A survey of the basics of 3D DDFV schemes, and the 2D CVFE (co-volume) scheme

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based on works of

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F. Hermeline ;
Y. Coudière & F. Hubert ;
M. Bendahmane & K. Karlsen & B.A. ;
F. Hubert & S. Krell & B.A. ;
J. Droniou & R. Eymard & R. Herbin & C. Guichard

Workshop “DDFV Method and Applications”
CIRM, Luminy, Oct. 2022

Main features of 2D DDFV

(having in mind the calculus of gradient schemes)

2D DDFV meshes and functional framework.

- Domain Ω partitioned twice by double mesh $\mathcal{T} = (\mathfrak{M}, \mathfrak{M}^\star)$ consisting of a primal mesh $\mathfrak{M}$ + its dual mesh $\overline{\mathfrak{M}^\star} = \mathfrak{M}^\star \cup \partial\mathfrak{M}^\star$;
  $\rightsquigarrow$ discrete function $u^{\mathcal{T}} = \left(u^{\mathfrak{M}}, u^{\mathfrak{M}^\star} \right)$, $u^{\mathfrak{M}} = (u_K)_{K\in\mathfrak{M}}$, idem $u^{\mathfrak{M}^\star}$
  $\rightsquigarrow$ space $\mathbb{R}^{\mathcal{T}}$ of discrete functions, scalar product

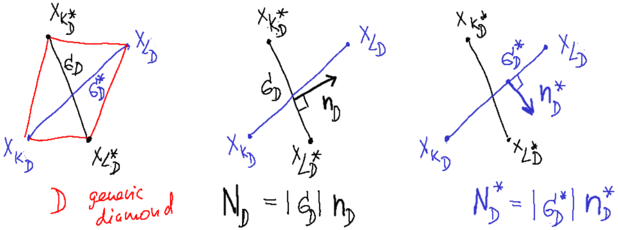
$$[\![u^{\mathcal{T}}, v^{\mathcal{T}}]\!] := \frac{1}{2} \sum_{K\in\mathfrak{M}} |K|\, u_K v_K + \frac{1}{2} \sum_{K^\star\in\mathfrak{M}^\star} |K^\star|\, u_{K^\star} v_{K^\star}$$

  $\rightsquigarrow$ zero boundary values: up-to-the-boundary space $\mathbb{R}_0^{\overline{\mathcal{T}}}$
- Diamond mesh $\mathfrak{D}$ $\rightsquigarrow$ discrete fields $\mathcal{F}^{\mathfrak{D}} = \left(\mathcal{F}_{\mathcal{D}} \right)_{\mathcal{D}\in\mathfrak{D}}$
  $\rightsquigarrow$ Space $(\mathbb{R}^2)^{\mathfrak{D}}$ of discrete fields, scalar product

$$\left\{ \mathcal{F}^{\mathfrak{D}}, \mathcal{G}^{\mathfrak{D}} \right\} := \sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}|\, \mathcal{F}_{\mathcal{D}} \cdot \mathcal{G}_{\mathcal{D}}$$

- discrete gradient $\nabla^{\mathfrak{D}} : \mathbb{R}_0^{\overline{\mathcal{T}}} \longrightarrow (\mathbb{R}^2)^{\mathfrak{D}}$
  discrete divergence $\mathrm{div}^{\mathcal{T}} : (\mathbb{R}^2)^{\mathfrak{D}} \longrightarrow \mathbb{R}^{\mathcal{T}}$
- projections of $C_c(\Omega)$ functions / of $C_c(\Omega; \mathbb{R}^2)$ fields:
  $\mathbb{P}^{\overline{\mathcal{T}}}$ (projection on $\mathbb{R}_0^{\overline{\mathcal{T}}}$), $\mathbb{P}^{\mathfrak{D}}$ (projection on $(\mathbb{R}^2)^{\mathfrak{D}}$)

Notation in a diamond. Discrete gradient. Exactness on affine functions.



