From the DDFV ideas to Nodal Discrete Duality (NDD) schemes on general meshes

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joint work with

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Remarks on standard DDFV

The core properties of DDFV recalled

Some conclusions from Part I:

- DDFV appears to be quite successful in approximating gradients and fluxes, but the advantage of aproximation of the solution itself via a double/triple set of DOFs seems doubtful
- the analytical framework of DDFV schemes mainly consists of
  - meshes, discrete spaces, gradient and divergence operators
  - duality calculus for these operators
  - strong consistency for $\nabla^{\mathcal{D}}$ (exactness on affine functions), a weak (dual) consistency property for $\mathrm{div}^{\mathcal{T}}$

A remark

- DDFV (like standard finite volume schemes) uses conservative fluxes at interfaces of volumes ("local conservativity"). The local conservativity leads to discrete duality and dual consistency of $\mathrm{div}^{\mathcal{T}}$, which are core properties... itself, it's not a core property!
- In turn, Discrete Duality is a kind of conservativity property

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Three drawbacks of standard DDFV

Drawbacks of standard DDFV:

- Too many unknowns involved
(cells + vertices of the primal mesh; even worse in 3D CeVeFE),
without apparent benefit for the solution approximation
- In case of jump-discontinuous diffusion tensors,
the more sophisticated m-DDFV scheme [Boyer, Hubert'08]
has a much better convergence order than standard DDFV.
m-DDFV requires resolution of extra equations, per interface.
- The convergence analysis requires proving that $\mathfrak{M}$ and $\mathfrak{M}^*$
components converge to the same limit... For this sake,
addition of a penalization operator looking like $-h^{\mathcal{T}} \Delta^{\mathfrak{M}\cup\mathfrak{M}^*} u^{\mathcal{T}}$
[A., Bendahmane, Karlsen'10], seems to be needed in general.

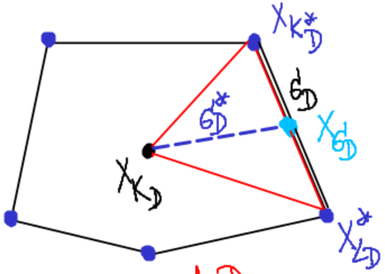
The 2D NDD scheme

...which possesses all core properties of DDFV...

(if fluxes' conservativity is omitted)

and which is free of three DDFV drawbacks!

2D NDD scheme [A., Quenjel preprint'22]



NDD diamond D = half of DDFV diamond

Primal DOFs u_{K_D} at x_{K_D}

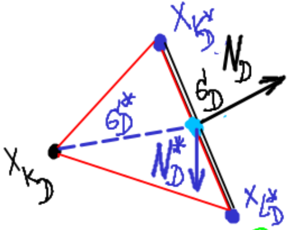
Dual DOFs $u_{K_D^*}, u_{L_D^*}$ at $x_{K_D^*}, x_{L_D^*}$

Interpolated DOF $u_{\sigma_D} = \dfrac{u_{K_D^*} + u_{L_D^*}}{2}$ at x_{σ_D}

$$\Longrightarrow \nabla_D u^{\tau} = \frac{1}{2|D|}\Big((u_{\sigma_D} - u_{K_D}) N_D + (u_{L_D^*} - u_{K_D^*}) N_D^* \Big)$$

Standard DDFV formula in NDD diamond

2D NDD scheme [A., Quenjel preprint'22]



$N_D = |\sigma_D| \, n_D$

$N_D^* = |\sigma_D^*| \, n_D^*$

Standard (?) FV discrete divergence

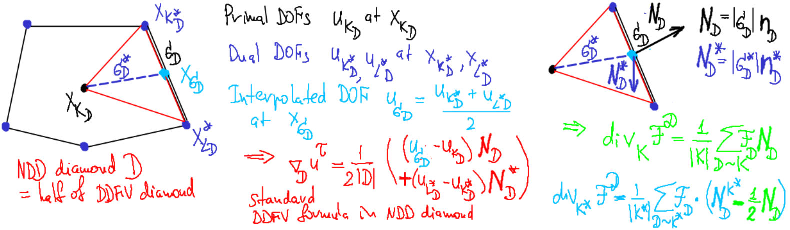
$$\Rightarrow \quad \mathrm{div}_K \mathcal{F}^{\mathcal{D}} = \frac{1}{|K|} \sum_{D \sim K} \mathcal{F}_D N_D$$

$$\mathrm{div}_{K^*} \mathcal{F}^{\mathcal{D}} = \frac{1}{|K^*|} \sum_{D \sim K^*} \mathcal{F}_D \cdot \underbrace{\left(N_D^{K^*} - \frac{1}{2} N_D \right)}$$

