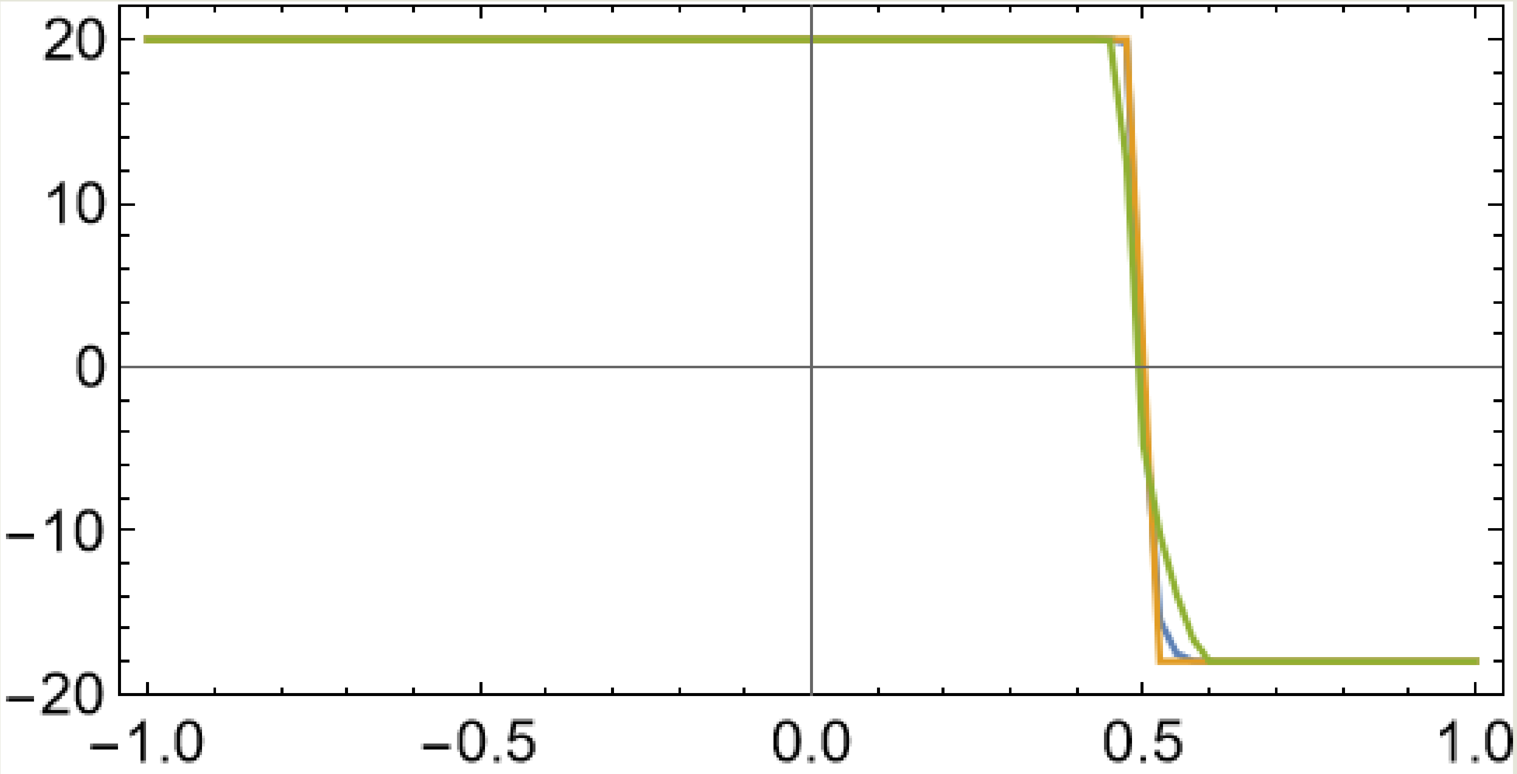


Numerical experiments

$$\partial_t u + \partial_x f(u) = 0\,, \qquad f(u) = \frac{u^2}{2}$$

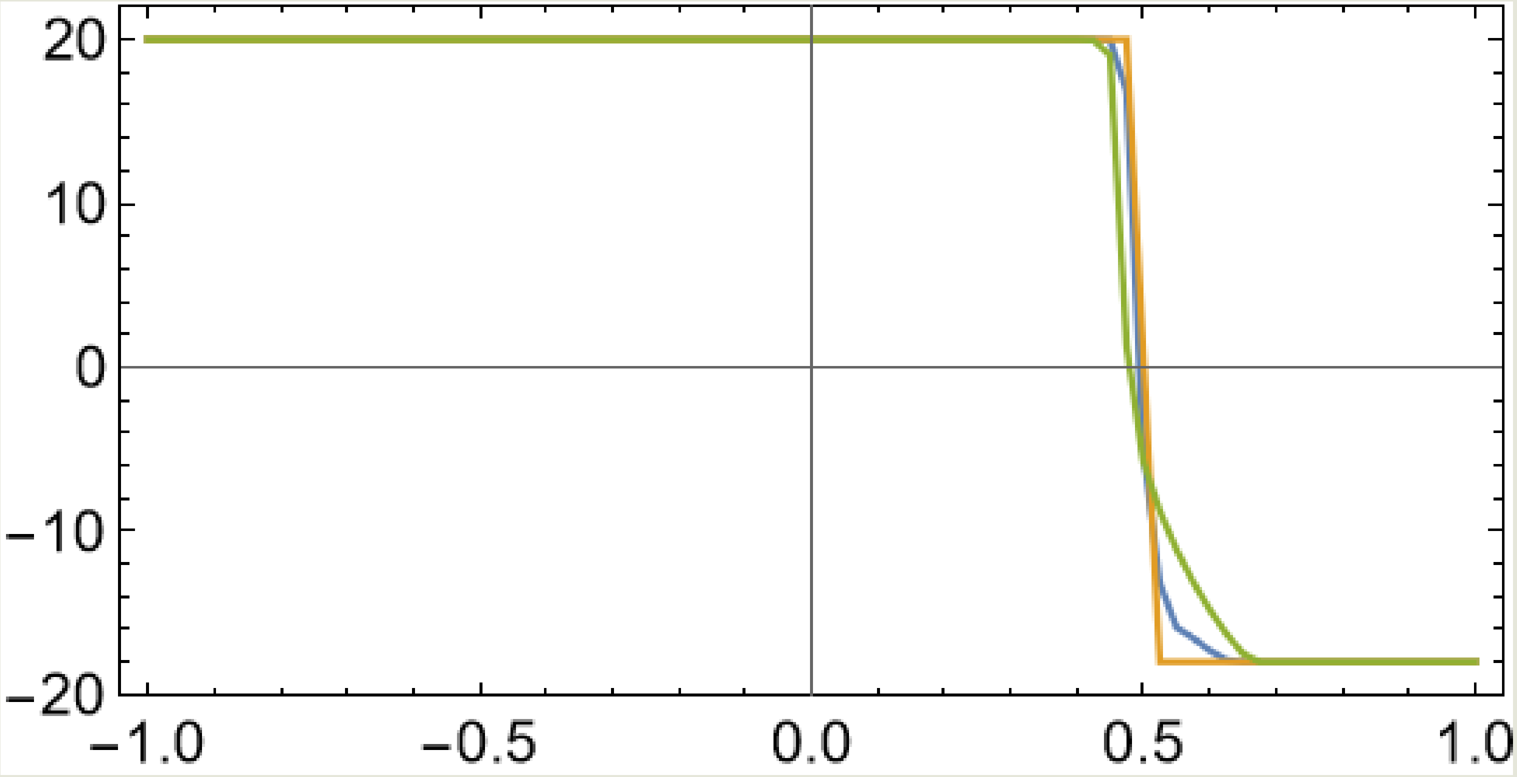
- **Slowly moving shock**
  - Max CN=10., T=1, h=2/40
  - orange=exact, blue=2nd order, green=1st order