- diamonds $\mathcal{D} \in \mathfrak{D}$ used as generic label.
  Formalism of per-diamond weighted normals: $\mathbf{N}_\mathcal{D}$,$\mathbf{N}^\star_\mathcal{D}$
- Discrete gradient defined per diamond by

$$\nabla_\mathcal{D} u^{\mathcal{T}} := \frac{1}{2|\mathcal{D}|}\Big((u_{L_\mathcal{D}} - u_{K_\mathcal{D}})\mathbf{N}_\mathcal{D} + (u_{L^\star_\mathcal{D}} - u_{K^\star_\mathcal{D}})\mathbf{N}^\star_\mathcal{D}\Big)$$

- Per-projection representation, explaining the definition:

$$\nabla_\mathcal{D} u^{\mathcal{T}} \cdot \overrightarrow{x_{K_\mathcal{D}} x_{L_\mathcal{D}}} = u_{L_\mathcal{D}} - u_{K_\mathcal{D}} \qquad \nabla_\mathcal{D} u^{\mathcal{T}} \cdot \overrightarrow{x_{K^\star_\mathcal{D}} x_{L^\star_\mathcal{D}}} = u_{L^\star_\mathcal{D}} - u_{K^\star_\mathcal{D}}$$

- Exactness: if $\forall x \in \mathcal{D}$, $\nabla u(x) \equiv \mathcal{G} = const$, then $\nabla_\mathcal{D} \mathbb{P}^{\mathcal{T}} u = \mathcal{G}$

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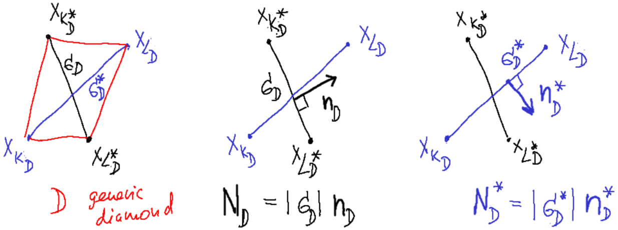

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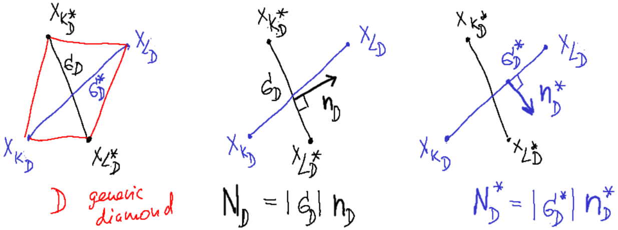

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Discrete divergence. Discrete Duality.

- Notation: $\mathcal{D} \sim K$ if $\sigma_{\mathcal{D}} \subset \partial K$; idem $\mathcal{D} \sim K^\star$ if $\sigma^\star_{\mathcal{D}} \subset \partial K^\star$
- Per-diamond weighted normals associated with cells:

  for $\mathcal{D} \sim K$, $\mathbf{N}^K_{\mathcal{D}} = \begin{cases} +\mathbf{N}_{\mathcal{D}} & \text{if } K = K_{\mathcal{D}} \\ -\mathbf{N}_{\mathcal{D}} & \text{if } K = L_{\mathcal{D}}. \end{cases}$

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- Discrete divergence defined per primal/dual cell in the FV way:

  $$\iint_K div\mathcal{F} = \int_{\partial K} \mathcal{F}\cdot\mathbf{n}^K = \sum_{\sigma\subset\partial K} \mathcal{F}|_\sigma \cdot \underbrace{|\sigma|\mathbf{n}^K}_{\mathbf{N}^K}, \quad \text{and idem for } K^\star$$

  $\Longrightarrow$

  $$\boxed{\mathrm{div}_K \mathcal{F}^{\mathfrak{D}} = \frac{1}{|K|}\sum_{\mathcal{D}\sim K} \mathcal{F}_{\mathcal{D}}\cdot\mathbf{N}^K_{\mathcal{D}}} \qquad \boxed{\mathrm{div}_{K^\star} \mathcal{F}^{\mathfrak{D}} = \frac{1}{|K^\star|}\sum_{\mathcal{D}\sim K^\star} \mathcal{F}_{\mathcal{D}}\cdot\mathbf{N}^{K^\star}_{\mathcal{D}}}$$
- Discrete Duality (the key feature):

  $$\boxed{\forall u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}_0 \ \forall \mathcal{F}^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}} \quad [\![\mathrm{div}^{\mathcal{T}}\mathcal{F}^{\mathfrak{D}}, u^{\mathcal{T}}]\!] = -\{\mathcal{F}^{\mathfrak{D}}, \nabla^{\mathfrak{D}} u^{\mathcal{T}}\}}$$