Non-standard due to "deviation" of $N_D^{K^*}$

deviated dual weighted normal $\tilde{N}_D^{K^*}$

2D NDD scheme [A., Quenjel preprint'22]



- NDD uses the DDFV meshes / discrete spaces / $[\![.,.]\!]$, $\{.,.\}$.
  NDD diamonds $\mathcal{D} \in \mathfrak{D}$ are half-diamonds, compared to DDFV.
- The NDD Discrete gradient in defined in the usual DDFV way,
  with the interpolated interface value $u_{\sigma_{\mathcal{D}}} := \frac{1}{2}\left(u_{K^\star_{\mathcal{D}}} + u_{L^\star_{\mathcal{D}}}\right)$,
- Per-diamond weighted normals: the usual ones $\mathbf{N}_{\mathcal{D}}$, $\mathbf{N}^{K^\star}_{\mathcal{D}} = \pm\mathbf{N}^\star_{\mathcal{D}}$
  but also the deviated dual normal $\widetilde{\mathbf{N}}^{K^\star}_{\mathcal{D}} := \mathbf{N}^{K^\star}_{\mathcal{D}} - \frac{1}{2}\mathbf{N}_{\mathcal{D}}$
- NDD Discrete Divergence is seemingly standard, up to two issues

$$\mathrm{div}_K \mathcal{F}^{\mathfrak{D}} := \frac{1}{|K|}\sum_{\mathcal{D}\sim K} \mathcal{F}_{\mathcal{D}} \cdot \mathbf{N}_{\mathcal{D}} \qquad \mathrm{div}_{K^\star} \mathcal{F}^{\mathfrak{D}} := \frac{1}{|K^\star|}\sum_{\mathcal{D}\sim K^\star} \mathcal{F}_{\mathcal{D}} \cdot \widetilde{\mathbf{N}}^{K^\star}_{\mathcal{D}}$$

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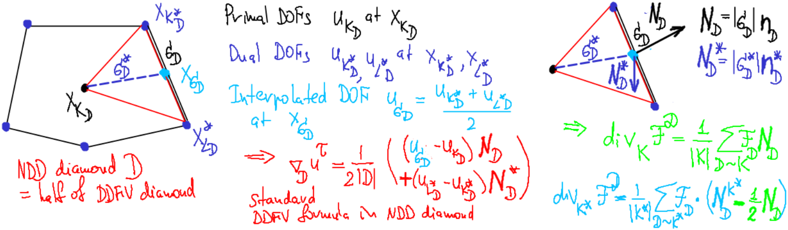

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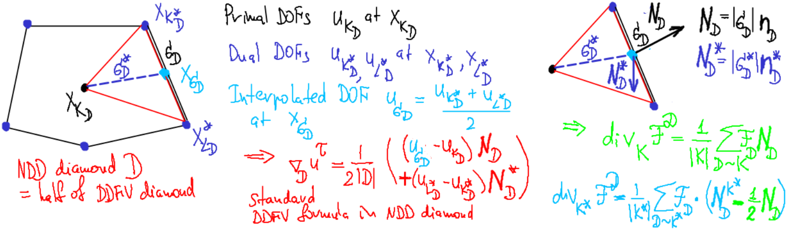

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Pecularities and advantages of NDD