Numerical experiments

$$\partial_t u + \partial_x f(u) = 0\,, \quad f(u) = \frac{u^2}{2}$$

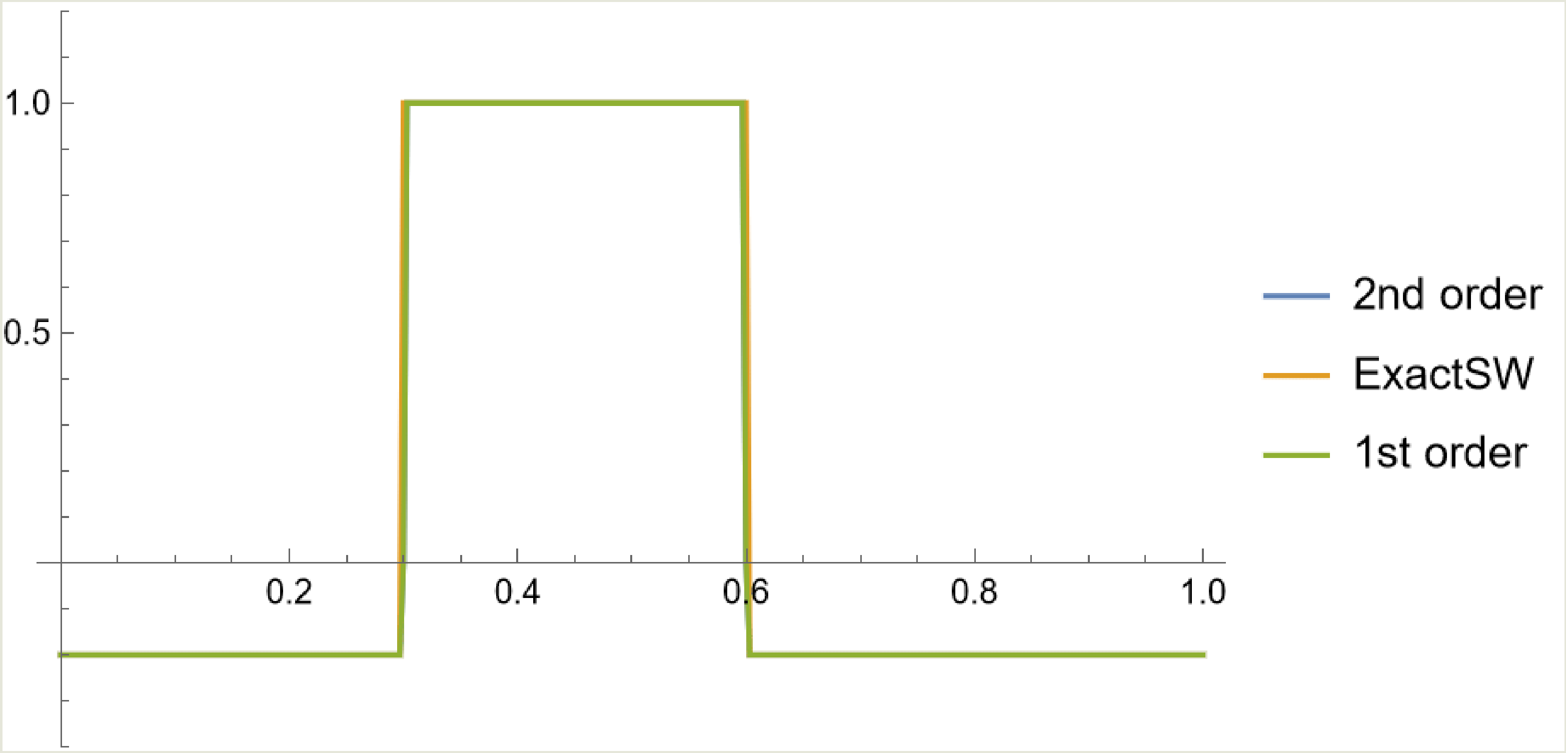
- **Slowly moving shock**
  - Max CN=20., T=1, h=2/40
  - orange=exact, blue=2nd order, green=1st order