  Proof: rearrange per diamond $[\![.,.]\!] = \sum_K \sum_{\mathcal{D}\sim K} + \sum_{K^\star}\sum_{\mathcal{D}\sim K^\star}$

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The core properties of DDFV (cf. "gradient schemes" calculus)

- Discrete Duality : $-\text{div}^{\mathcal{T}}$ in duality with $\nabla^{\mathfrak{D}}$
- (Strong) consistency of $\nabla^{\mathfrak{D}} \circ \mathbb{P}^{\overline{\mathcal{T}}}$

$$\forall \phi \in C_c^1(\Omega), \quad \left\| \nabla^{\mathfrak{D}} \mathbb{P}^{\overline{\mathcal{T}}} \phi - \nabla \phi \right\|_{\infty} \longrightarrow 0 \text{ as } h_{\mathcal{T}} \to 0$$

Proof: exactness of $\nabla^{\mathfrak{D}}$ on affine functions + Taylor

- Weak consistency of $\text{div}^{\mathcal{T}} \circ \mathbb{P}^{\mathfrak{D}}$ (cf. [Droniou M3AS'14] survey):

$$\forall \psi \in C_c^1(\Omega; \mathbb{R}^2), \text{ provided } \|\nabla^{\mathfrak{D}} w^{\overline{\mathcal{T}}}\|_{L^1(\Omega)} \leq const,$$

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Proof: calculations similar to the proof of Discrete Duality
+ approx. of ψ by averages over σ / $\sigma^\star$ + uniform mesh regularity

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Compactness, reconstruction, penalization. DDFV vs gradient schemes.

A toy scheme Discretization of the toy pb. $u \in H_0^1(\Omega),\ -\Delta u = f(u)$:

$$\text{find } u^{\overline{\mathcal{T}}} \in \mathbb{R}_0^{\overline{\mathcal{T}}} \text{ s.t. } \forall v^{\overline{\mathcal{T}}} \in \mathbb{R}_0^{\overline{\mathcal{T}}} \ \{\nabla^{\mathfrak{D}} u^{\overline{\mathcal{T}}}, \nabla^{\mathfrak{D}} v^{\overline{\mathcal{T}}}\} \underbrace{+ h_{\mathcal{T}} \sum_{K \cap K^\star} \dots}_{\text{penalization}} = [\![f(u^{\overline{\mathcal{T}}}), v^{\overline{\mathcal{T}}}]\!]$$

A compactness feature

Given $(\mathcal{T}_n)_n$ a family of DDFV meshes with $h_{\mathcal{T}_n} \to 0$ as $n \to \infty$,
assume the meshes obey a uniform regularity property.
Assume that discrete functions $u^{\overline{\mathcal{T}}_n}$ have bounded $\left\| \nabla^{\mathfrak{D}_n} u^{\overline{\mathcal{T}}_n} \right\|_{L^2(\Omega)}$.
Then, up to a subsequence,

$$u^{\mathcal{T}_n} : x \mapsto \underbrace{\tfrac{1}{2}\left(u^{\mathfrak{M}_n}(x) + u^{\mathfrak{M}_n^\star}(x)\right)}_{\text{the DDFV reconstruction}} \text{ converges to } u, \quad \text{with } u \in H_0^1(\Omega)$$

in the sense: $u^{\mathcal{T}_n} \to u$ strongly in L^2 and $\nabla^{\mathfrak{D}_n} u^{\mathcal{T}_n} \to \nabla u$ weakly in L^2.
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$\rightsquigarrow$ **a question: DDFV is it a gradient scheme?**
NO... but YES, if $[\![f(u^{\mathcal{T}}), v^{\mathcal{T}}]\!]$ replaced by $\sum_{K,K^\star} |K \cap K^\star| f\left(\frac{u_K + u_{K^\star}}{2}\right) \frac{v_K + v_{K^\star}}{2}$