- Obvious exactness of Discrete Gradient (it is the FE gradient!)
- Discrete fields have jumps across primal interfaces
  $\rightsquigarrow$ Entries div_K of NDD Discrete Divergence look standard
  but the fluxes are non-conservative (against the FV orthodoxy !)
- Entries $\mathrm{div}_{K^\star}$ of the NDD Discrete Divergence are non-obvious:
  a "deviated normal" $\widetilde{\mathbf{N}}_{\mathcal{D}}^{K^\star}$ appears in the place of expected $\mathbf{N}_{\mathcal{D}}^{K^\star}$
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- Actually the expression of "deviated normal" $\widetilde{\mathbf{N}}_{\mathcal{D}}^{K^\star}$
  is taylored for proving Discrete Duality (a mimetic inspiration!).
  Indirectly, weak consistency of Discrete Divergence follows.
- NDD is free from the three above indicated drawbacks of DDFV!
  - the cell DOFs $(u_K)_{K\in\mathfrak{M}}$ are not coupled $\rightsquigarrow$ eliminated algebraically via Schur complement (within each Newton iteration, if nonlinearity)
  - discontinuity of diffusion tensor across $K|L$ does not impact convergence orders because each $\mathcal{D}$ is contained within one cell
  - compactness claim ($u^{\mathfrak{M}}, u^{\mathfrak{M}^\star} \to u$) is true without penalization because $\nabla_{\mathcal{D}} u^{\overline{\mathcal{T}}}$ controls $u_K - \frac{1}{2}(u_{K^\star} + u_{L^\star})$

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Discrete Duality proved [A., Quenjel preprint'22]

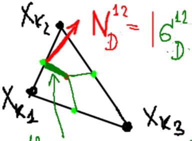
...

The 3D full-diamond NDD
(uses 3D CeVe-DDFV gradient)

2D CVFE (co-volume) scheme gradient & 3D CeVe-DDFV gradient recalled.

Recall CVFE gradient

$$\nabla_D u^{\mathcal{T}} = \frac{2}{|D|}\Big((u_{K_2}-u_{K_1})N_D^{12} + (u_{K_3}-u_{K_2})N_D^{23} + (u_{K_1}-u_{K_3})N_D^{31} \Big)$$



$N_D^{12} = |\sigma_D^{12}|\, n_D^{12}$

σ_D^{12} part of the interface $K_1|K_2$ contained within diamond D

Claims (the 2nd "magical formula") [A., Bendahmane, Hubert, Krell IMAJNA'12]

- this formula represents the gradient of the affine per D (Finite Element) reconstruction from DOFs $(u_K)_{K\in\mathcal{T}}$
  ⇝ exactness of $\nabla^{\mathcal{D}}$
- exactness remains true if $\mathcal{D}$ has >3 vertices
- this formula is used for 3D CeVe-DDFV, it ensures the DD

Discrete Gradient for the 3D full-diamond NDD scheme

Recall CVFE gradient

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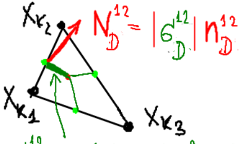

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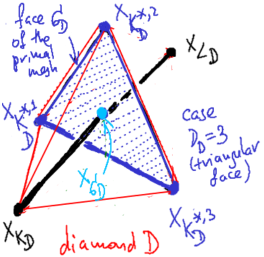
3D fd-NDD gradient



$x_{K_{\mathcal{D}}}$, $x_{L_{\mathcal{D}}}$ cell centers

$x_{K_{\mathcal{D}}^{*,i}}$, $i=1,\dots,\not{D}_{\mathcal{D}}$ neighbour vertices of the cell face $\sigma_{\mathcal{D}}$ $\equiv$ dual cell centers

$x_{\sigma_{\mathcal{D}}}$ the face barycenter

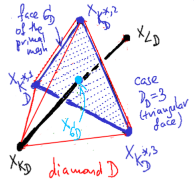
$u_{\sigma_{\mathcal{D}}} = \frac{1}{\not{D}_{\mathcal{D}}} \sum_{i=1}^{\not{D}_{\mathcal{D}}} u_{K_{\mathcal{D}}^{*,i}}$ interpolated value

$\nabla_{\mathcal{D}} u^{\mathcal{T}}$ = the 3D CeVe-DDFV gradient

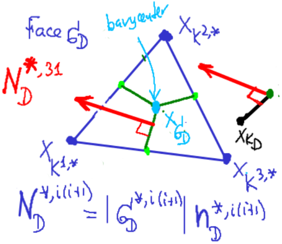
Discrete Divergence for the 3D full-diamond NDD scheme

The 3D fd-NDD gradient
is obtained like 3D-CeVe-DDFV

NB: coercivity OK if triangular faces
or topologically cartesian mesh



3D fd-NDD divergence adapted to $[\![\cdot,\cdot]\!] = \frac{1}{3}\sum_K \cdots + \frac{2}{3}\sum_K \cdots$