Numerical experiments

$$\partial_t u + \partial_x f(u) = 0\,, \qquad f(u) = \frac{u^2}{2}$$

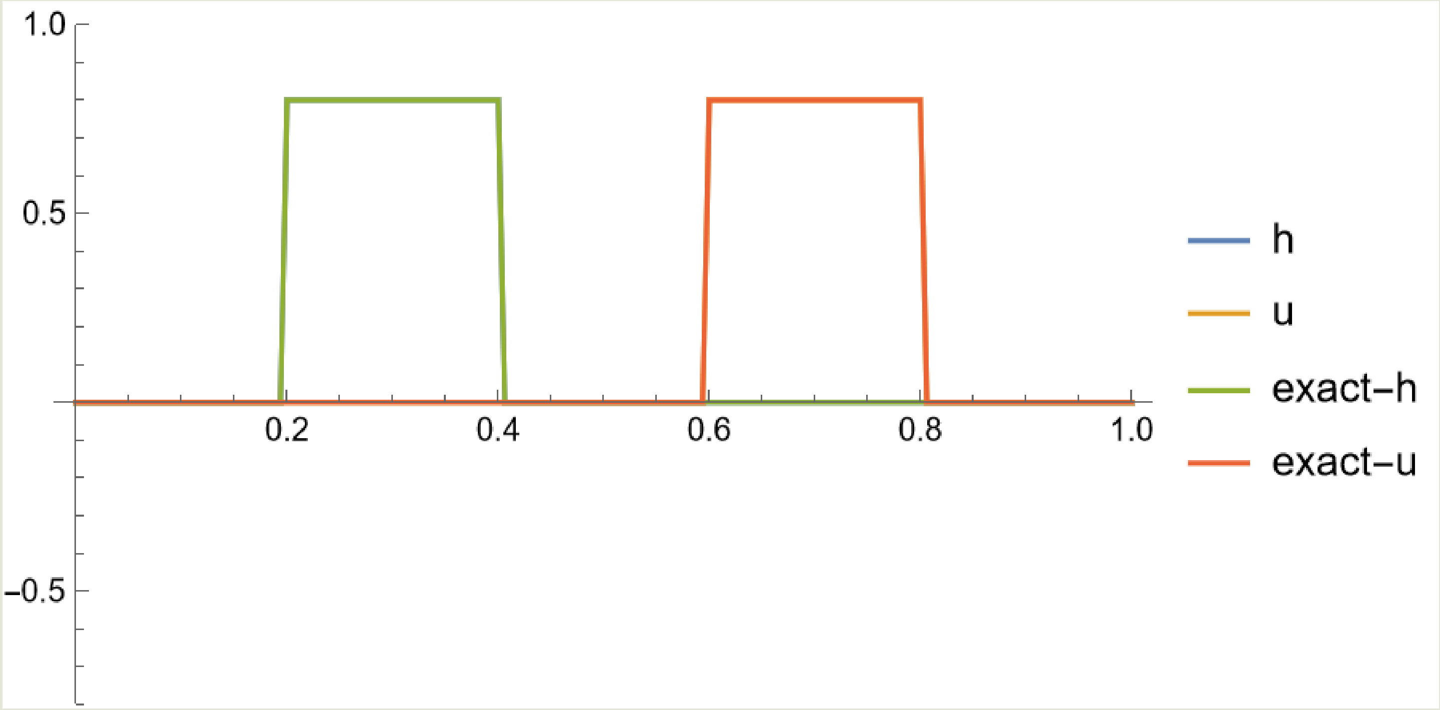
- **"Complex" example [Lozano, Aslam, 2021]**
  - Max CN=4., T=1, h=1/320
  - orange=exact, blue=2nd order, green=1st order



Numerical experiments

$$\partial_t u + \partial_x f(h,u) = 0\,, \qquad f = A\begin{pmatrix} h \\ u \end{pmatrix}$$

- **Linear system**
  - Max CN=8., eigenvalues 1 and 0.1
  - high-resolution 2nd order correction in characteristic variables [LeVeque]



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- **<u>Linear system</u>**
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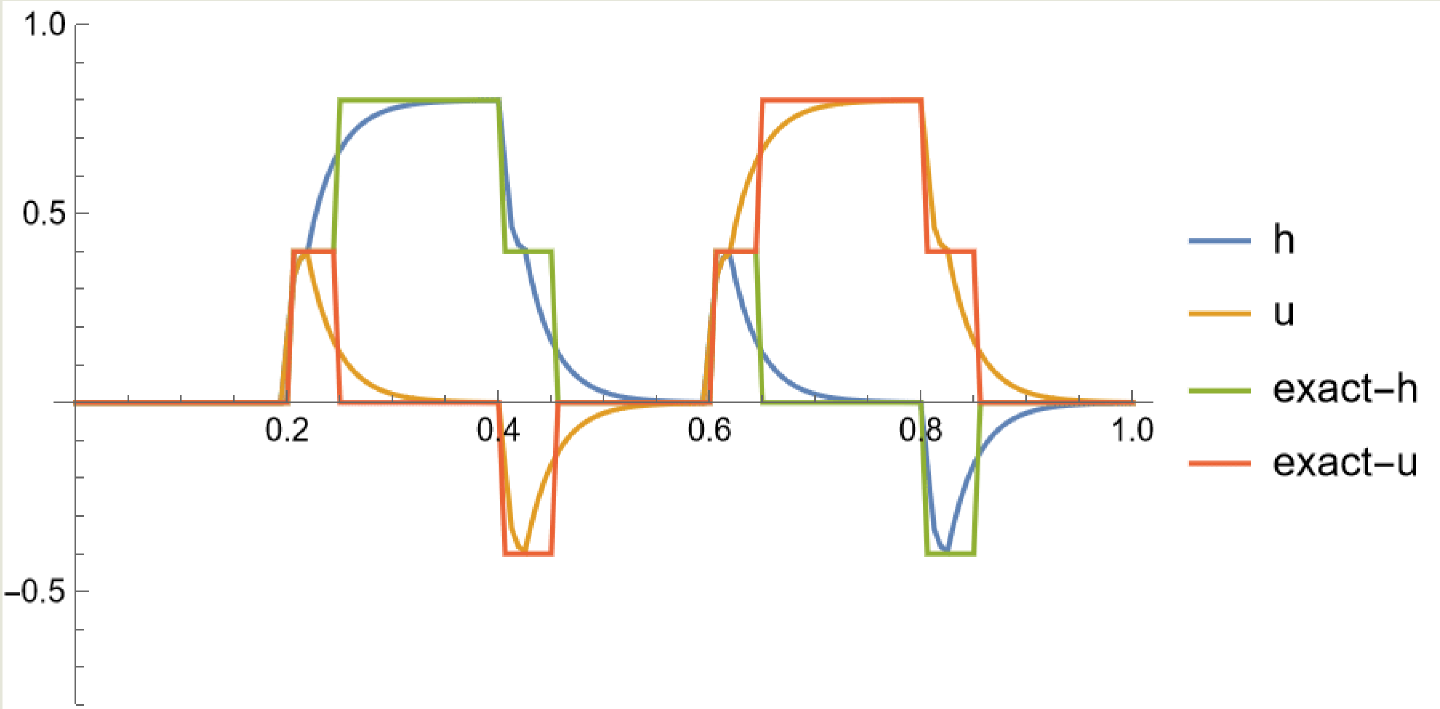