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$$\underbrace{u^{\mathcal{T}_n} : x \mapsto \tfrac{1}{2}\left(u^{\mathfrak{M}_n}(x) + u^{\mathfrak{M}^\star_n}(x)\right)}_{\text{the DDFV reconstruction}} \text{ converges to } u, \quad \text{with } u \in H^1_0(\Omega)$$

in the sense: $u^{\mathcal{T}_n} \to u$ strongly in L^2 and $\nabla^{\mathfrak{D}_n} u^{\mathcal{T}_n} \to \nabla u$ weakly in L^2.
Moreover, if $u^{\mathfrak{M}_n} - u^{\mathfrak{M}^\star_n} \to 0$ in L^2, then $u^{\mathfrak{M}_n}, u^{\mathfrak{M}^\star_n} \to u$
(needed to get $[\![f(u^{\mathcal{T}_n}), v^{\mathcal{T}_n}]\!] \to \int_\Omega f(u)v$, in passage to the limit)

$\rightsquigarrow$ **a question: DDFV is it a gradient scheme?**
NO... but YES, if $[\![f(u^{\mathcal{T}}), v^{\mathcal{T}}]\!]$ replaced by $\sum_{K,K^\star} |K\cap K^\star| f\left(\frac{u_K+u_{K^\star}}{2}\right) \frac{v_K+v_{K^\star}}{2}$

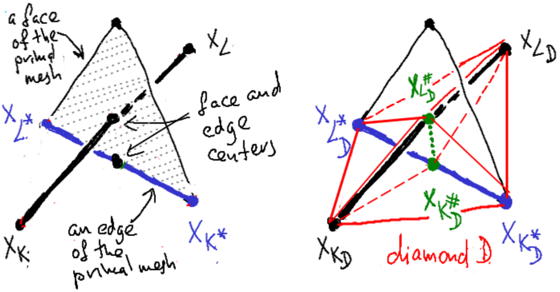
The 3D CeVeFE-DDFV scheme

precursor: [Hermeline CMAME'07]

main: [Coudière, Hubert '09, puis SIAM JSC'11]

variant: [Eymard, Herbin, Guichard M2AN'12]
[Droniou, Eymard, Herbin M2AN'16]

Construction of 3D CeVeFE-DDFV [Coudière, Hubert '09,'11]



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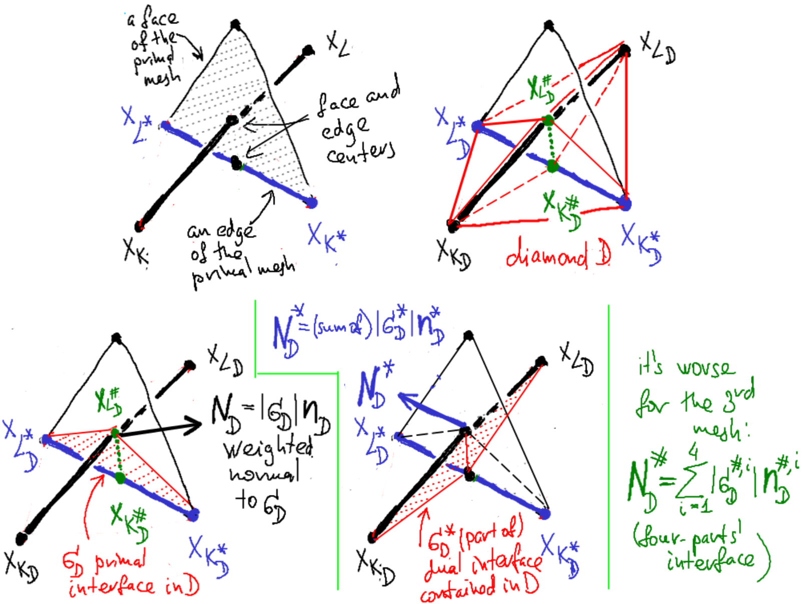
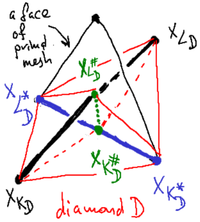