$N_D\,,\; N_D^{K^*} = \pm N_D^{*,i(i+1)}$

standard weighted normals

$\widetilde{N}_D^{K^*} = N_D^{K^*} - \frac{1}{4 D_D} N_D$

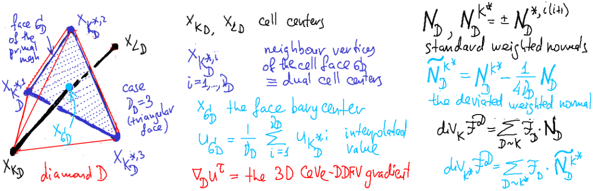
the deviated weighted normal

$div_K \mathcal{F}^D = \sum_{D\sim K} \mathcal{F}_D \cdot N_D$

$div_{K^*} \mathcal{F}^D = \sum_{D\sim K^*} \mathcal{F}_D \cdot \widetilde{N}_D^{K^*}$

Claim: all the core features of DDFV hold for this "3D fd-NDD" scheme

3D full-diamond NDD construction



Summary of 3D full-diamond NDD

- A 3D scheme with no face/edge unknowns and with rapid algebraic elimination of cell unknowns $\rightsquigarrow$ in practice, a Nodal Scheme complexity
- A concecrated discrete gradient but exotic dicrete divergence; loss of flux conservation (but conservativity via Dicrete Duality)
- Exotic divergence is a mimetic one $\rightsquigarrow$ Discrete Duality OK!
- Consistencies, compactness OK / "*m*-NDD", penalization not needed
- Coercivity not OK if faces have general shape

Q. Can one find for a 3D NDD scheme with unconditional coercivity ?

Q. 3D fd-NDD uses the CeVe-DDFV (Pierre/Hermeline/ABHK) gradient. What about the CeVeFE (Coudière-Hubert) gradient idea?

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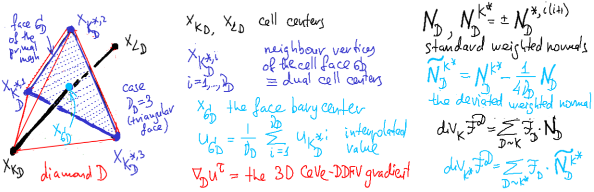

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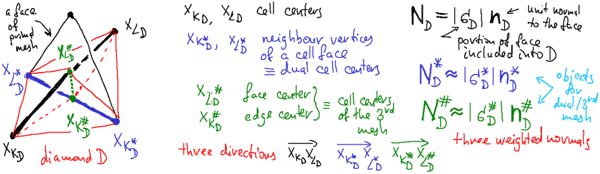
The 3D split-diamond NDD
(uses 3D CeVeFE-DDFV gradient)

...here Discrete Divergence gets wildly exotic...

The 3D CeVeFE-DDFV gradient recalled

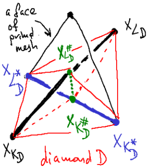
CeVeFE-DDFV gradient



$$\nabla_{\mathcal{D}} u^{\mathcal{T}} := \frac{1}{3|\mathcal{D}|}\Big((u_{L_{\mathcal{D}}} - u_{K_{\mathcal{D}}})\mathbf{N}_{\mathcal{D}} + (u_{L^{\star}_{\mathcal{D}}} - u_{K^{\star}_{\mathcal{D}}})\mathbf{N}^{\star}_{\mathcal{D}} + (u_{L^{\#}_{\mathcal{D}}} - u_{K^{\#}_{\mathcal{D}}})\mathbf{N}^{\#}_{\mathcal{D}} \Big)$$

Discrete Gradient for the 3D split-diamond NDD scheme

CeVeFE-DDFV gradient



X_{K_D}, X_{L_D} cell centers

$X_{K_D}^*$, $X_{L_D}^*$ neighbour vertices of a cell face ≡ dual cell centers

$X_{L_D}^{\#}$ face center, $X_{K_D}^{\#}$ edge center } = cell centers of the 3rd mesh

three directions $\overrightarrow{X_{K_D}X_{L_D}}$ $\overrightarrow{X_{K_D}^*X_{L_D}^*}$ $\overrightarrow{X_{K_D}^{\#}X_{L_D}^{\#}}$

$N_D = |\sigma_D|\, n_D$ ← unit normal to the face; $|\sigma_D|$: portion of face included into D