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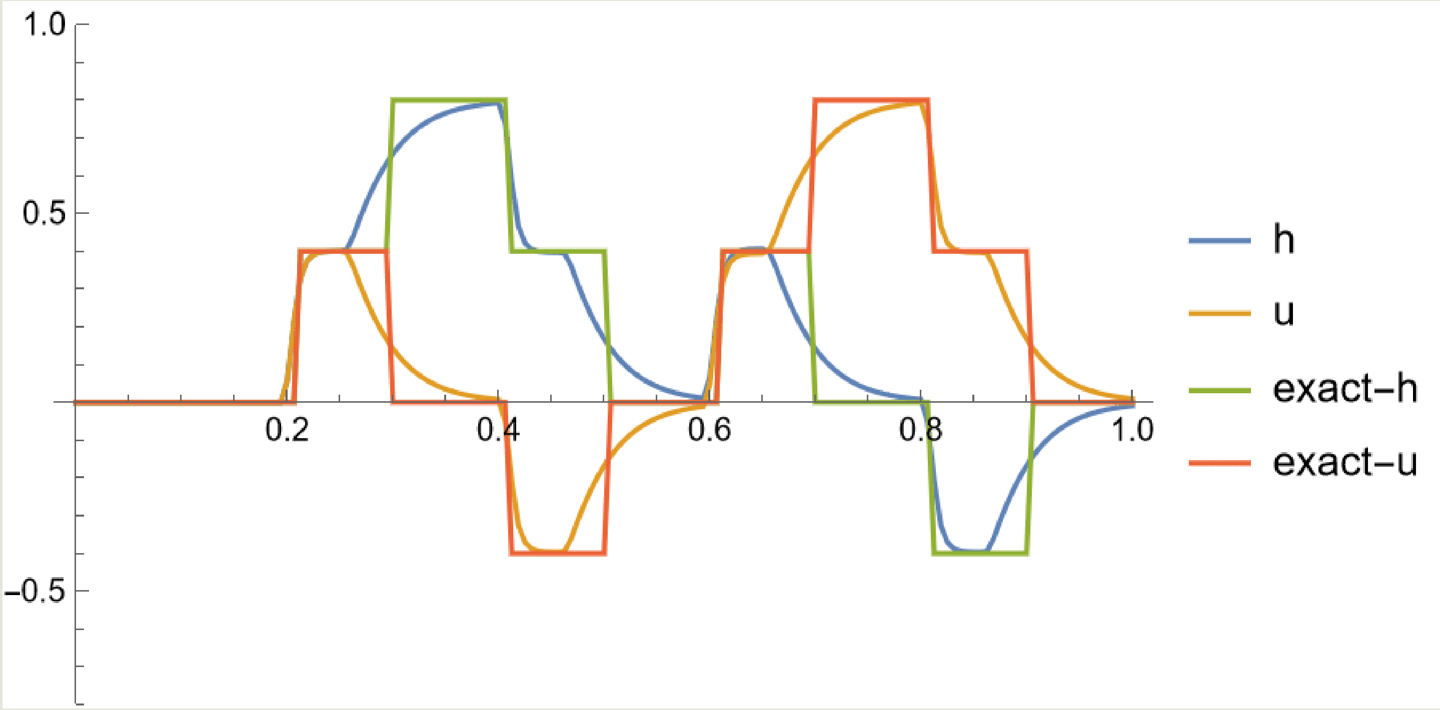

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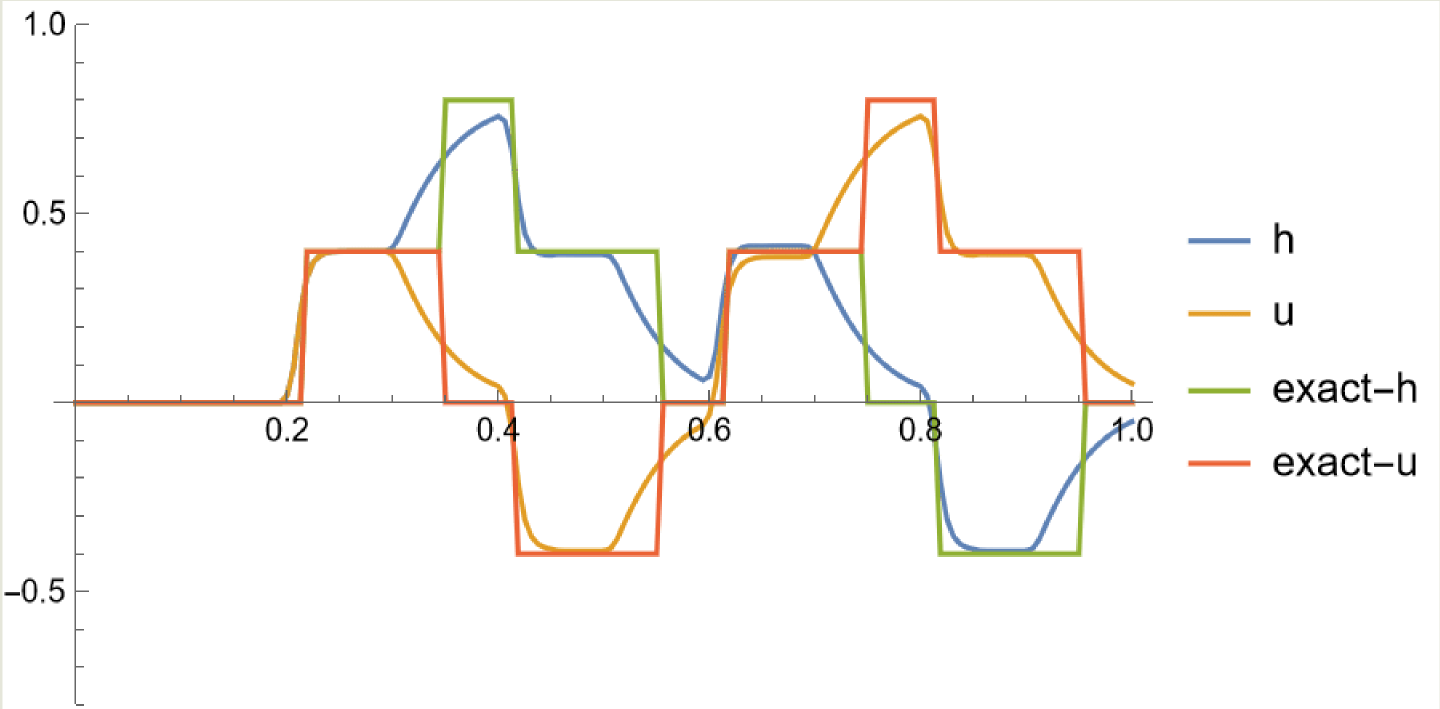