Summary of 3D CeVeFE-DDFV [Coudière, Hubert '09,'11]



x_{K_D}, x_{L_D} cell centers

$x_{K^*_D}$, $x_{L^*_D}$ neighbour vertices of a cell face = dual cell centers

$x_{L^\#_D}$ face center, $x_{K^\#_D}$ edge center } = cell centers of the 3rd mesh

three directions $\overrightarrow{x_{K_D}x_{L_D}}$ $\overrightarrow{x_{K^*_D}x_{L^*_D}}$ $\overrightarrow{x_{K^\#_D}x_{L^\#_D}}$

$N_D = |\sigma_D|\, n_D$ ← unit normal to the face; $|\sigma_D|$: portion of face included into D

$N^*_D \approx |\sigma^*_D|\, n^*_D$

$N^\#_D \approx |\sigma^\#_D|\, n^\#_D$ } objects for dual/3rd mesh

three weighted normals

NB: Precise shape of volumes $K^\#$ does not count (invariance of $\mathbf{N}^\#$)

- Discrete gradient defined per diamond by

$$\nabla_{\mathcal{D}} u^{\mathcal{T}} := \frac{1}{3|\mathcal{D}|}\Big((u_{L_{\mathcal{D}}} - u_{K_{\mathcal{D}}})\mathbf{N}_{\mathcal{D}} + (u_{L^\star_{\mathcal{D}}} - u_{K^\star_{\mathcal{D}}})\mathbf{N}^\star_{\mathcal{D}} + (u_{L^\#_{\mathcal{D}}} - u_{K^\#_{\mathcal{D}}})\mathbf{N}^\#_{\mathcal{D}}\Big)$$

  Per-projection representation analogous to 2D implies exactness:

  $\nabla_{\mathcal{D}} u^{\mathcal{T}} \cdot \overrightarrow{x_{K_{\mathcal{D}}} x_{L_{\mathcal{D}}}} = u_{L_{\mathcal{D}}} - u_{K_{\mathcal{D}}}$, idem for projections on $\overrightarrow{x_{K^\star_{\mathcal{D}}} x_{L^\star_{\mathcal{D}}}}$, $\overrightarrow{x_{K^\#_{\mathcal{D}}} x_{L^\#_{\mathcal{D}}}}$

- Finite Volume discrete divergence, per primal/dual/3rd mesh cell:

$$\mathrm{div}_K \mathcal{F}^{\mathfrak{D}} = \frac{1}{|K|}\sum_{\mathcal{D}\sim K} \mathcal{F}_{\mathcal{D}} \cdot \mathbf{N}^K_{\mathcal{D}} \quad (\mathbf{N}^K_{\mathcal{D}} = \pm \mathbf{N}_{\mathcal{D}}), \quad \text{idem } \mathrm{div}_{K^\star}\mathcal{F}^{\mathfrak{D}},\ \mathrm{div}_{K^\#}\mathcal{F}^{\mathfrak{D}}$$

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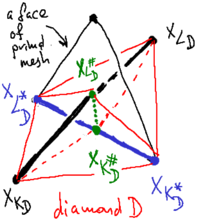

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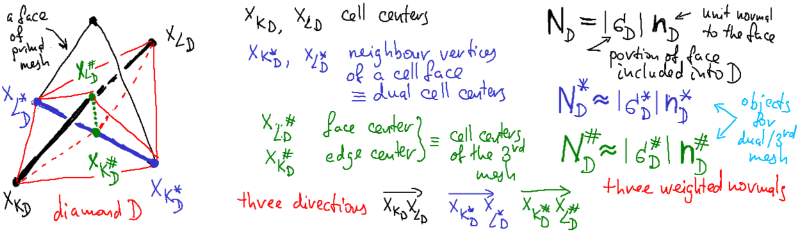

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3D CeVeFE-DDFV shares the core properties of 2D DDFV