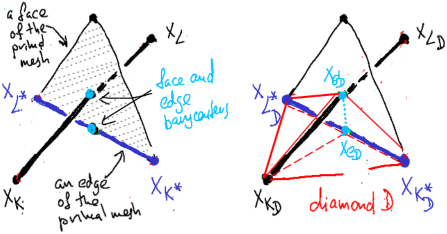
$N_D^* \approx |\sigma_D^*|\, n_D^*$

$N_D^{\#} \approx |\sigma_D^{\#}|\, n_D^{\#}$ } objects for dual / 3rd mesh

three weighted normals

$$\nabla_{\mathcal{D}} u^{\mathcal{T}} := \frac{1}{3|\mathcal{D}|}\Big((u_{L_{\mathcal{D}}} - u_{K_{\mathcal{D}}})\mathbf{N}_{\mathcal{D}} + (u_{L^\star_{\mathcal{D}}} - u_{K^\star_{\mathcal{D}}})\mathbf{N}^\star_{\mathcal{D}} + (u_{L^\#_{\mathcal{D}}} - u_{K^\#_{\mathcal{D}}})\mathbf{N}^\#_{\mathcal{D}} \Big)$$

3D sd-NDD gradient



Interpolated values $u_{e_D} = \frac{1}{2}(u_{K_D^*} + u_{L_D^*})$

$u_{\sigma_D} = \frac{1}{|K_D|} \sum_{K^* \sim K_D} u_{K^*}$

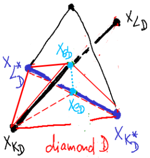
⇒ 3D split-diamond NDD gradient

$$\nabla_D u^{\mathcal{T}} = \frac{1}{3|D|}\Big((u_{\sigma_D} - u_{K_D}) N_D + (u_{L_D^*} - u_{K_D^*}) N_D^* + (u_{e_D} - u_{\sigma_D}) N_D^{\#} \Big)$$

Ready for the 3D split-diamond NDD Discrete Divergence?

The 3D sd-NDD gradient
is obtained like 3D CeVeFE-DDFV

NB: coercivity always OK



3D sd-NDD divergence (!) adapted to $[\![.,.]\!] = \frac{1}{3}\sum_K \cdots + \frac{2}{3}\sum_K$ (!)

N_D, $N_D^{K^*} = \pm N_D^*$, $N_D^\#$ are the usual CeVeFE weighted normals

two deviated weighted normals:
$\tilde{N}_D^{K^*} = \frac{1}{2} N_D^{K^*} - \frac{1}{4} N_D$, $\hat{N} = \frac{1}{2 D_D}\left(-N_D + N_D^\#\right)$

Usual $\mathrm{div}_K \mathcal{F}^D = \frac{1}{|K|}\sum_{D\sim K} \mathcal{F}_D \cdot N_D$

Tricky $\mathrm{div}_{K^*} \mathcal{F}^D =$
$= \frac{1}{|K^*|}\left(\sum_{D\sim K^*} \mathcal{F}_D \cdot \tilde{N}_D^{K^*} + \sum_{\substack{\text{all diamonds}\\ D'\sim K_D}} \mathcal{F}_{D'} \cdot \hat{N}_{D'}\right)$

Claim: all the core features of DDFV hold for this "3D sd-NDD" scheme

Conclusions

- NDD enters the analytical framework of DDFV schemes:
  - meshes, discrete spaces, gradient and divergence operators
  - duality calculus for these operators
  - strong consistency for $\nabla^{\mathcal{D}}$, dual consistency for $\mathrm{div}^{\mathcal{T}}$
    $\rightsquigarrow$ compactness, reconstruction

  but it lacks local conservation
- NDD has the following advantages over DDFV
  - rapid elimination of cell unknowns; in 3D, no face+edge unknowns
    $\rightsquigarrow$ in practice, NDD can be seen as a *nodal* scheme
  - direct use on piecewise continuous diffusion tensors (no need for "m-NDD")
  - no need to penalize $u^{\mathfrak{M}} - u^{\mathfrak{M}^*}$
- there are two different kinds of 3D DDFV schemes
  $\rightsquigarrow$ 3D fd-NDD, coercive if faces have 3 or 4 vertices
  $\rightsquigarrow$ 3D sd-NDD, more cumbersome but unconditionally coercive
- the 2D co-volume scheme on general meshes is non-coercive; the split-diamond approach offers a coercive variant (in progress)

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Thank you / Merci !