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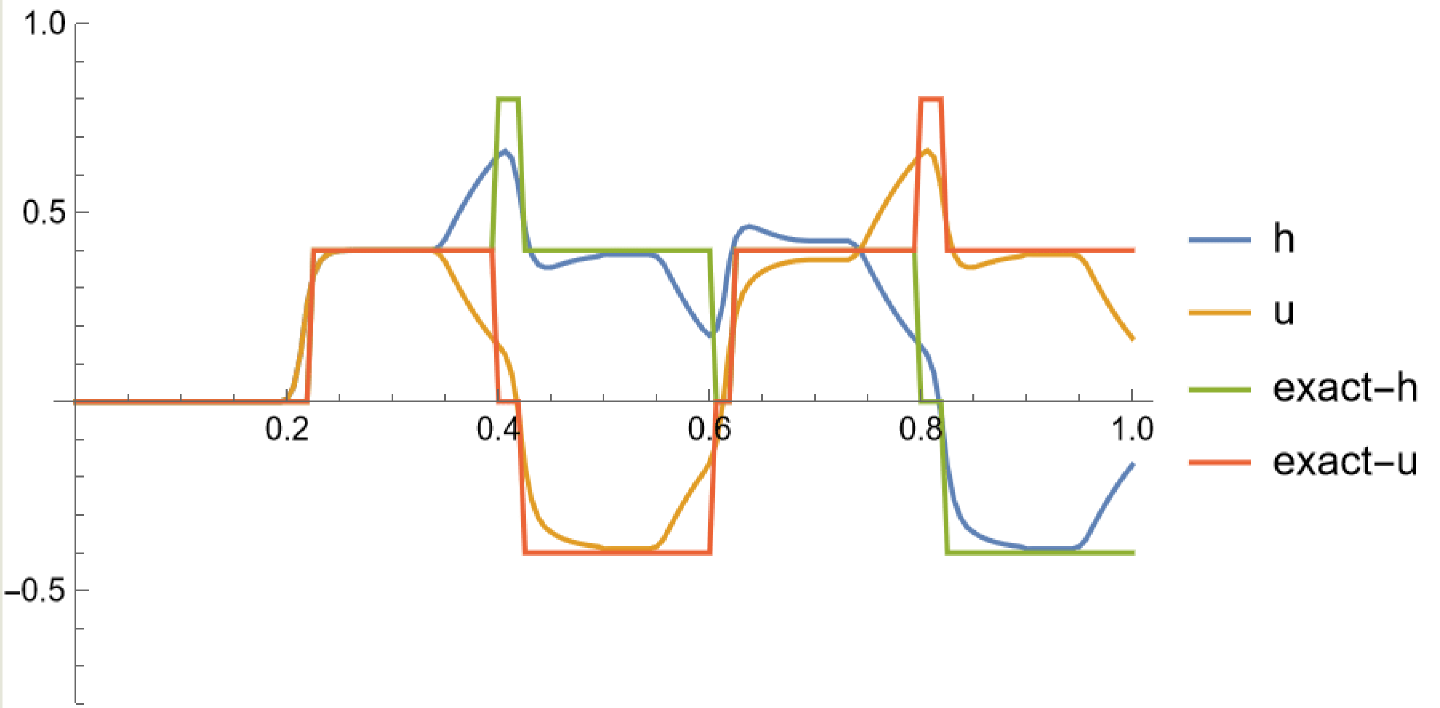

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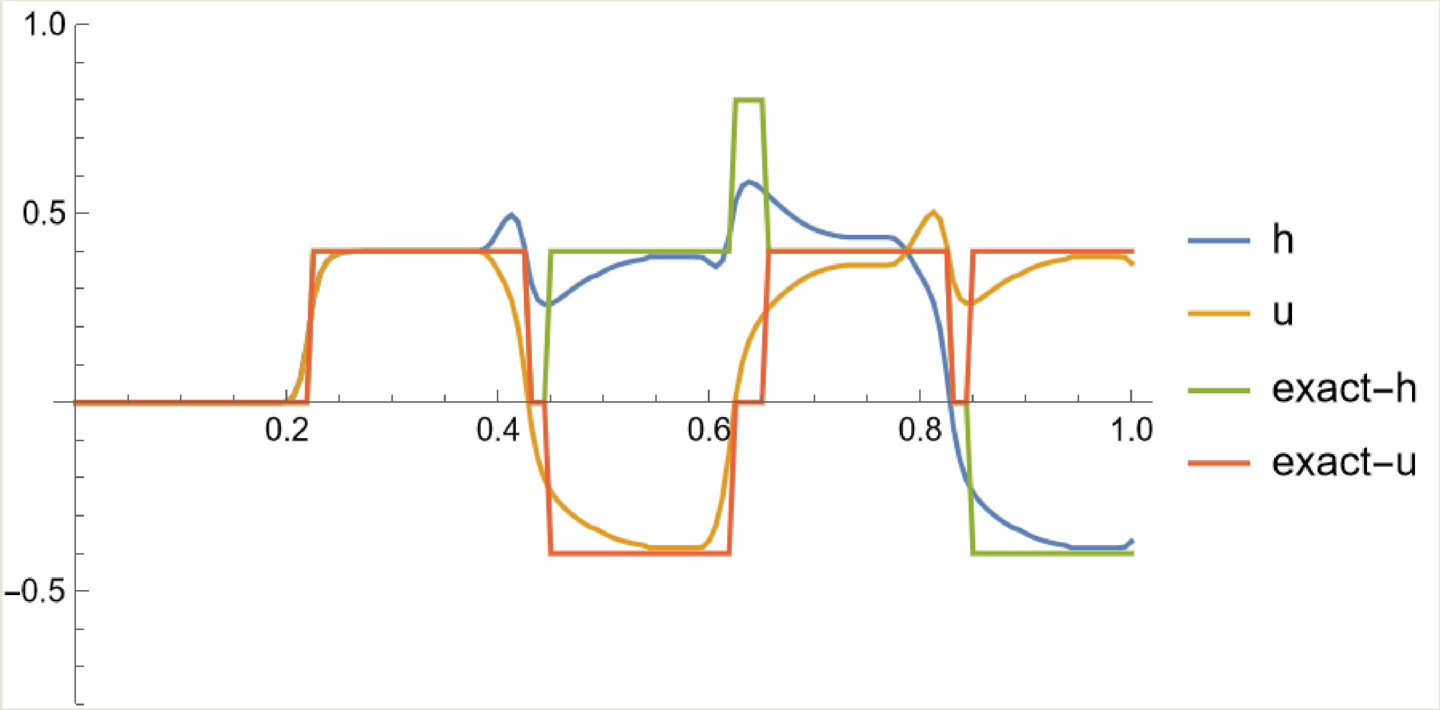