- Discrete Duality is proved by elementary rearrangements: operators $-\mathrm{div}^{\mathcal{T}}\colon (\mathbb{R}^3)^{\mathfrak{D}} \to \mathbb{R}^{\mathcal{T}}$ and $\nabla^{\mathfrak{D}} : \mathbb{R}^{\mathcal{T}} \to (\mathbb{R}^3)^{\mathfrak{D}}$ fulfill

$$\forall u^{\mathcal{T}} \in \mathbb{R}_0^{\mathcal{T}}\ \forall \mathcal{F}^{\mathfrak{D}} \in (\mathbb{R}^3)^{\mathfrak{D}} \quad [\![\mathrm{div}^{\mathcal{T}} \mathcal{F}^{\mathfrak{D}}, u^{\mathcal{T}}]\!] = -\left\{ \mathcal{F}^{\mathfrak{D}}, \nabla^{\mathfrak{D}} u^{\mathcal{T}} \right\}$$

  if one takes $\left\{\mathcal{F}, \mathcal{G}\right\} = \sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| \mathcal{F}_{\mathcal{D}} \cdot \mathcal{G}_{\mathcal{D}}$ and defines $[\![., .]\!]$ by:

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- Weak consistency of $\mathrm{div}^{\mathcal{T}} \circ \mathbb{P}^{\mathfrak{D}}$ is also ok, as well as compactness claim, under uniform mesh regularity and employing penalization
- coercivity is true for any mesh geometry

Variant: [Eymard, Herbin, Guichard M2AN'12 / Droniou, E., H. M2AN'16]
Like for 2D DDFV, one can construct the 3D CeVeFE-DDFV meshes starting from a diamond mesh consisting of octahedra.
NB: The method is a gradient scheme, up to nonlinearities $f(u)$ etc.

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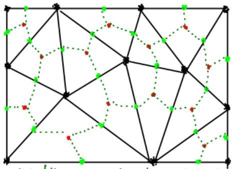
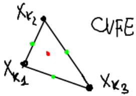
The 2D co-volume (CVFE) scheme

2D co-volume (CVFE) scheme [Cai, Num. Math.'91], [Mikula's team]



Donald / "median dual" mesh $\mathcal{T}$



CVFE diamond D . Space $(\mathbb{R}^2)^{\mathfrak{D}}$, $\{\!\{ \mathcal{F}^{\mathfrak{D}}, \mathcal{G}^{\mathfrak{D}} \}\!\} = \sum_{D\in\mathfrak{D}} |D| \mathcal{F}_D \cdot \mathcal{G}_D$

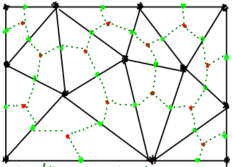
CVFE cell K Spaces $\mathbb{R}^{\mathcal{T}}$, $\mathbb{R}_0^{\mathcal{T}}$ $[\![u^{\mathcal{T}}, v^{\mathcal{T}}]\!] = \sum_{K\in\mathfrak{M}} |K| u_K v_K$

- Discrete gradient $\nabla^{\mathfrak{D}} u^{\mathcal{T}}$: Finite Element reconstruction per D
- Discrete divergence $\operatorname{div}^{\mathcal{T}} \mathcal{F}^{\mathfrak{D}}$: Finite Volume expression per K

Claim: $[\![-\operatorname{div}^{\mathcal{T}} \mathcal{F}^{\mathfrak{D}}, u^{\mathcal{T}}]\!] = \{\!\{ \mathcal{F}^{\mathfrak{D}}, \nabla^{\mathfrak{D}} u^{\mathcal{T}} \}\!\}$

[A., Bendahmane, Karlsen '08]

The 2D co-volume (CVFE) scheme is a Discrete Duality scheme



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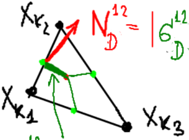
[A., Bendahmane, Karlsen '08]

Key ingredient: $\nabla_D u^{\mathcal{T}} = \frac{2}{|D|}\Big((u_{K_2}-u_{K_1}) N_D^{12} + (u_{K_3}-u_{K_2}) N_D^{23} + (u_{K_1}-u_{K_3}) N_D^{31} \Big)$



$N_D^{12} = |\sigma_D^{12}|\, n_D^{12}$

σ_D^{12} part of the interface $K_1|K_2$ contained within diamond D

Claims (the 2nd "magical formula") [A., Bendahmane, Hubert, Krell IMAJNA '12]

- this formula represents the gradient of the affine per D (Finite Element) reconstruction from DOFs $(u_K)_{K\in\mathcal{T}}$ ⇝ exactness of $\nabla^{\mathfrak{D}}$
- exactness remains true if D has >3 vertices (example: scheme of [A., Boyer, Hubert M^{2}AN '04] on rectangles)
- this formula leads to Discrete Duality

Properties of the co-volume (CVFE) scheme

The co-volume scheme of [Cai '91], aka Afif-Amaziane scheme, aka Control Volume Finite Element scheme (CVFE, sometimes FVE), aka lumped FE scheme, is a gradient scheme [Droniou et al.]...