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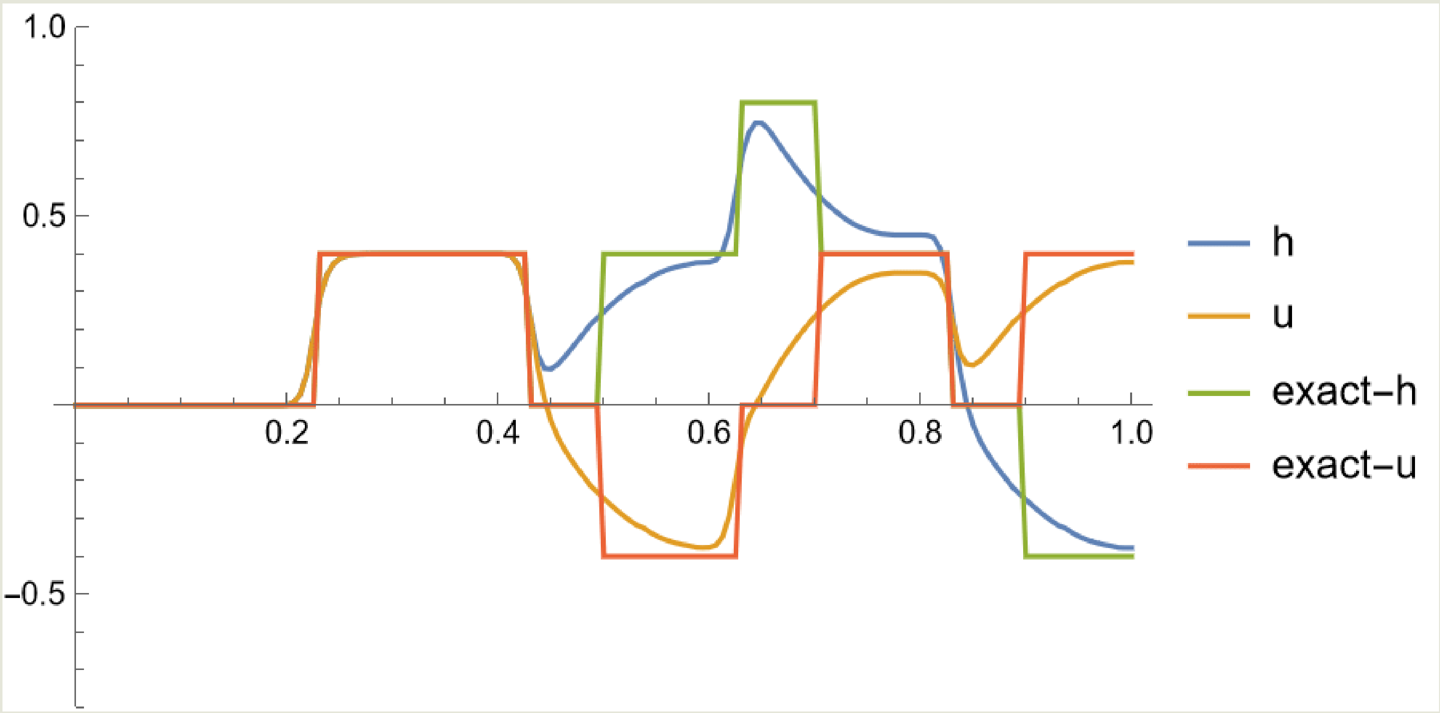