Moreover, regarding the core properties of DDFV, co-volume/CVFE can also be seen as a Discrete Duality Finite Volume scheme !

Indeed,

- discrete spaces, operators analogous to DDFV (though simpler)
- Discrete Duality is OK
- Strong consistency of Discret Gradient is OK
- Weak (dual) consistency of Discrete Divergence is OK
- coercivity is ok on triangular/quadrangular diamond meshes
  $\leadsto$ however, on general diamond meshes coercivity fails
  $\leadsto$ cf. the 2nd part of the talk [A., Quenjel in preparation]

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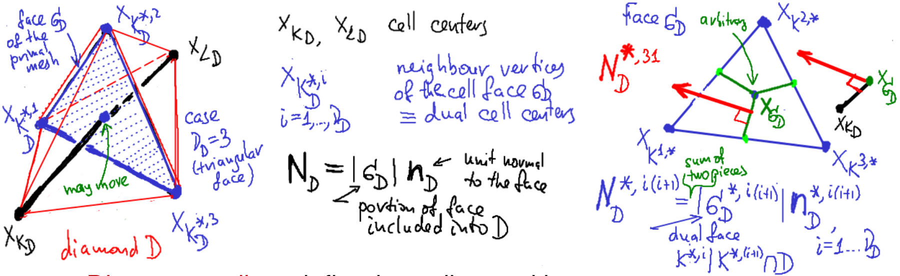
The 3D CeVe-DDFV scheme

[Pierre PhD'05], [Coudière, Pierre et al. FVCA'08, IJFV'09]

[Hermeline JCP'09]

[A., Bendahmane, Karlsen FVCA'08, JHDE'10]
[A., Bendahmane, Hubert, Krell IMAJNA'12, CMAM'13]

Summary of 3D CeVe-DDFV [A., Bendahmane, Hubert, Krell IMAJNA'12]



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Projections view ($\Rightarrow$ exactness): $\nabla_{\mathcal{D}} u^{\mathcal{T}} \cdot \overrightarrow{x_{K_{\mathcal{D}}} x_{L_{\mathcal{D}}}} = u_{L_{\mathcal{D}}} - u_{K_{\mathcal{D}}}$ and

$$\mathrm{Proj}_{\text{face } \sigma_{\mathcal{D}}}(\nabla_{\mathcal{D}} u^{\mathcal{T}}) \text{ is the (exact) 2D CVFE reconstruction}$$

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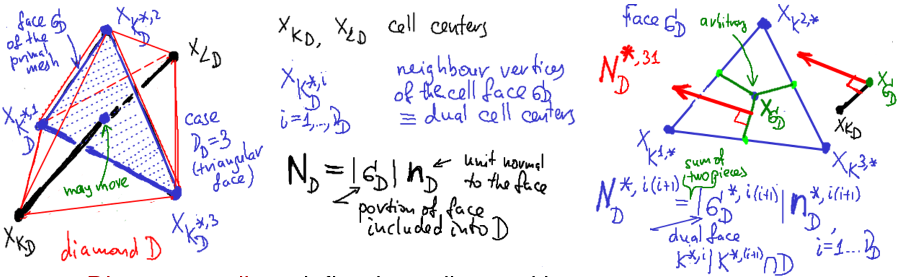
$$\mathrm{div}_K \mathcal{F}^{\mathfrak{D}} = \frac{1}{|K|}\sum_{\mathcal{D}\sim K} \mathcal{F}_{\mathcal{D}} \cdot \mathbf{N}_{\mathcal{D}}^K \quad (\mathbf{N}_{\mathcal{D}}^K = \pm \mathbf{N}_{\mathcal{D}}), \quad \text{idem } \mathrm{div}_{K^\star} \mathcal{F}^{\mathfrak{D}}$$