Numerical experiments

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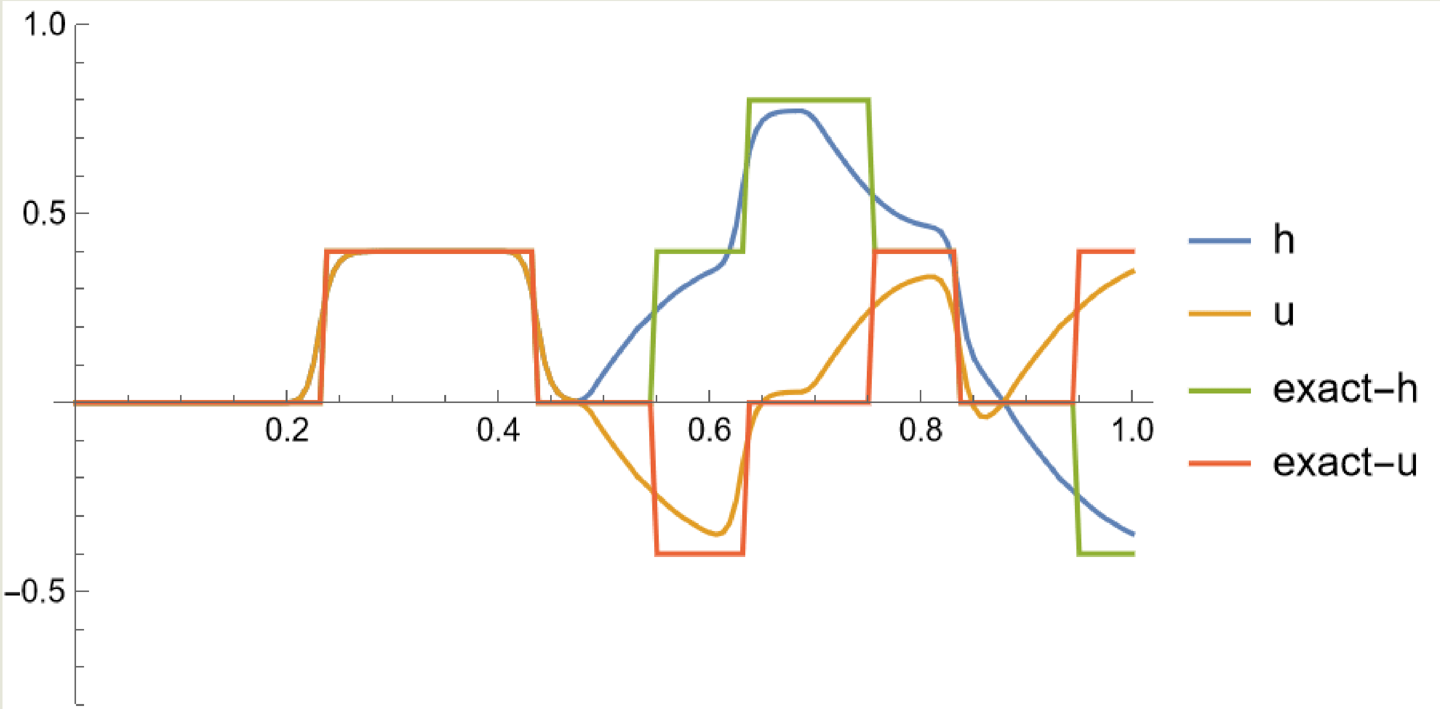
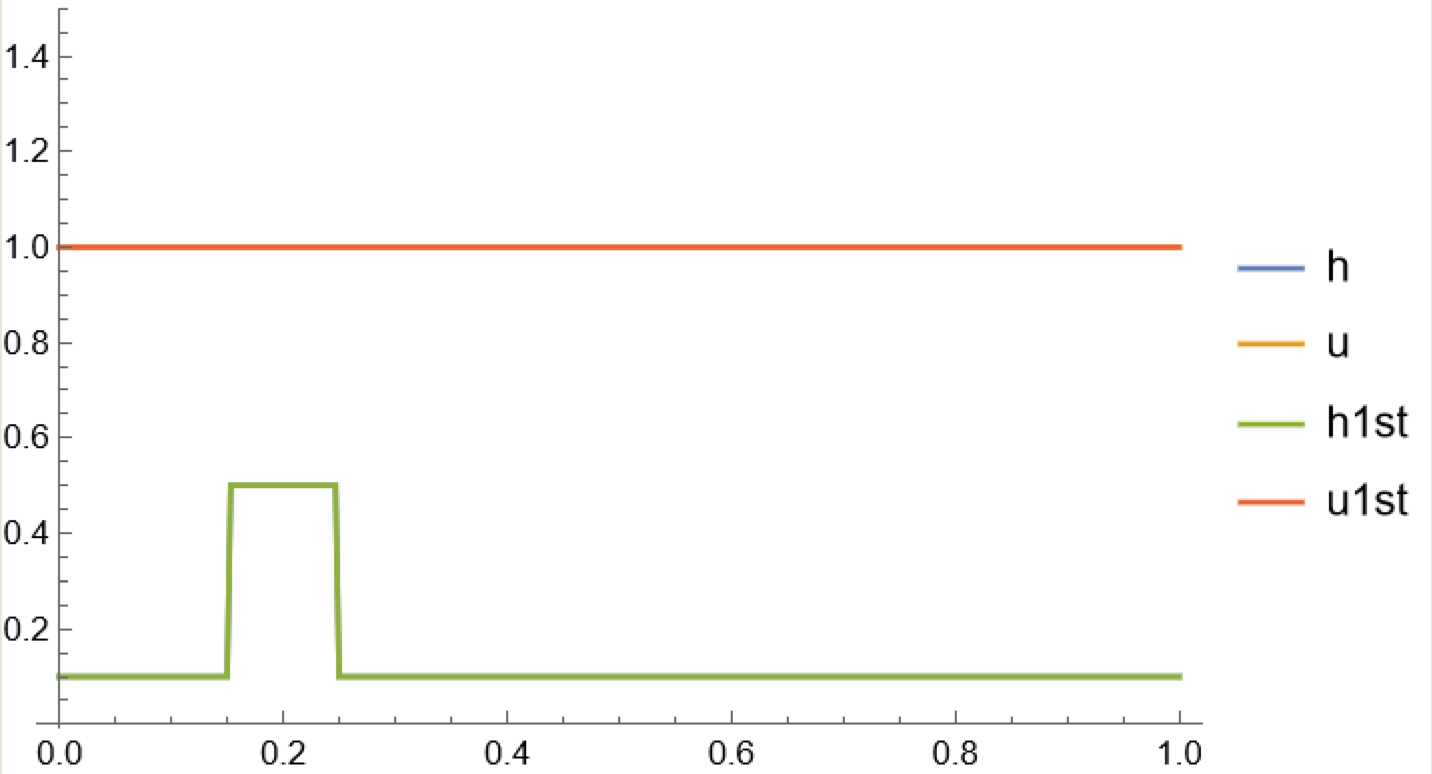


Numerical experiments

$$\partial_t u + \partial_x f(h,u) = 0\,, \qquad f = \left(hu, hu^2 + h^2/2\right)$$

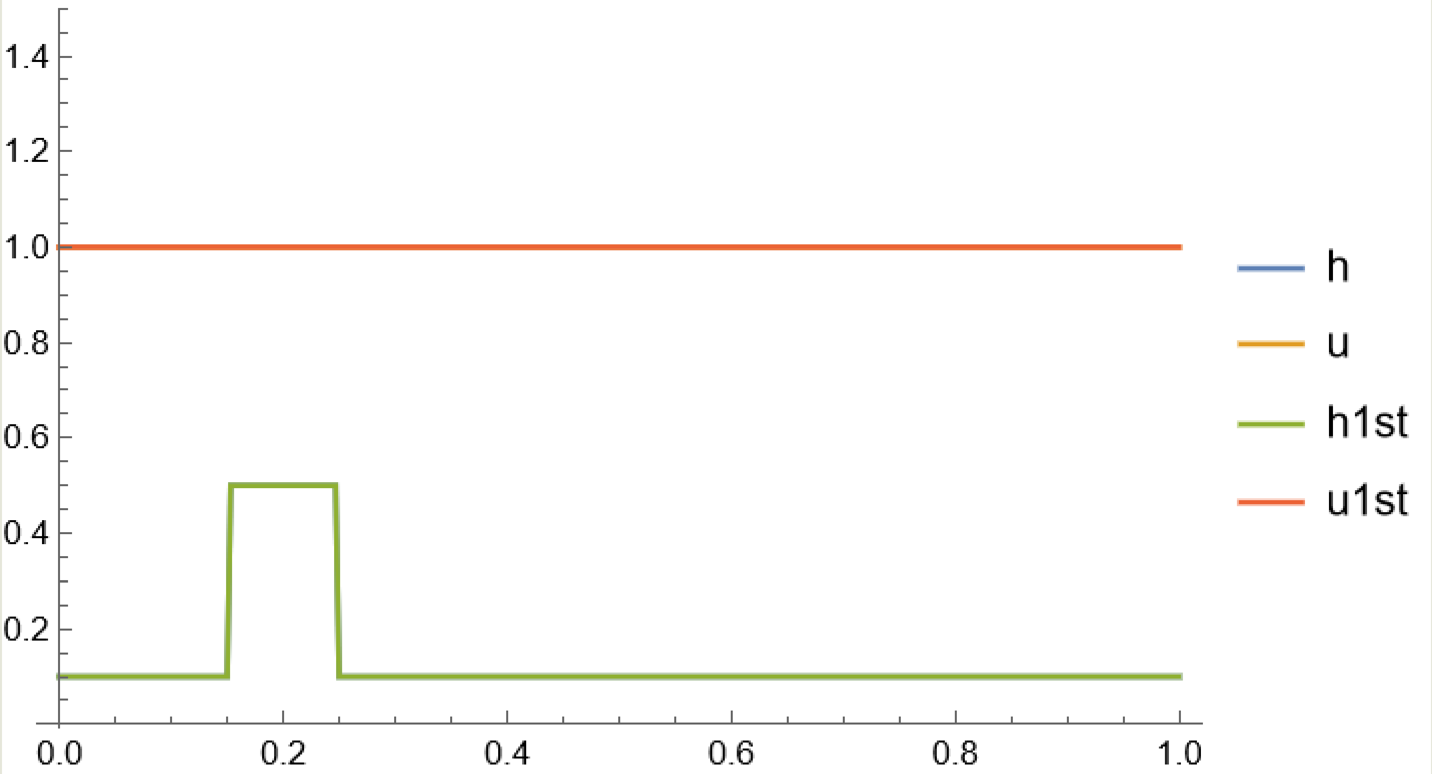
- <u>**Simple nonlinear system (SWE)**</u>
  - Max CN=1, h=1/320, comparison 1st and 2nd order scheme



Numerical experiments

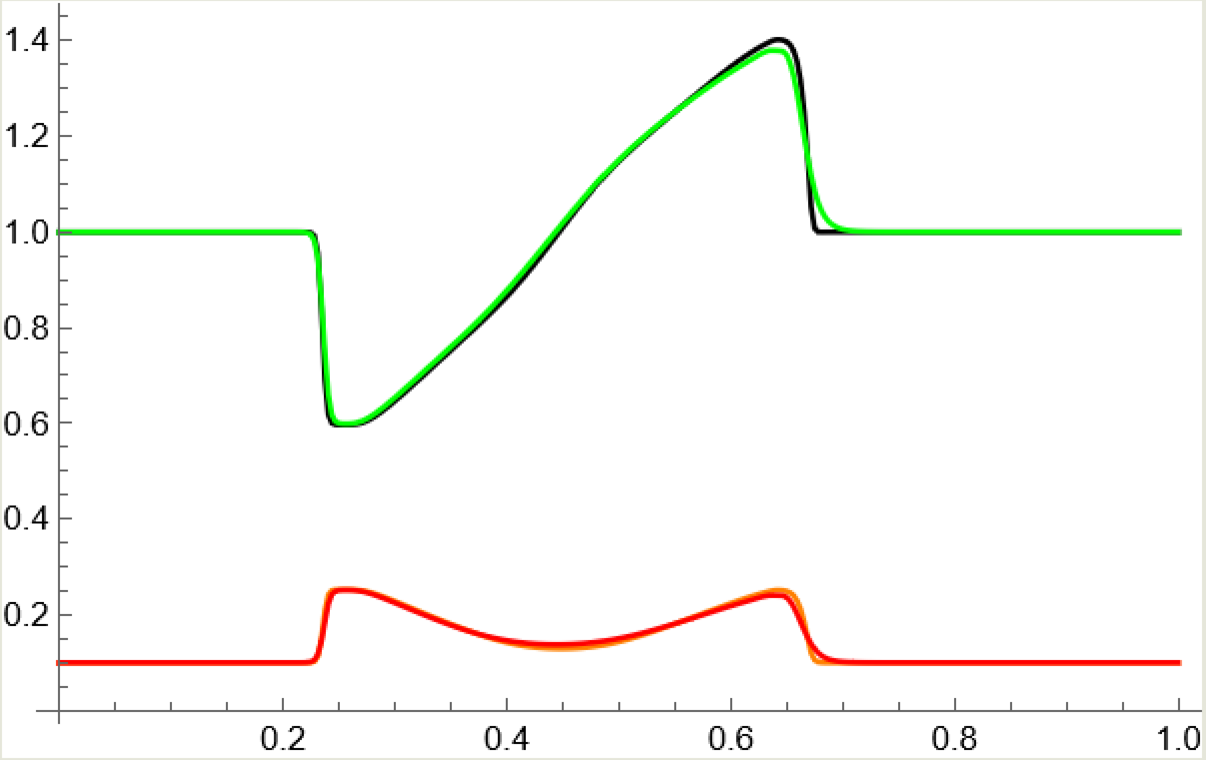
$$\partial_t u + \partial_x f(h,u) = 0\,, \qquad f = \left(hu, hu^2 + h^2/2\right)$$

- **Simple nonlinear system (SWE)**
  - Max CN=4, h=1/320, comparison 1st and 2nd order scheme
  - high-resolution 2nd order correction in characteristic variables [LeVeque]



Numerical experiments

- <u>**Simple nonlinear system (SWE)**</u>
  - Max CN=1 or 4, h=1/320, comparison at final time for the 2nd order scheme
  - high-resolution 2nd order correction in characteristic variables [LeVeque]



Conclusions, future plans, references

- one may prefer semi-implicit schemes over fully-implicit ones!
- we plan to extend it to higher order WENO type methods

Frolkovič, Mikula: *Semi-implicit second order schemes for numerical solution of level set advection equation on Cartesian grids*. Appl. Math. Comp. 2018
Frolkovič, Krišková, Rohová, Žeravý: *Semi-implicit methods for advection equations with explicit forms of numerical solution*. arXiv preprint, 2021, accepted to JJIAM
Frolkovič & Žeravý: *High-resolution compact implicit numerical schemes for conservation laws.* Arxiv

THANK YOU FOR YOUR ATTENTION

Numerical experiments

- **SOD - fresh from the owen**
  - CN=1.2, h=1/160,