Summary of 3D CeVe-DDFV [A., Bendahmane, Hubert, Krell IMAJNA'12]



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- coercivity is true at least for
  - meshes with faces that are triangles (including simplicial meshes)
  - cartesian meshes and meshes topologically equivalent to cartesian

A variant [Ch. Pierre PhD'05, Coudière et al. '08,'09]
The construction where $K^\star$ overlap, to cover the domain twice, is equivalent to the above CeVe construction

3D CeVe-DDFV shares the core properties of 2D DDFV

- Discrete Duality is proved by elementary rearrangements: operators $-\mathrm{div}^{\mathcal{T}}\colon (\mathbb{R}^3)^{\mathfrak{D}} \to \mathbb{R}^{\mathcal{T}}$ and $\nabla^{\mathfrak{D}} : \mathbb{R}^{\mathcal{T}} \to (\mathbb{R}^3)^{\mathfrak{D}}$ fulfill

$$\forall u^{\mathcal{T}} \in \mathbb{R}_0^{\mathcal{T}}\ \forall \mathcal{F}^{\mathfrak{D}} \in (\mathbb{R}^3)^{\mathfrak{D}} \quad [\![\mathrm{div}^{\mathcal{T}} \mathcal{F}^{\mathfrak{D}}, u^{\mathcal{T}}]\!] = -\{\mathcal{F}^{\mathfrak{D}}, \nabla^{\mathfrak{D}} u^{\mathcal{T}}\}$$

if one takes $\{\mathcal{F}^{\mathfrak{D}}, \mathcal{G}^{\mathfrak{D}}\} = \sum_{\mathcal{D}\in\mathfrak{D}} |\mathcal{D}| \mathcal{F}_{\mathcal{D}} \cdot \mathcal{G}_{\mathcal{D}}$ and defines $[\![.,.]\!]$ by:

$$[\![u^{\mathcal{T}}, v^{\mathcal{T}}]\!] := \frac{1}{3} \sum_{K\in\mathfrak{M}} |K|\, u_K v_K + \frac{2}{3} \sum_{K^\star\in\mathfrak{M}^\star} |K^\star|\, u_{K^\star} v_{K^\star}$$

- (Strong) consistency of $\nabla^{\mathfrak{D}}\circ\mathbb{P}^{\mathcal{T}}$ comes from exactness
- Weak consistency of $\mathrm{div}^{\mathcal{T}}\circ\, \mathbb{P}^{\mathfrak{D}}$ is also ok, as well as compactness claim, under uniform mesh regularity and employing penalization
- coercivity is true at least for
  - meshes with faces that are triangles (including simplicial meshes)
  - cartesian meshes and meshes topologically equivalent to cartesian

A variant [Ch. Pierre PhD'05, Coudière et al. '08,'09]
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Conclusions:

- the analytical framework of DDFV schemes mainly consists of
  - meshes, discrete spaces, gradient and divergence operators
  - duality calculus for these operators
  - strong consistency for $\nabla^{\mathcal{D}}$, dual consistency for $\mathrm{div}^{\mathcal{T}}$
    $\rightsquigarrow$ compactness, reconstruction
- there are two different kinds of 3D DDFV schemes
  $\rightsquigarrow$ CeVeFE-DDFV (Cell+Vertex+Face+Edge unknowns): 3D=1D+1D+1D
  $\rightsquigarrow$ CeVe-DDFV (Cell+Vertex unknowns): 3D=1D+2D
- the 2D co-volume scheme (aka Control Volume Finite Element,...) can/should also be seen as a Discrete Duality scheme

Perspectives:

- other Discrete Duality schemes $\rightsquigarrow$ Quenjel's nodal NDD scheme
- Benefit from the DDFV reconstruction $u^{\mathcal{T}} = \frac{1}{2}(u^{\mathfrak{M}} + u^{\mathfrak{M}^*})$?
- 4D adaptation of CeVeFE idea looks natural...
  $\rightsquigarrow$ what about 4D CeVe-DDFV (and 3D co-volume) schemes?

Thank you / Merci !

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